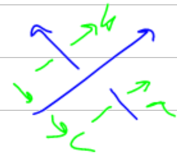
Hour 26, Monday November 16: More on Knot Colourings.

The Wirtinger presentation

$\pi_1(K) = \langle a_i \,;\, a = c^b \ldots \rangle$



$\rightarrow a = c^{-1} b c$
$= b^c$

Instead pick a finite group G, & colour by

$\mathrm{Hom}(\pi_1, G)$

Rels $a = b^c$

Improve further

$g \in G$ $\quad \mathrm{Hom}(\pi_1, g^G) := \{ \phi : \pi_1 \to G :$ 1. Homo... 2. For every generator of π_1, $\phi(a)$ is a conj. of g $\}$

Ex: $\mathbb{C} \to \mathbb{C}$ $\quad z \mapsto e^{\frac{2\pi i}{n}} z$

D_{2n} $\quad z \mapsto \bar{z}$

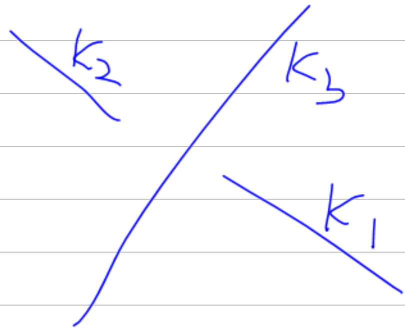
$= \left\{ (s,k) : \begin{array}{l} s \in \{\pm 1\} \\ k \in \mathbb{Z}/n \end{array} \right\}$

w/ $(s_1, k_1)(s_2, k_2) = (s_1 s_2, s_2 k_1 + k_2)$

$(s,k)^{-1} = (s, -sk)$

$(-1,0)^G = \{(-1,k)\}$ unless $n=2$

$$(-1,k_1)^{(-1,k_3)} = (-1,\, 2k_3 - k_1)$$



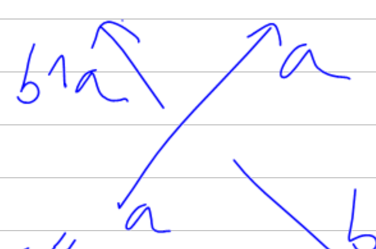
s.t. $k_2 = 2k_3 - k_1$

$k_1 + k_2 = 2k_3$

$k_i \in \mathbb{Z}/n$

If $n=3$, $\Rightarrow$ $k_1 + k_2 + k_3 = 0$.

Coloring rules:

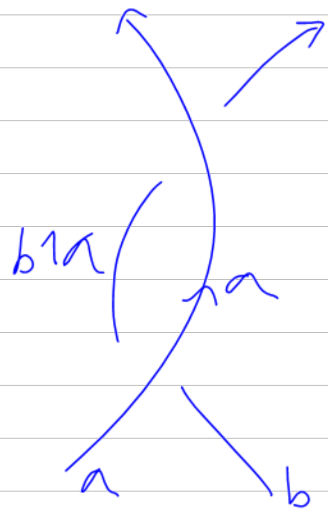


A set Q "of colours"

an operation $\wedge : Q \times Q \to Q$, s.t.

