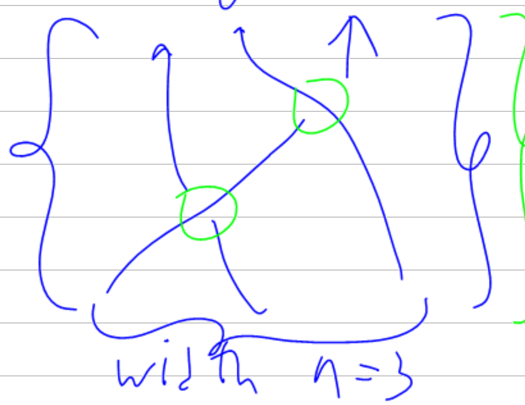
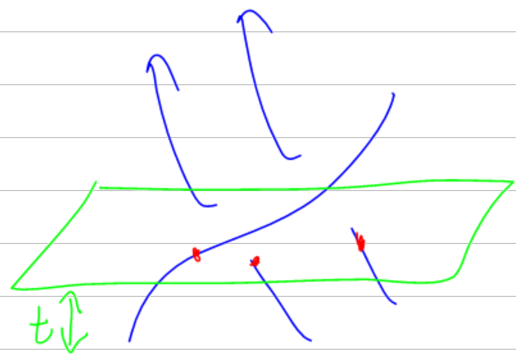
Hour 32, Monday November 30: Braids and knots, combing.
Should we have a "HW discussion" class?

B_n = The braid group on n strands:

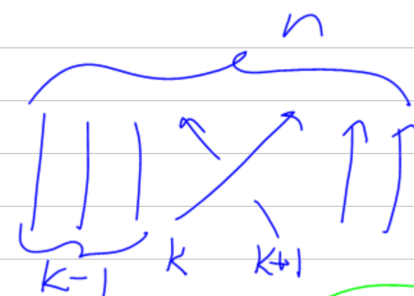
$B_n = \Big\{$  $\Big\}$ / length.



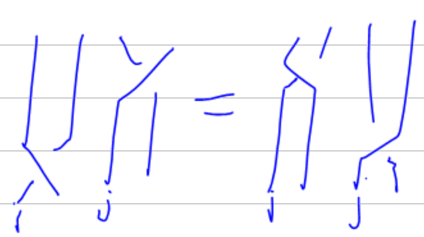
Braid $\mapsto \beta : [0,1] \to \tilde{C}_n$

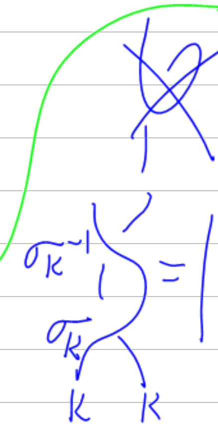
$$C_n = \{(z_1 \dots z_n) \in \mathbb{C}^n : i \neq j \Rightarrow z_i \neq z_j\}$$

$$B_n = \pi_1(\tilde{C}_n) \qquad \tilde{C}_n = C_n / S_n$$

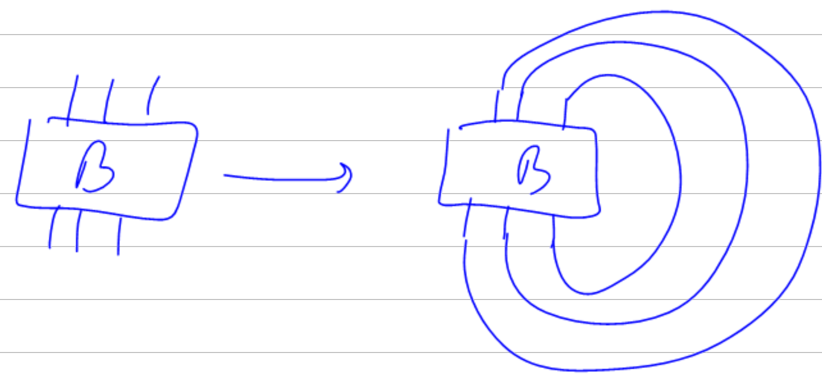
$\sigma_k =$ 

$$B_n = \langle \sigma_1 \dots \sigma_{n-1} : \sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j|>1 \rangle$$

