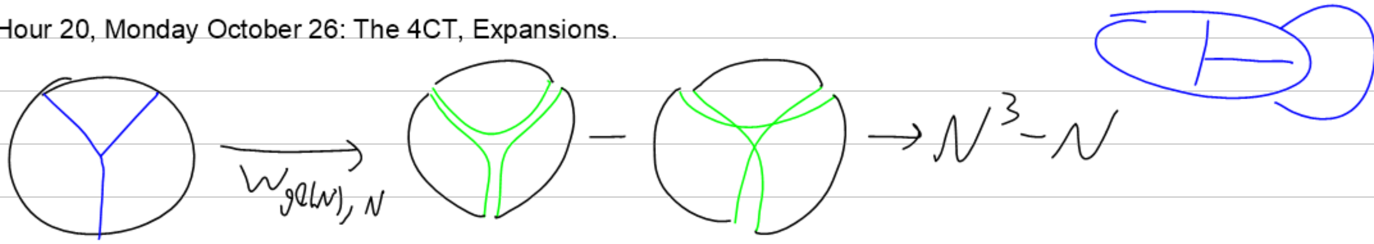
$$W_\emptyset : \mathcal{A} \longrightarrow \mathbb{Q} \qquad W_\emptyset(D) = \begin{cases} 1 & \text{if } D = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$w_1, w_2 \in \mathcal{A}^{\#} \qquad (w_1 \cdot w_2)(a) = (w_1 \otimes w_2)(\square a)$$

$$w(\partial_\emptyset a) = (w \cdot w_\emptyset)(a) = (w \otimes w_\emptyset)(\square a)$$

Hour 20, Monday October 26: The 4CT, Expansions.



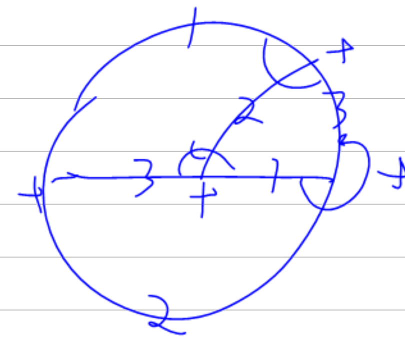
HW6 Q2 If D is blue, $|W^{top}_{gl(N)}(D)| = \#$ planar embeddings of D

Last bit of H19: If D is blue & planar, $|W_{sl(2)}(D)| \propto \#$ edge 3-colorings of D.

$sl_2 \sim so(3)$ $\quad X_1, X_2, X_3$

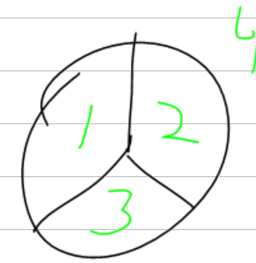
$$[X_i, X_j] = \sum_k \epsilon_{ijk} X_k$$

$t^{ab} \propto f^{ab}$



If D is planar

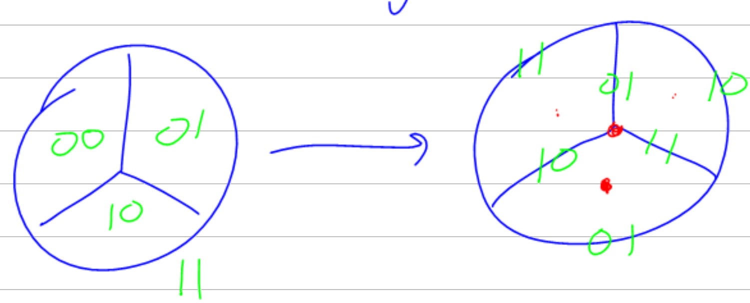
$$4(\#\text{ edge 3-colorings of D}) = \#\text{ face 4-colorings of D}$$

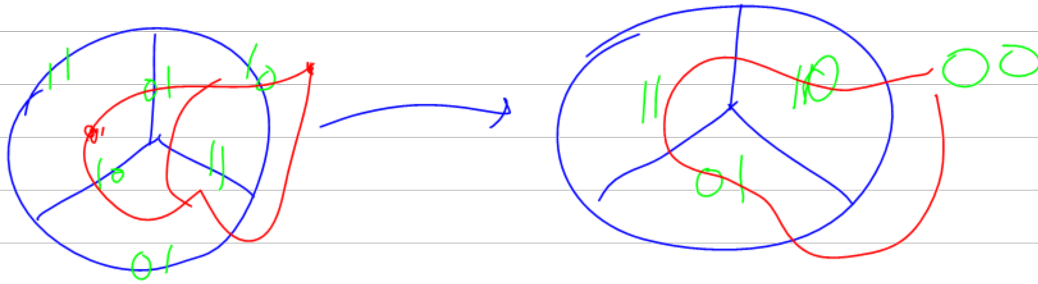


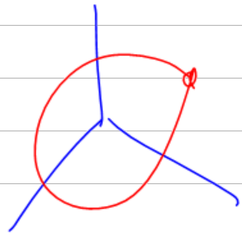
use face colours $\mathbb{Z}/2 \times \mathbb{Z}/2$ 00, 01, 10, 11

