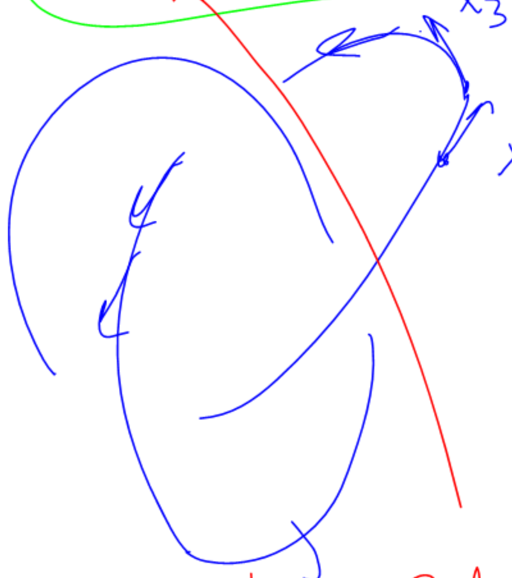
Class of November 2: A Quick Introduction to Feynman Diagrams

We wish to understand

Witten-Chern-Simons:

$$\int_{A\in\Omega^1(\mathbb{R}^3,\mathfrak{g})} \mathcal{D}A\, \mathrm{hol}_\gamma(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right]$$



does not depend on metric properties

⇓

topological invts.

$A = F_1\,dx + F_2\,dy + F_3\,dz$

$\mathbb{R}^N \quad \mathbb{R}^{\mathbb{R}}$

$$\int \mathcal{D}A\, e^{\frac{ik}{4\pi}\int_{\mathbb{R}^3} tr(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)}\ \overline{hol_\gamma(A)}$$

↓

$\sum$ (messy combinatorics) $\iint_{\mathbb{R}^3}$ (cubic combinatorial) (integral kernel)

FD:

D:

As a warm up, suppose (λ_{ij}) is a symmetric positive definite matrix and (λ^{ij}) is its inverse, and (λ_{ijk}) are the coefficients of some cubic form. Denote by $(x^i)_{i=1}^n$ the coordinates of $\mathbb{R}^n$, let $(t_i)_{i=1}^n$ be a set of "dual" variables, and let ∂^i denote $\frac{\partial}{\partial t_i}$. Also let $C := \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\lambda_{ij}x^ix^j + \frac{1}{6}\lambda_{ijk}x^ix^jx^k\right)$$

$$= \int_{\mathbb{R}^n} \exp\left(\frac{1}{6}\lambda_{ijk}x^ix^jx^k\right)\exp\left(-\frac{1}{2}\lambda_{ij}x^ix^j\right)$$

The Fourier Transform.

$$(F\colon V \to \mathbb{C}) \Rightarrow (\tilde{f}\colon V^* \to \mathbb{C})$$

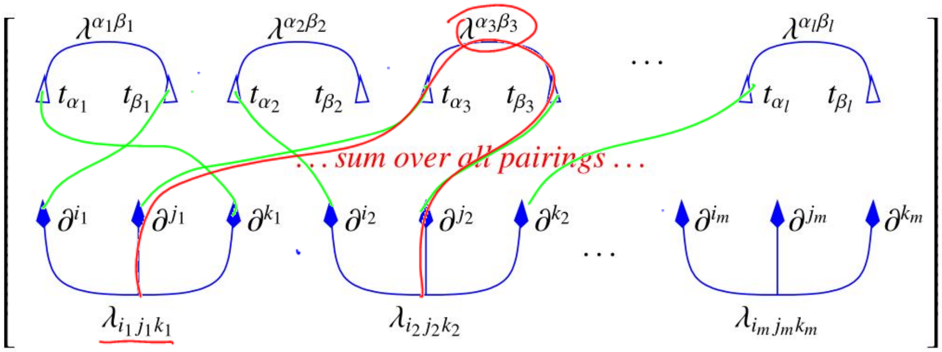
via $\tilde{F}(\varphi) := \int_V f(v)e^{-i\langle\varphi,v\rangle}dv$. Some facts:

- $\tilde{f}(0) = \int_V f(v)dv$.
- $\frac{\partial}{\partial\varphi_i}\tilde{f} \sim \widetilde{v^i f}$.
- $\widetilde{(e^{Q/2})} \sim e^{Q^{-1}/2}$, where Q is quadratic, $Q(v) = \langle Lv, v\rangle$ for $L\colon V \to V^*$, and $Q^{-1}(\varphi) := \langle\varphi, L^{-1}\varphi\rangle$. (This is the key point in one of the proofs of the Fourier inversion formula!)