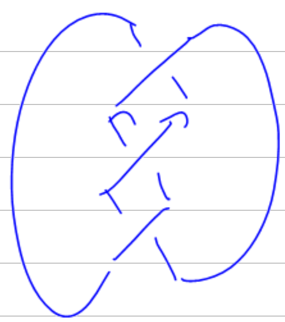




Map $\overset{\text{Alexander}}{A} : B_n \longrightarrow \left\{ \begin{array}{l} \text{knots} \\ \text{\& links} \end{array} \right\}$

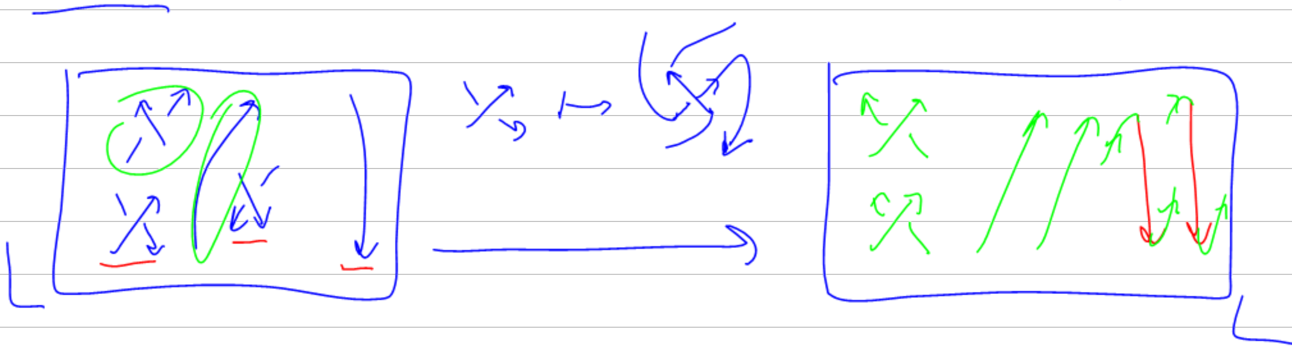


$\sigma_1^{\,3} \in B_2 \xmapsto{\quad A \quad}$ Trefoil

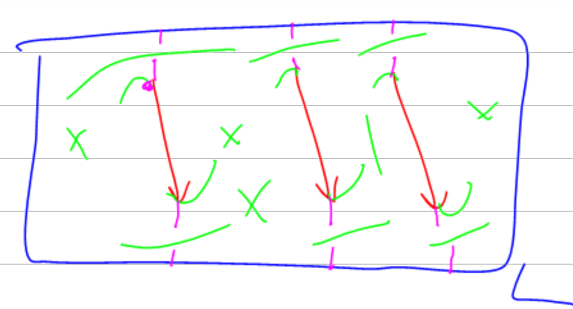


Alexander's thm:
A is onto.
Any knot or link is a closure of a braid.

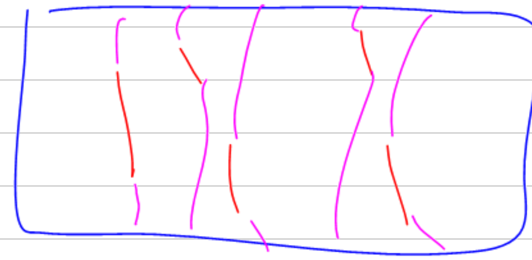
Pf schematic pictorial pf



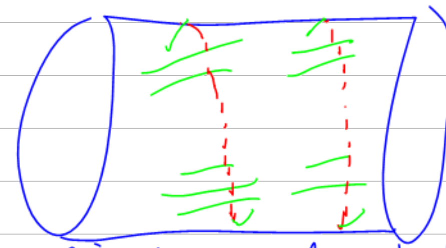
Connect all mins to bottom & all maxes to top with purple lines that run under everything, &



s.t. red & purple never cross.