use edge colours $\mathbb{Z}/2 \times \mathbb{Z}/2 \setminus \{00\}$: 01, 10, 11

Face 4-coloring $\longrightarrow$ edge 3-colorings





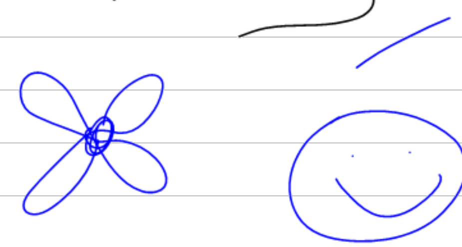


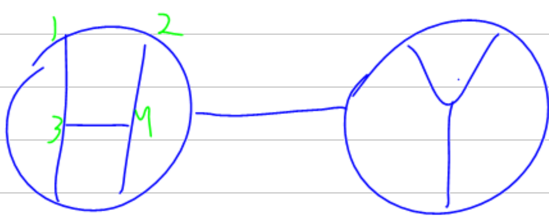
$$\begin{array}{r} 01 \\ 10 \\ 11 \\ \hline 00 \end{array}$$

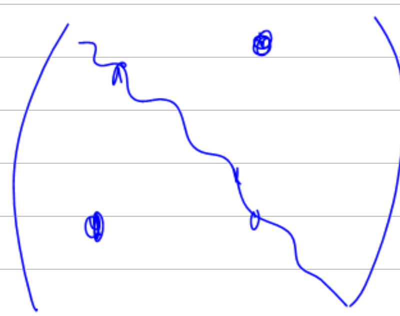
So 4CT $\iff$ $\left(\begin{array}{c}\#\text{ planar}\\ \text{embeddings}\\ \text{of } D\end{array}\right) \neq 0 \implies \left(\begin{array}{c}\#\text{ face}\\ \text{4-colourings}\\ \text{of } D\end{array}\right) \neq 0$

$$\iff \left[W_{sl(2)}(D) = 0 \implies W^{tor}_{gl(N)}(D) = 0 \right]$$

Sounds reasonable







$W_{g,R} : \mathcal{A} \longrightarrow \mathbb{Q}$

HW6Q1

$$\begin{tikzcd} \mathcal{A}^{\frown} \arrow[r, "P"] \arrow[rr, bend right, "\overline{W}_{g,R}"'] & \mathcal{A} \arrow[r, "W_{g,R}"] & \mathbb{Q} \end{tikzcd}$$

<u>Prop</u> $\begin{pmatrix}\text{Fund. thm} \\ \text{For framed} \\ \text{knots}\end{pmatrix} \Longleftrightarrow \exists\, Z : \underset{\displaystyle \mathcal{K}}{\underset{\shortparallel}{\{\text{knots}\}}} \to \hat{\mathcal{A}}$

s.t.

$$Z(K) = D_K + \text{higher degrees}$$

m-singular   chord diag of K

<u>Pf</u> $\Longleftarrow$: Suppose such Z is given, suppose $W \in \mathcal{A}_m^*$

$$\begin{tikzcd} \mathcal{K} \arrow[r, "Z"] \arrow[rr, bend right, "V"'] & \mathcal{A} \arrow[r, "W"', "\text{supported on deg } m"] & \mathbb{Q} \end{tikzcd}$$

$$W_V(D) \underset{\text{s.t. } D_K = D}{\overset{\text{Fine } K}{=\!=\!=}} V(K) = W(Z(K))$$

$$= W(D_K + h.o.) = W(D) + 0 \quad \checkmark$$

$\Longrightarrow$ Let $(a_{m,i})_{i=1}^{d_m}$ be a basis of $\mathcal{A}_m$

Let $W_{m,i}$ be the dual basis of $\mathcal{A}_m^*$

$$W_{m,i}(a_{m,j}) = \delta_{ij}$$

By fund thm, find $V_{m,i}$ s.t. $W_{V_{m,i}} = W_{m,i}$

Define $Z : \mathcal{K} \longrightarrow \mathcal{A}$ as follows:

$$Z(K) = \sum_{m \geq 0} \sum_{i=1}^{d_i} V_{m,i}(K)\, a_{m,i} \in \mathcal{A}.$$

Suppose that K is m-singular.