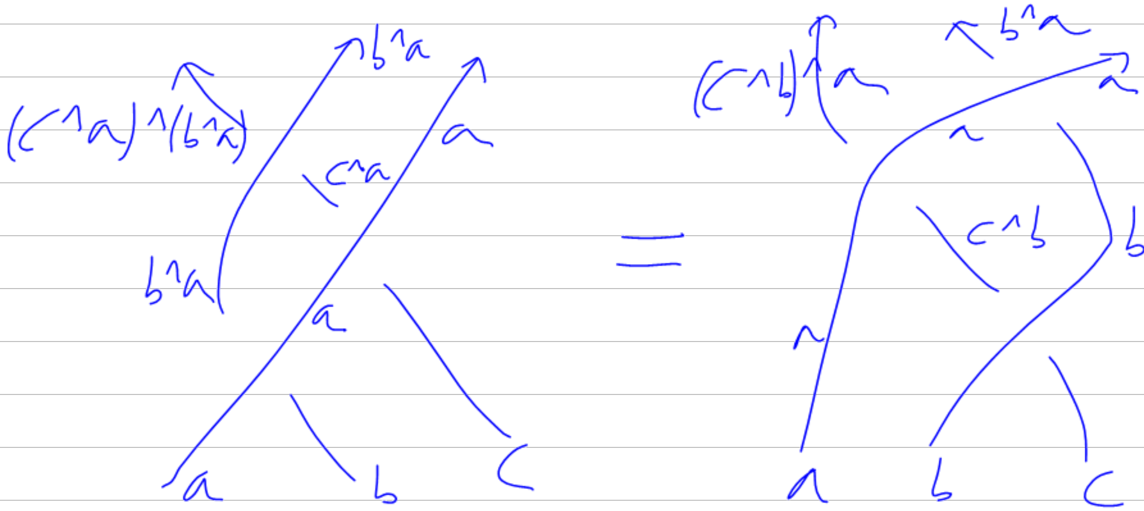
R1

1. $a \wedge a = a$



$(a \bar{\wedge} b)^\wedge b = a \quad \left(\text{think: } a^{(b^{-1})} \right)$

2. $\forall b$, $a \mapsto a^\wedge b$ is an invertible map $Q \to Q$



Easy exercise $a^\wedge b = a^b$ in G, satisfies this

3. $\forall a, b, c \in Q$
$(c^\wedge b)^\wedge a = (c^\wedge a)^\wedge (b^\wedge a)$
$(a^\wedge b)^\wedge c = (a^\wedge c)^\wedge (b^\wedge c)$

<u>Def</u> A quandle is a set Q w/ op
$^\wedge : Q \times Q \to Q$, s.t.

1. $x^\wedge x = x$
2. $\forall y$ $x \mapsto x^\wedge y$ is invertible
3. $(x^\wedge y)^\wedge z = (x^\wedge z)^\wedge (y^\wedge z)$

Axiom 3 is sometimes called
"$\wedge$ is self-distributive"

$$(x+y)\cdot z = x\cdot z + y\cdot z$$

$$(x\wedge y)\wedge z = (x\wedge z)\wedge(y\wedge z)$$

If $M, *$ is a set w/ binary op.
then every $m\in M$ defines

$$T_m : M \longrightarrow M$$

by $T_m(x) = x * m$

<u>Axiom</u> $\forall m$ T_m is an auto.

$$T_m(x*y) = (T_m x)*(T_m y)$$

$$(x*y)*m = (x*m)*(y*m)$$

At the infinitesimal level:

$$T_m : Q\times Q \longrightarrow Q\times Q$$

$L : Q \otimes Q \longrightarrow Q \otimes Q \qquad L_z x = x * z$

$$L(x \otimes y) = (Lx \otimes y) + (x \otimes Ly)$$

$$L(x * y) = (Lx) * y + x * (Ly)$$

$$(x * y) * z = (x * z) * y + x * (y * z)$$

$$x * y = [x, y]$$

$$[[x, y], z] = \_ \; \_ \; \_ \; \_ = \text{Jacobi Id.}$$

$$\mathrm{ad}_z : \mathfrak{g} \longrightarrow \mathfrak{g}$$

is a ~~homomorphism of Lie alg.~~

Added after class: I was in time pressure, so I said something wrong. The correct statement is "$[\,,\,] : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$ is an intertwiner of reps, when $\mathfrak{g}$ is a representation using the adjoint action".

Hour 27, Wednesday November 18: A bit more on quandles, Seifert surfaces.
HW8 on web by midnight tomorrow!

$\mathrm{ad}_z x = [x, z]$

Correction I said "$\mathrm{ad}_z : L \to L$ is a morphism of Lie Alg's". Nonsense!

$Q \times Q \xrightarrow{\wedge} Q$ is equivariant, meaning

$$\begin{tikzcd} Q\times Q \arrow[r, "\wedge"] \arrow[d, "\wedge z"] \arrow[dr, phantom, "\circlearrowright"] & Q \arrow[d, "\wedge z"] \\ Q \times Q \arrow[r, "\wedge"] & Q \end{tikzcd} \qquad \begin{tikzcd} L\otimes L \arrow[r, "{[\,,\,]}"] \arrow[d, "\mathrm{ad}_z"] \arrow[dr, phantom, "\circlearrowright"] & L \arrow[d, "\mathrm{ad}_z"] \\ L \otimes L \arrow[r, "{[\,,\,]}"] & L \end{tikzcd}$$

$\mathrm{ad}_z(x \otimes y)$
$\mathrm{ad}_z x \otimes y$
$+ x \otimes \mathrm{ad}_z y$

Quandles from groups:

1. $x \wedge y := y^{-1} x y$
2. $x \wedge y := y^{-n} x y^n$ } (can restrict to a conjugacy class)
3. $x \wedge y := y x^{-1} y$

Vendramin:

TABLE 2. The number of non-isomorphic indecomposable quandles

n	1	2	3	4	5	6	7	8	9	10	11	12
$q(n)$	1	0	1	1	3	2	5	3	8	1	9	10
n	13	14	15	16	17	18	19	20	21	22	23	24
$q(n)$	11	0	7	9	15	12	17	10	9	0	21	42
n	25	26	27	28	29	30	31	32	33	34	35	
$q(n)$	34	0	65	13	27	24	29	17	11	0	15	

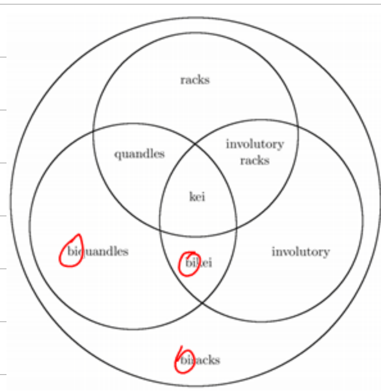
Conjeture 3.5. *Let p be an odd prime number and let Q be an indecomposable quandle of $2p$ elements. Then $p \in \{3, 5\}$.*

$Q(K) =$
$Q\langle a_i : \underbrace{a = b \wedge c}_{\text{for each xing}} \rangle$

Elhamdadi/Nelson: A whole book.

The Fundamental quandle of a knot (Joyce)

On beyond quandles!



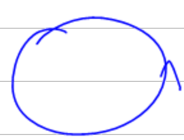
Aksoy, Nelson arXiv: 1102.1473.

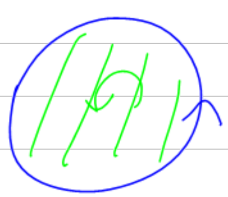
$(B, \wedge, \vee)$

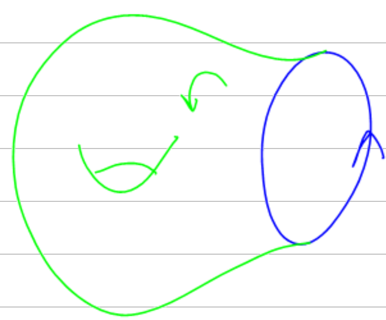
"biquandle"

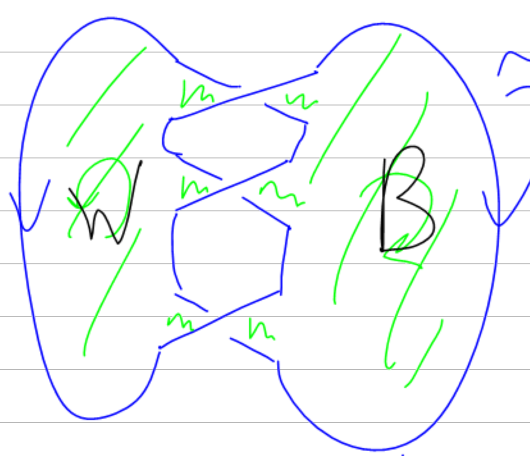
Def A "Seifert surface" for a knot K is a smooth ~~connected~~ oriented 2D surface with bndry Σ, whose oriented bndry is the knot:

$$\partial \Sigma = K$$

$K =$ 

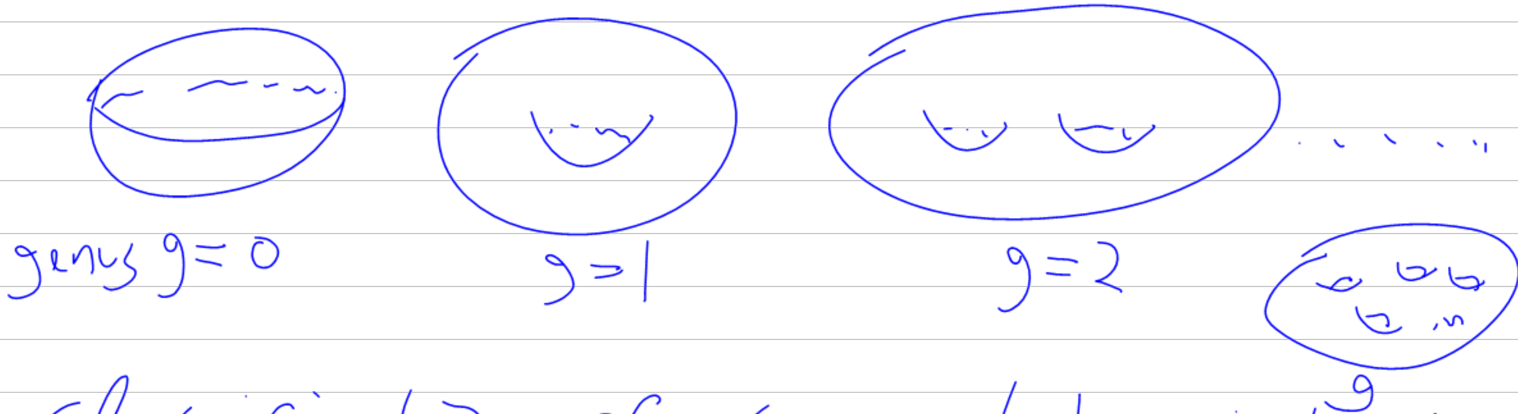






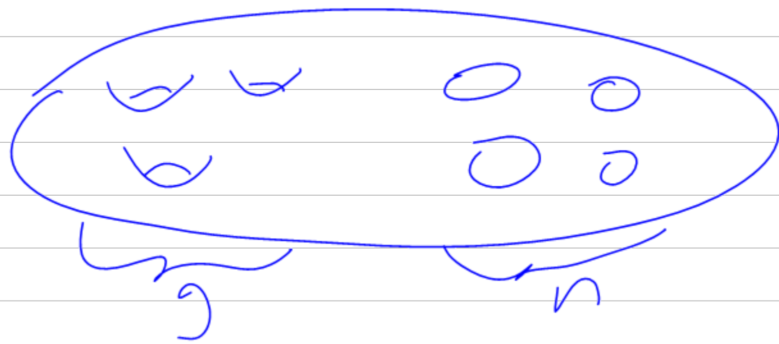
$\simeq \Sigma_{g,n}$

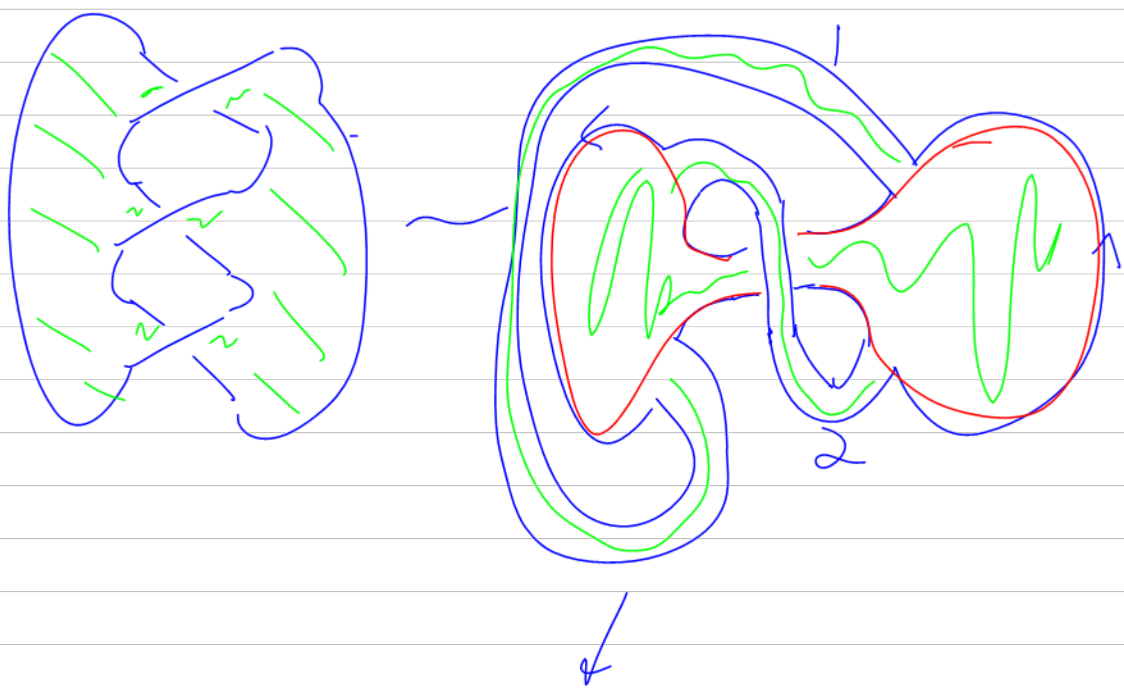
The classification of connected orientable surfaces: Every connected orientable surface is homeomorphic to one of the following:

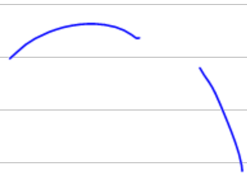


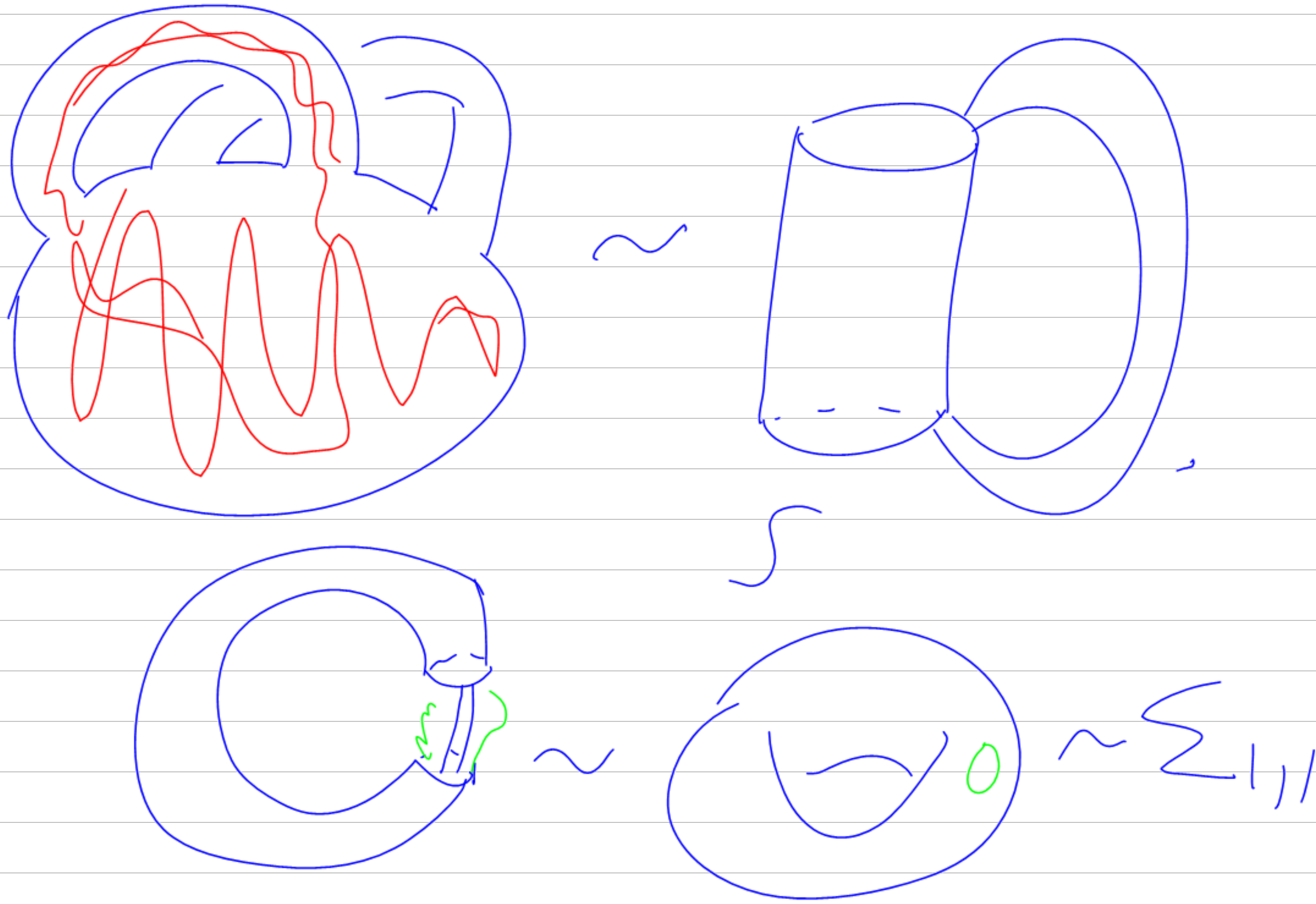
Classification of connected orientable surfaces w/ bndry: Every such surface is homeo. to a $\Sigma_{g,n}$