pull along purple

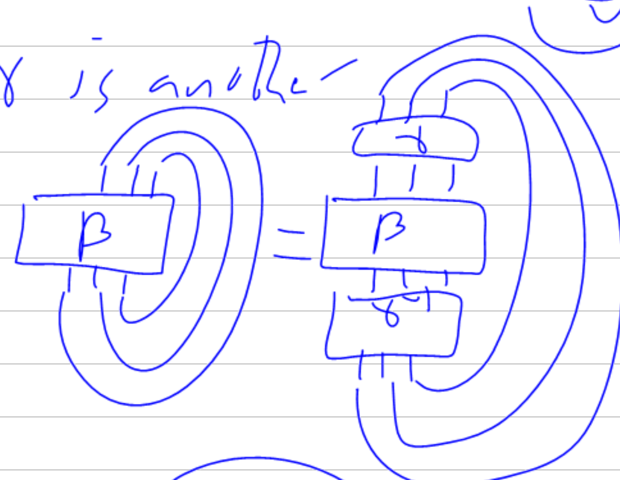


This is a closed braid

Markov's Thm β_1 & β_2 are braids, then $A(\beta_1) = A(\beta_2)$ iff β_1 & β_2 differ by a sequence of moves of the following kinds:

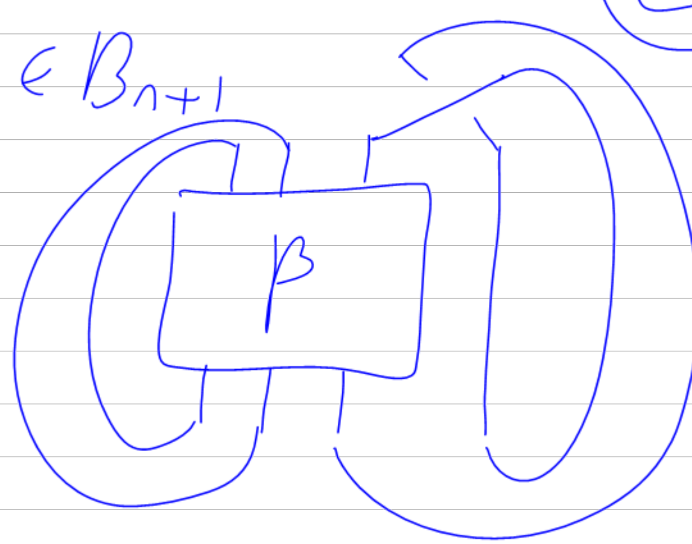
0. R-moves on braids

1. $\beta \mapsto \gamma^{-1}\beta\gamma$ where γ is another braid.



2. if $\beta \in B_n$ then

$B_n \ni \beta \sim \beta \cdot \sigma_n^{\pm 1} \in B_{n+1}$

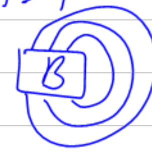


Harder if $A(\beta_1) = A(\beta_2)$
$\Rightarrow \beta_1 \sim \beta_2$

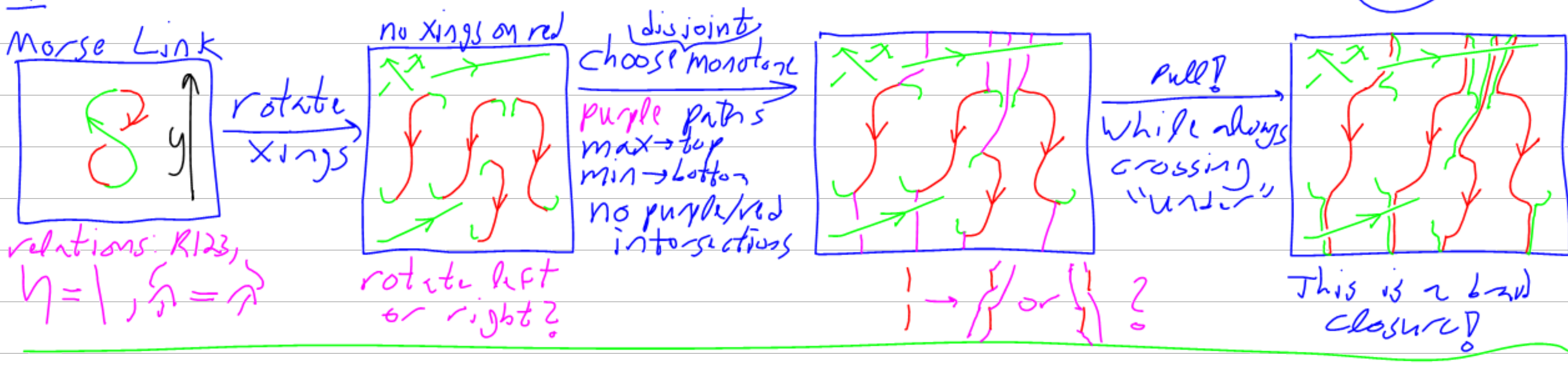
Hour 33, Wednesday December 2: Combing braids.
IOU a correction for the last bit of the proof of unique factorization.
HW10 on web by Thu midnight!