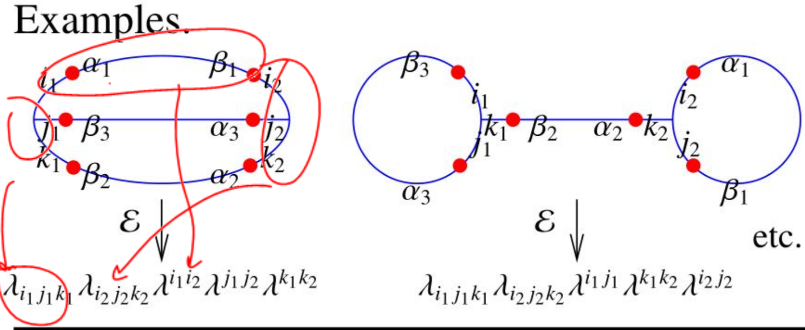
$$= C\exp\left(\frac{1}{6}\lambda_{ijk}\partial^i\partial^j\partial^k\right)\exp\left(\frac{1}{2}\lambda^{\alpha\beta}t_\alpha t_\beta\right)\Bigg|_{t_\alpha=0}$$

$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}} \frac{C}{6^m m! 2^l l!}\left(\lambda_{ijk}\partial^i\partial^j\partial^k\right)^m\left(\lambda^{\alpha\beta}t_\alpha t_\beta\right)^l$$

$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}} \frac{C}{6^m m! 2^l l!}$$



Examples.

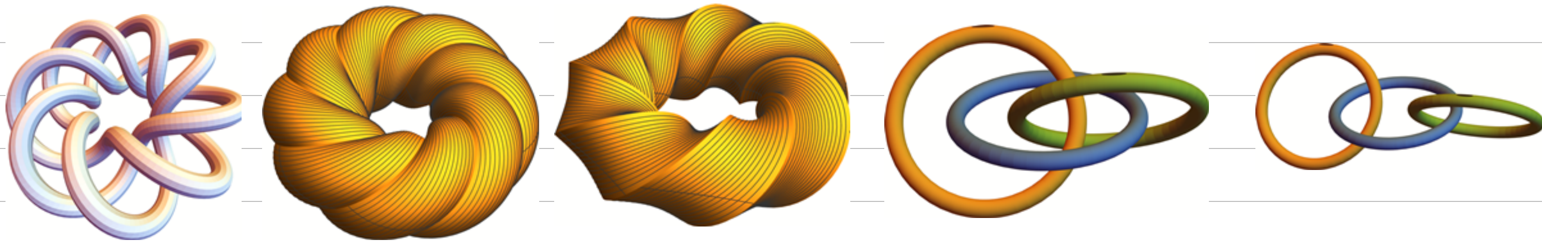


$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}} \frac{C}{6^m m! 2^l l!} \sum_{\substack{m\text{-vertex fully marked}\\ \text{Feynman diagrams } D}} \mathcal{E}(D)$$

$\rightarrow \sum_{i_1 i_2 j_1 j_2 k_1 k_2}$ Prod of cubics & inverse quadratics

Hour 24, Wednesday November 4: The Fundamental Group / Knot Group.
HW7 on web by midnight! (And I hope to clear my marking backlog soon).
Finite type / Lie Algebras and Reps omissions:
* The KZ proof of the Fundamental Theorem.
* The "Associators" proof of the Fundamental Theorem (also, "Knotted Trivalent Graphs").
* The step-by-step-integration non-proof of the Fundamental Theorem.
* Computing FT Invariants using "Gauss Diagram Formulas".
* Computations of invariants for specific Lie algebras and reps ("Quantum Groups").
* Finite type invariants of other types knotted objects.
* Finite type invariants of 3-manifolds.
* Vogel's work on non-Lie-algebraic weight systems.
* And more....