$\Sigma_{g,n}$








$\sim \Sigma_{1,1}$

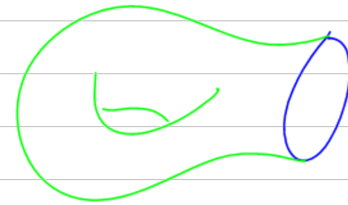
$\boxed{\text{Everything is smooth}}$

Def A Seifert surface for an oriented link K is a connected oriented surface $\Sigma \subset \mathbb{R}^3$ s.t. $\partial \Sigma = K$.

Example

$v=2 \quad e=3$

topologically this is a $\Sigma_{1,1}$ (g, n)



$\{\Sigma_{g,n}\}$

$$\chi(X) = \sum (-1)^r \dim H_r(X)$$

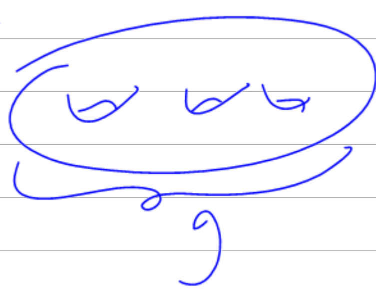
$$= \sum (-1)^r \dim C_r(X)$$

$C_0 = \{ \cdot \ \cdot \}$, $C_1 = \{ \diagdown \ | \}$, $C_2 = \{ \triangle \ \square \ \diamond \}$

in 2D $= V - e + F$ (verts, edges, faces)

in 1D $= V - e$

homotopy invts.

$\chi($  $) = 2 - 2g$

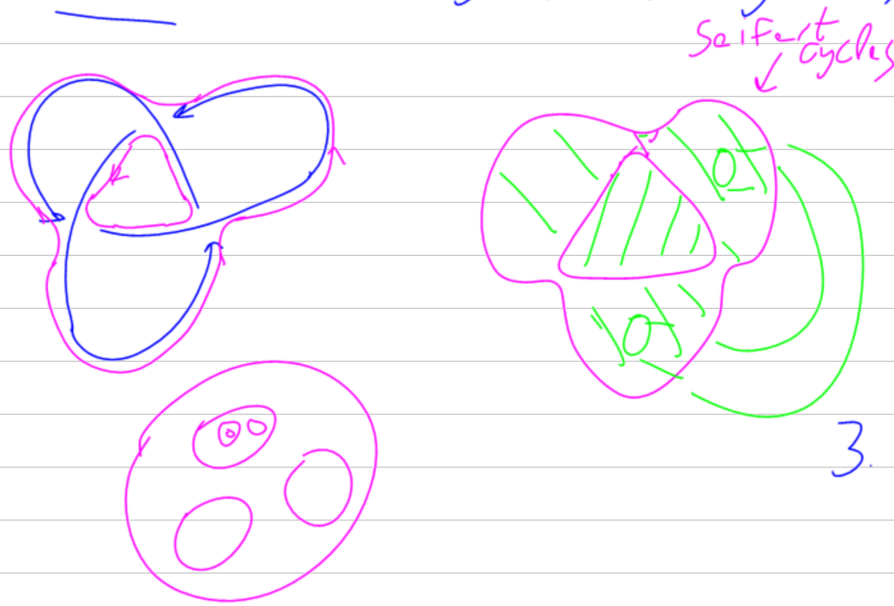
$$\chi(\Sigma_{g,n}) = 2 - 2g - n$$

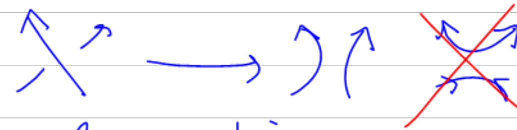
$$\chi = 2 - 3 = -1 = 2 - 2g - 1$$

$$g = 1$$

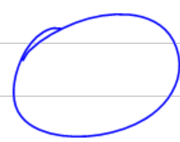
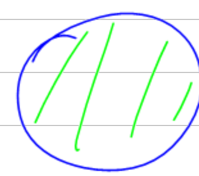
Thm Every K has a Σ.