$C_n = \{z \in \mathbb{C}^n : z_i \neq z_j \text{ for } i \neq j\}$ $\quad \tilde{C}_n = C_n / S_n$

$$B_n := \pi_1(\tilde{C}_n) = \left\langle \sigma_i,\ 1 \le i \le n-1 : \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j|>1 \end{array} \right\rangle$$

Alexander's Thm. Every knot/link is a braid closure: 

Pf

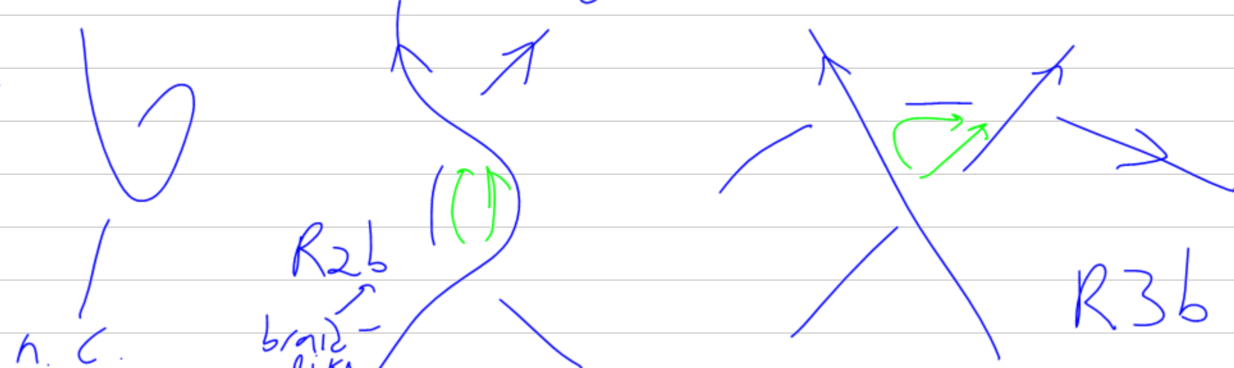


Comments 1. Computationally inefficient. 

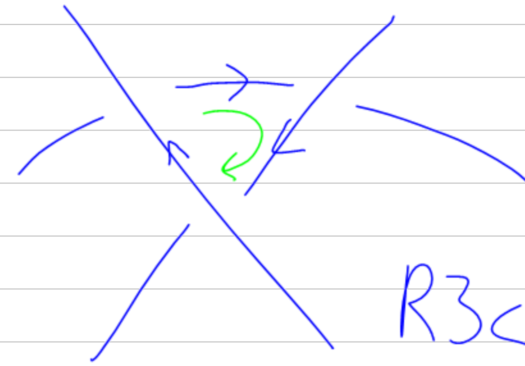
knot of complexity n $\longleftarrow$ braid of complexity $\cancel{n^2}$ $n^{3/2}$

✓ locally

Open problem. Find a sequence K_n of n-xing knots such that the minimal length $l(K_n)$ of a braid whose closure is K_n grows faster than Cn.

2. 

R2c
cyclic



R3c

knots · diagrams

$$\mathcal{K} = \mathcal{D} \Big/ R1 \quad R2_{b\&c} \quad R3_{b\&c}$$

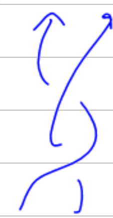
$$\mathcal{K}' = \mathcal{D} \Big/ R1, R2b, R3b$$

Markov's thm implies that $\mathcal{K}'$ contains a copy of $\mathcal{K}$.