A Gallery of Pictures from BlownTorus.nb at http://drorbn.net/AcademicPensieve/Classes/20-1350-KnotTheory:



Conjecture 1 Finite type invts separate knots.

Conjecture 2 $\langle W_{g,R} \rangle = \mathcal{A}^*$ (all F.t. invts come from Lie Algebras)

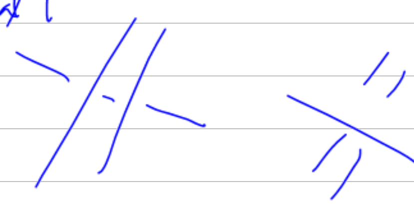
That one is FALSE.

dim $\mathcal{A}_m$ ... match to $m=17$ diverge at $m=18$

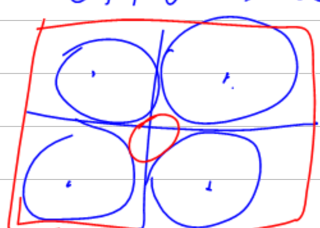
dim $\langle W_{g,R} \rangle_m$

Alt.

non-alt



Riddle: Let B_n be the largest n-dim ball centered at 0 and bound by the 2^n unit balls centered at $\{\pm 1\}^n$



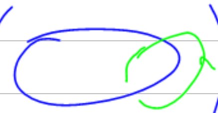
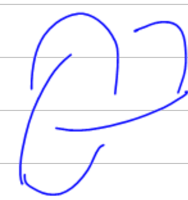
Let C_n be the smallest ~~box~~ cube containing all of the above

$$\lim_{n\to\infty} \frac{Vol(B_n)}{Vol(C_n)}$$

Things get interesting in dim 1206.

<u>Def</u> If K is a knot, $\pi_1(K)$, or the fundamental group of K, the group of K, is

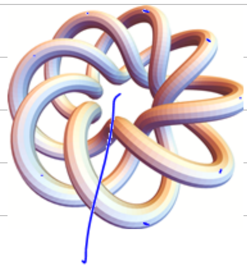
$$\pi_1(K) = \pi_1(\mathbb{R}^3 \setminus K)$$

$\pi_1($  $) = \mathbb{Z}$ $\quad \pi_1($  $)$

<u>Example</u> $\pi_1(T_{p,q})$ $\quad T_{p,q}$: (p,q) Torus knots (where (p,q) are rel. prime).

$\mathbb{R}^2/\mathbb{Z}^2$

$\gamma(t) = \tau(pt, qt)$ $\quad \gamma : \underbrace{[0,1]/_{0\sim1}}_{S^1} \xrightarrow{1-1} \mathbb{R}^3$



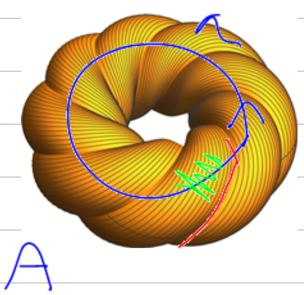
$T_{8,3}$ $\quad \pi_1(T_{8,3})$

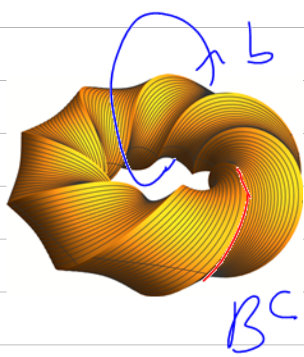
Van Kampen: $A, B, A \cap B$ are connected, $b \in A \cap B$

$$\pi_1(A \cup B) = \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)$$

$\pi_1(A \cap B)$, with $\alpha: \pi_1(A \cap B) \to \pi_1(A)$ and $\beta: \pi_1(A \cap B) \to \pi_1(B)$

$$= \pi_1(A) * \pi_1(B) \Big/ \forall \gamma \in \pi_1(A \cap B)\ \ \alpha(\gamma) = \beta(\gamma)$$