$$Z(K) = \sum_{m'=0}^{\infty} \sum_{i=1}^{d_i} V_{m',i}(K)\, a_{m',i}$$

$$= (h.o.) + \sum_{i=1}^{d_i} V_{m,i}(K)\, a_{m,i}$$

$$= (h.o.) + \sum_{i} \cancel{W_{m,i}}(D_K)\, a_{m,i}$$

$W_{m,i}$

$$= (h.o.) + D_{K} \qquad \square$$

Homework Assignment 6

$$\frac{\mathbb{Q}[x,y]}{\langle x \rangle} \xrightarrow{ev_{x=0}} \mathbb{Q}[x,y]$$

Question 1. Let $\Theta : \mathcal{A} \to \mathcal{A}$ be the multiplication operator by the 1-chord diagram θ, and let $\partial_\theta = \frac{d}{d\theta}$ be the adjoint of multiplication by W_θ on $\mathcal{A}^\star$, where W_θ is the obvious dual of θ in $\mathcal{A}^\star$. Let $P : \mathcal{A} \to \mathcal{A}$ be defined by

$$P = \sum_{n=0}^{\infty} \frac{(-\Theta)^n}{n!} \partial_\theta^n.$$

$$\frac{\mathcal{A}}{\langle \theta \rangle} \longrightarrow \mathcal{A}$$

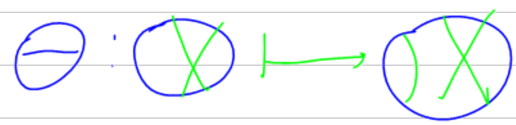
Verify the following assertions, but submit only your work on assertions 4,5,7,11:

1. $[\partial_\theta, \Theta] = 1$, where $1 : \mathcal{A} \to \mathcal{A}$ is the identity map and where $[A, B] := AB - BA$ for any two operators.
2. P is a degree 0 operator; that is, $\deg Pa = \deg a$ for all $a \in \mathcal{A}$.
3. ∂_θ satisfies Leibnitz' law: $\partial_\theta(ab) = (\partial_\theta a)b + a(\partial_\theta b)$ for any $a, b \in \mathcal{A}$.
4. P is an algebra morphism: $P1 = 1$ and $P(ab) = (Pa)(Pb)$.
5. Θ satisfies the co-Leibnitz law: $\square \circ \Theta = (\Theta \otimes 1 + 1 \otimes \Theta) \circ \square$ (why does this deserve the name "the co-Leibnitz law"?).
6. P is a co-algebra morphism: $\eta \circ P = \eta$ (where η is the co-unit of $\mathcal{A}$) and $\square \circ P = (P \otimes P) \circ \square$.
7. $P\theta = 0$ and hence $P\langle\theta\rangle = 0$, where $\langle\theta\rangle$ is the ideal generated by θ in the algebra $\mathcal{A}$.
8. If $Q : \mathcal{A} \to \mathcal{A}$ is defined by
$$Q = \sum_{n=0}^{\infty} \frac{(-\Theta)^n}{(n+1)!} \partial_\theta^{(n+1)}$$
then $a = \theta Q a + P a$ for all $a \in \mathcal{A}$.
9. $\ker P = \langle\theta\rangle$.
10. P descends to a Hopf algebra morphism $\mathcal{A}^r \to \mathcal{A}$, and if $\pi : \mathcal{A} \to \mathcal{A}^r$ is the obvious projection, then $\pi \circ P$ is the identity of $\mathcal{A}^r$. (Recall that $\mathcal{A}^r = \mathcal{A}/\langle\theta\rangle$).
11. $P^2 = P$.

$f \mapsto f|\hat{x} - \hat{x})$

$$= \sum_{n>0} \frac{(-x)^n}{n!} (\partial_x)^n f$$

a



$$\partial_\theta = W_\theta^*$$

$W_\theta(\theta) = 1$ $\quad W_\theta(\text{anything not of deg } 1) = 0$.

$$\varphi(\partial_\theta D) = (W_\theta \cdot \varphi)(D)$$

$\varphi \in \mathcal{A}^*$ $\quad = (W_\theta \otimes \varphi)(\underbrace{\square D}_{\mathcal{A}\otimes\mathcal{A}})$

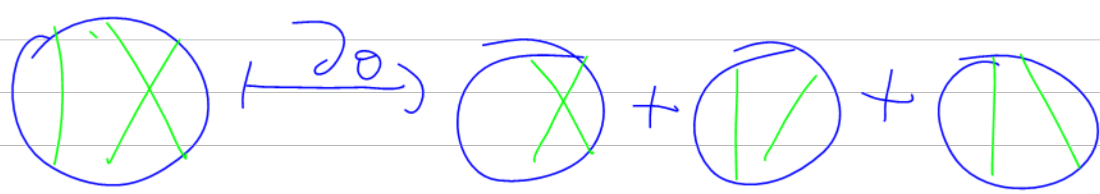
$= \varphi$(all diag obtained from D by dropping one chord)