* There are elements of $\mathcal{K}'$ that are not braid closures!
* If you regard knots as braid closures, there are fewer moves to check.

  I know just one example where this is useful!

(*)
$$1 \to PB_n \to B_n \xrightarrow{\quad} S_n \to 1 \qquad B_n = \pi_1(\tilde{C}_n)$$

"Pure braids": braids that induce the identity perm. E.g.  $=\sigma_1^2$

$PB_n = \pi_1(C_n)$

1. studying B_n & PB_n is more or less the same. $\beta_1 \cdot \beta_2^{-1}$
2. Yet (*) does not "split".

Aside on split exact sequence:

what means where $p \circ s = Id = s // p$

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] & C \arrow[r] \arrow[l, bend left, "s"] & 1 \end{tikzcd} \quad (\text{Groups})$$

$b \qquad c = pb$

0. $B > A$ $(B > iA)$ $\quad B > C$ $(B > sC)$

1. $B = A \cdot C$ (not $B = A \times C$)

Also need $A \cap B = \{1\}$!

$b = \underbrace{\alpha}\, s(p(b))$ $\quad \alpha = b \cdot (spb)^{-1}$ $\quad p\alpha = 1$ $\quad$ So $\alpha \in iA$.

2. $A \lhd B$ as $A = \ker p$.

So $B = A \rtimes C$. $\quad B_n = \cancel{PB_n \rtimes S_n}$

Hour 34, Friday December 4: Combing braids.
IOU a correction for the last bit of the proof of unique factorization.
HW10 on web!

$$1 \to PB_n \to B_n \xrightarrow{\sigma} S_n \to 1$$

1. Studying B_n & PB_n is more or less the same.
2. Not split?

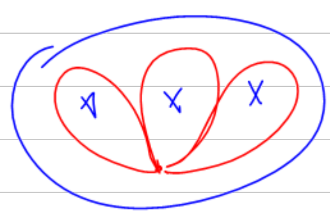
Aside on split exact: $1 \to A \xrightarrow{i} B \underset{s}{\overset{p}{\rightleftarrows}} C \to 1$ $\quad s/\!/p = Id_C$

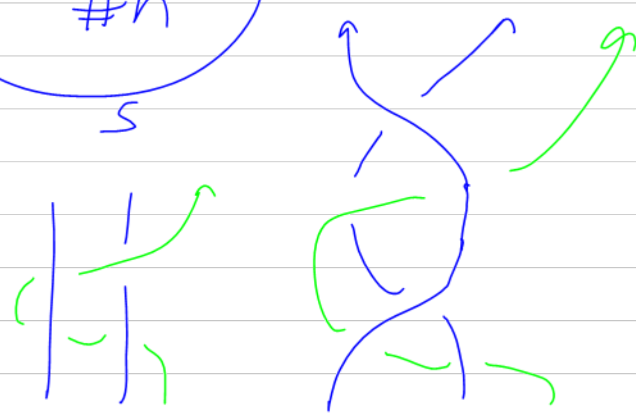
1. $B = AC$ and $A \cap C = \{1\}$ $\qquad$ So
2. $A \lhd B$ hence C acts on A. $\qquad B = A \rtimes C$

$\pi_1(D \setminus {n-1 \atop \text{pts}})$

$$1 \to F_{n-1} \to PB_n \underset{\text{drop strand \#}n}{\overset{d}{\rightleftarrows}} PB_{n-1} \to 1$$