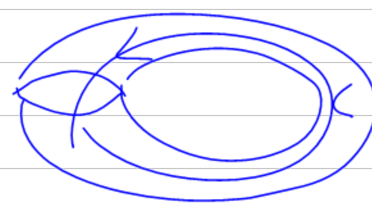




A = solid torus

$\pi_1(A) = \mathbb{Z} = \langle a \rangle$

$\pi_1(B) = \mathbb{Z} = \langle b \rangle$

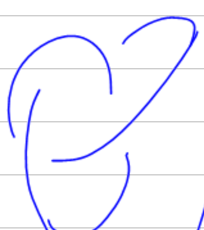
$A \cap B =$ 

$\pi_1(A \cap B) = \mathbb{Z}$

$\alpha(c) = a^3$

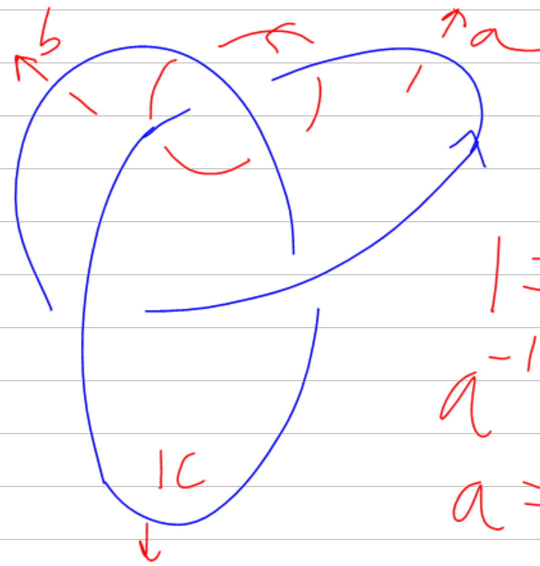
$\beta(c) = b^8$

$$\pi_1(T_{8,3}) = \langle a, b \rangle \big/ a^3 = b^8$$

$\pi_1($  $) = \langle a, b \rangle \big/ a^3 = b^2$

$\pi_1\Big($  $\Big)$

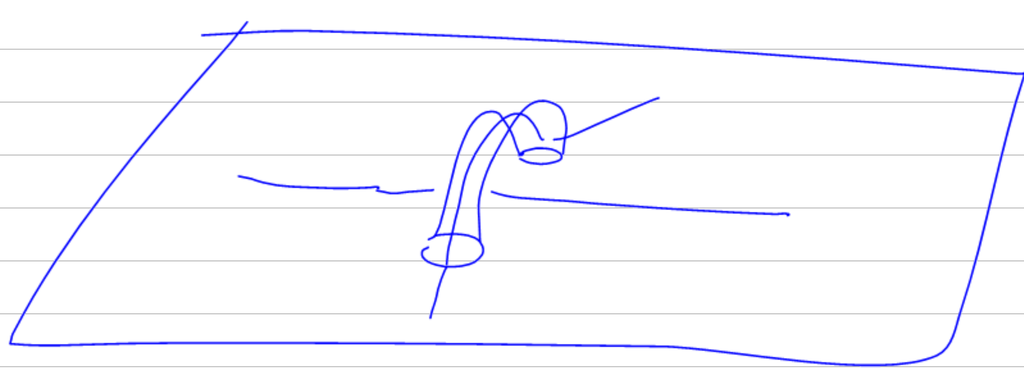
$$= \langle a, b, c \mid a = c^b,\ b = a^c,\ c = b^a \rangle$$