Pf 1 "Use Seifert cycles"



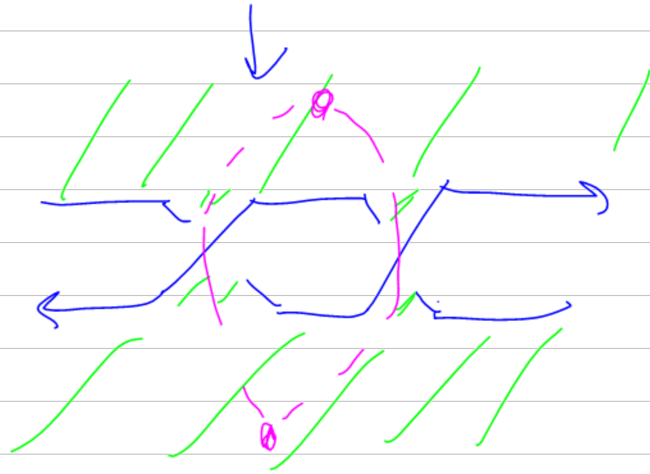
1. 
2. Place disks in $\mathbb{R}^3$ w/ boundaries the Seifert cycles
3. Wherever there was a xing, put a twisted band

Pf 2 By induction on unknotting number = # of xing you have to flip to get to the unknot.

$u=0$ $\quad K=$  $\quad \Sigma=$ 

If $K=\partial\Sigma$ and K' is obtained from K by flipping one xing, then $\exists\,\Sigma'$ s.t. $\partial\Sigma'=K'$.

K:  

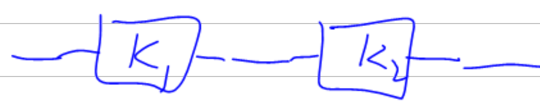


$\square$

Def $g(K)$ = "the genus of K" is the minimal genus of a s.s. for K.

Obs. $K = O \iff g(K) = 0$

$g(K) \leq u(K)$ $\quad O + K = K$



Thm $g(K_1 + K_2) = g(K_1) + g(K_2)$

Cor $\mathcal{K}$ isn't a group as if $K_1 + K_2 = 0$ then $K_1 = K_2 = 0$

Pf $\Downarrow$ $g(K_1 + K_2) = 0$ $= g(K_1) + g(K_2) \Rightarrow g(K_1) = g(K_2) = 0$ $\Uparrow$

Def K is "prime" if $K_1 + K_2 = K$

$\Rightarrow$ either $K_1 = 0$ $K_2 = K$
or $K_2 = 0$ $K_1 = K$.

Cor 2 A knot of genus 1 is prime. In particular the trefoil is prime.

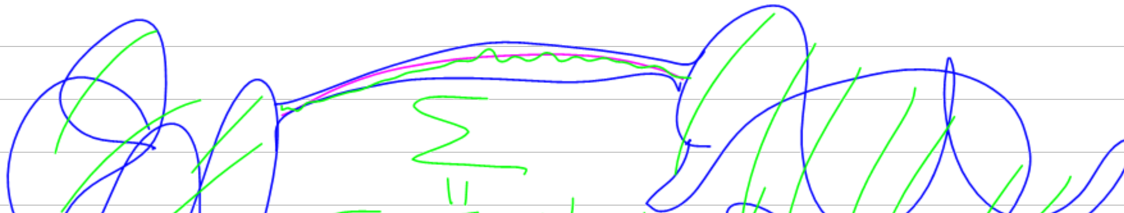
Cor 3 Every K can be decomposed as a sum of primes. [Later: uniquely]

$$K = K_1 + K_2$$

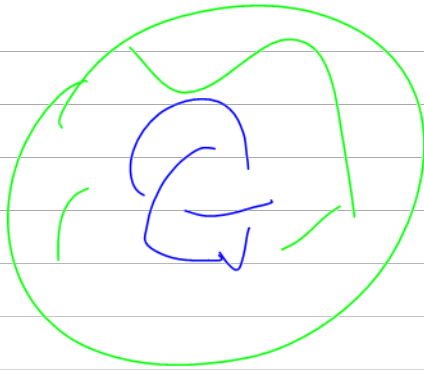
Pf of Thm (modulo ~~all~~ lots of diff geom & top of $\mathbb{R}^2$.

$$g(K_1+K_2) \leq g(K_1) + g(K_2)$$

Suppose $\partial \Sigma_i = K_i$ $g(\Sigma_i) = g(K_i)$



$$\Sigma = \Sigma_1 \cup \Sigma_2 \cup \text{band}$$



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