s

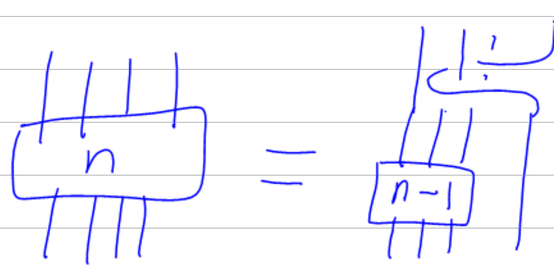


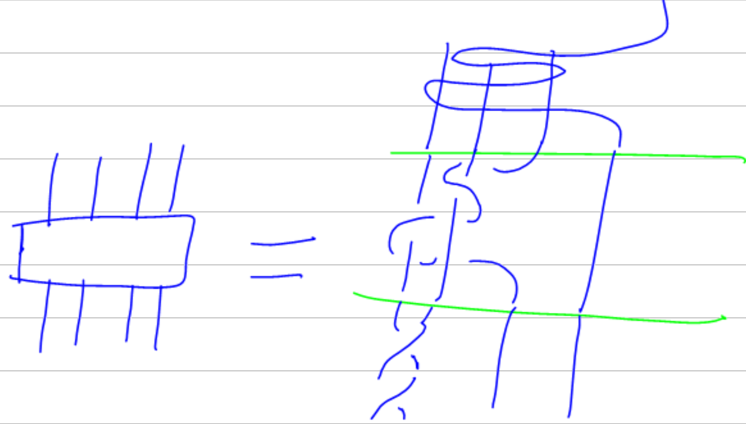


$$PB_n = PB_{n-1} \ltimes F_{n-1}$$
$$= (PB_{n-2} \ltimes F_{n-2}) \ltimes F_{n-1}$$
$$= ((F_1 \ltimes F_2) \ltimes F_3 \cdots) \ltimes F_{n-1}$$

$PB_1 = \{1\}$

$PB_2 = PB_1 \ltimes F_1$
$= \mathbb{Z}$

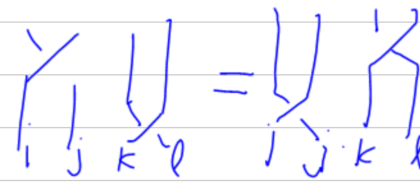




"combing the braid!"

$$B_n = \left\langle \sigma_i \quad 1 \le i \le n-1 \ : \ \begin{matrix} \sigma_i \sigma_j = \sigma_j \sigma_i & |i-j|>1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \end{matrix} \right\rangle$$

$\longrightarrow \sigma_{12}\sigma_{13}\sigma_{23}$



$=$

$\longrightarrow \sigma_{23}\sigma_{13}\sigma_{12}$

$$\left\langle \sigma_{ij} \quad 1 \le i \ne j \le n \ : \ \begin{matrix} \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} & \text{if } |\{i,j,k,l\}|=4 \\ \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} & \text{if } |\{i,j,k\}|=3 \end{matrix} \right\rangle$$

$\parallel$

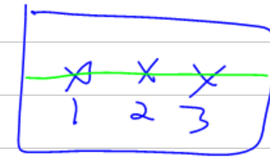
PvB_n "Pure virtual Braids"

$$PwB_n = PvB_n \Big/ \underbrace{\sigma_{ij}\sigma_{ik} = \sigma_{ik}\sigma_{ij}}_{\text{"overcrossings commute" } OC} \quad |\{i,j,k\}|=3$$

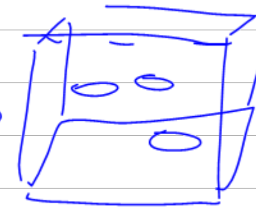
weakly-virtual
welded
warmup

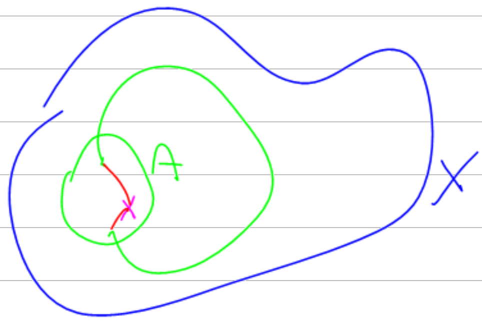
$PB_n = P_u B_n$ = "motion group of n pts in $\mathbb{R}^2$"

↑ usual



"motion group of n horiz rings in $\mathbb{R}^3$"





"group of horizontal flying rings"

$\pi_1(X, A)$ (A simply connected)

$P_w B_n = \pi_1$(horiz. flying rings, external to each other)

↑ claim

σ_{ij} = j flies through i in the positive dir.