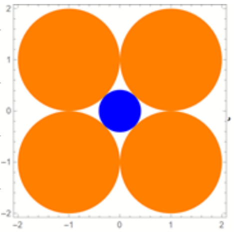
$$1 = a b^{-1} c^{-1} b$$

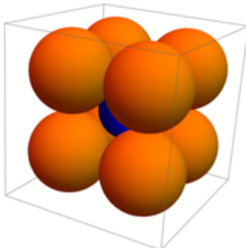
$$a^{-1} = (c^{-1})^b$$

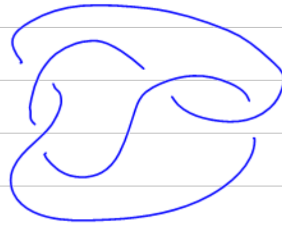
$$a = c^b$$

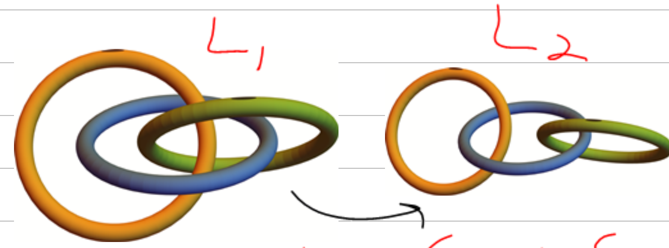


Hour 25, Friday November 6: More on Knot Groups





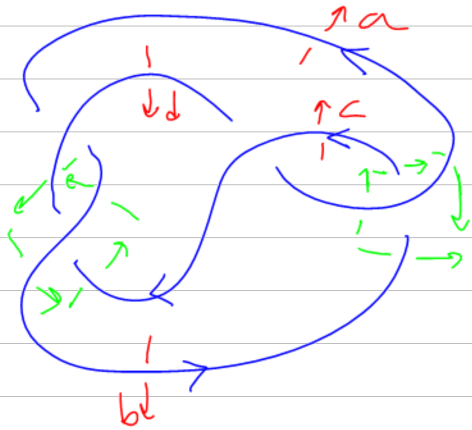




$L_1 \neq L_2$ yet $L_1^c \cong L_2^c$

$$\lim_{n\to\infty} \frac{Vol(B_n)}{Vol(C_n)}$$

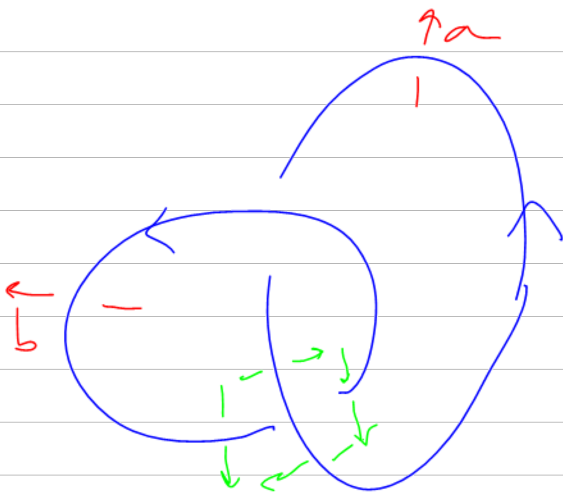
at $n=9$
radius $\sqrt{9}-1=2$



$b = a^{-1}ca = c^a$

$c = bdb^{-1} = d^{(b^{-1})}$

"Wirtinger presentation"



$b = a^{-1}ba$

$\pi_1(\text{Hopf}) = \langle a,b\rangle / ab=ba$

$= \mathbb{Z}^2 = \pi_1(\mathbb{T}^2)$

$\mathbb{T}^2 = S^1 \times S^1$

Riddle show that
$(S^3 \setminus \text{Hopf}) \sim S^1 \times S^1$

π_1 is very strong <u>but</u>

0. "Word problem for groups is insoluble"

$K_1 \longrightarrow \langle g_i^1 \rangle / r_j^1$

$K_2 \longrightarrow \langle g_i^2 \rangle / r_j^2$

Lickorish's book GTM 175, P115

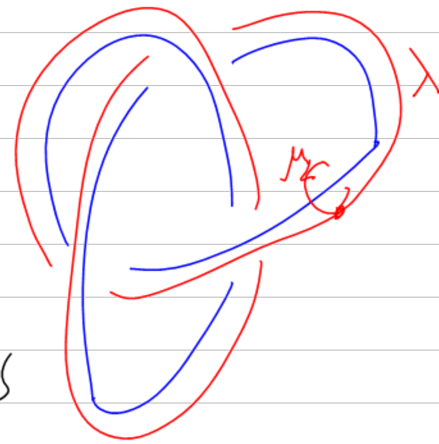
1. Waldhausen 66':

(π, λ, μ) determines the knot

$\downarrow \quad \downarrow \quad \downarrow$

(π', λ', μ')

Also true for links

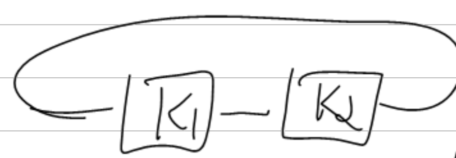


λongtitude
μerdian

2. Whitten/Gonzales-Acuña 87'

If K is prime, $\pi_1(K)$ determines K^c (as a manifold)

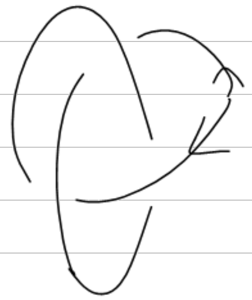
K is prime not

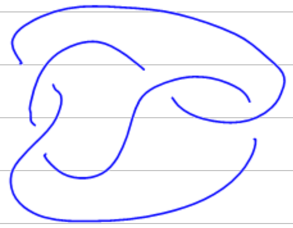


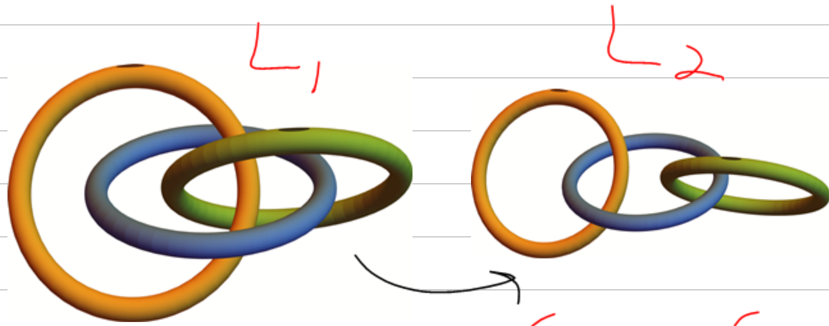
K_1 & K_2 non-trivial.

3. Gordon-Luecke 89'

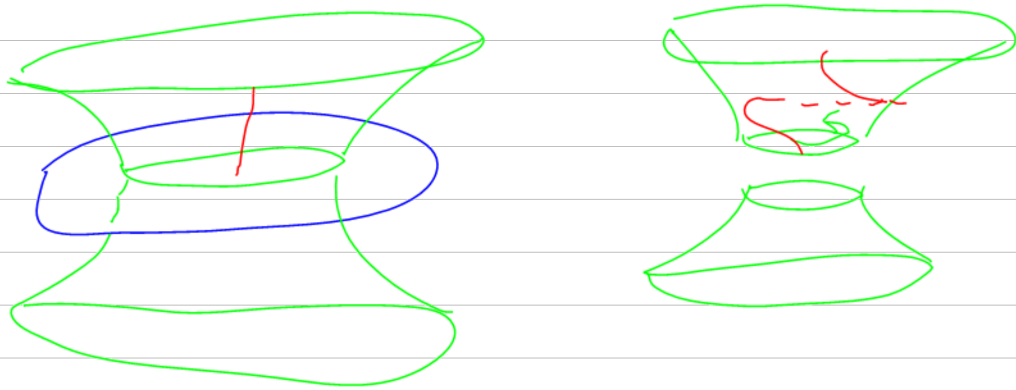
The complement of an <u>unoriented</u> knot determines it.

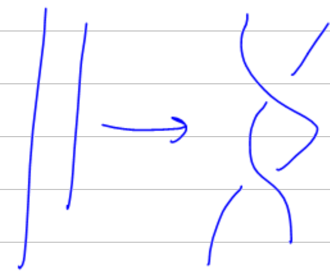






$L_1 \neq L_2$ yet $L_1^c \cong L_2^c$





Pick some finite group G (S_5) ($\underline{\underline{D_6}}$)

$$\left|Rep(\pi_1(K) \longrightarrow G)\right|$$

a knot invariant.