$T : \overset{u}{V} \to W$

$T^* : \underset{w}{W^*} \to V^*$

$w(Tu) = (T^* w)(u)$

$\Rightarrow \partial_\theta D =$ (Sum of all ways of dropping one chord from D)



$$\partial_x x^5 = \sum x \cancel{x} x x x = 5x^4$$

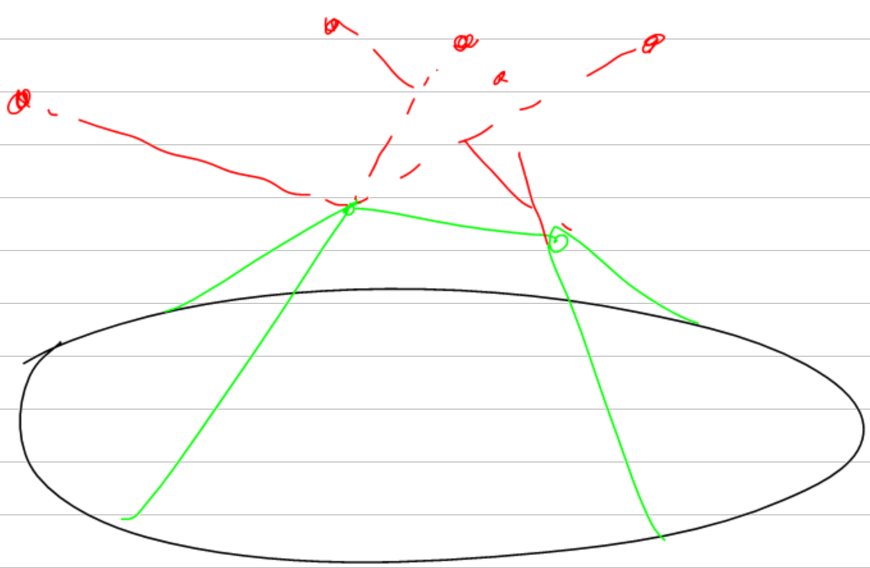
$$[\partial_x, \hat{x}] = I$$

heisenberg commutation relation

$$[\partial_\theta, \hat{\theta}] = I$$

N^3

$W_{gl(N)}\Big($  $\Big)$

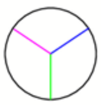
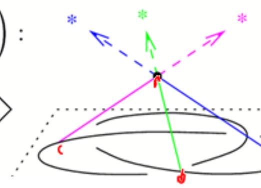


Hour 21, Wednesday October 28: Expansions and Cosmic Coincidences.
No HW assigned this week! But HW6 is due. Should we discuss it?

Recall: We are seeking an "expansion/UFTI";

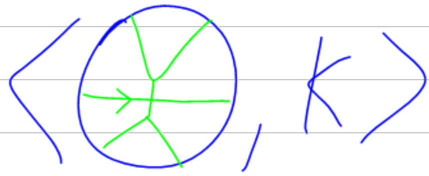
$Z: \mathcal{K} \to \mathcal{A}$ s.t. $Z(K) = D_K + h.o.$

$\langle D, K\rangle_{\text{stonehenge}} := \left(\text{The signed Stonehenge pairing of } D \text{ and } K\right)$:

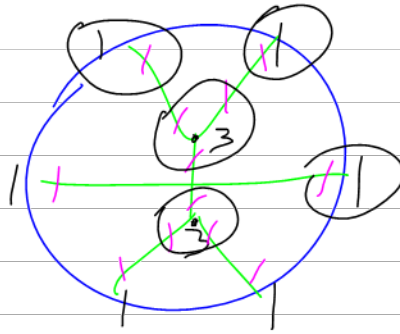
$D=$  $K=$  $\Rightarrow$ 

unknowns: $S^1 \times S^1 \times S^1 \times \mathbb{R}^3 \sim 6$

eqns: $S^2 \times S^2 \times S^2 \sim 6$



$1+1=2$

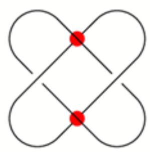


$F: M^k \to (N^k, n)$

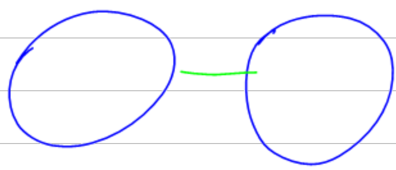
$x \mapsto n$

The Gaussian linking number

$lk(\text{⊙⊙}) = \sum_{\text{vertical chopsticks}} (\text{signs}) = \langle \text{O—O}, \text{⊙⊙} \rangle_{\text{stonehenge}}$



C.F. Gauss



The generating function of all cosmic coincidences:

$$Z(K) := \lim_{N\to\infty} \sum_{\text{3-valent } D} \frac{\langle D, K\rangle_{\text{stonehenge}} D}{2^c c! \binom{N}{e}} \cdot \cancel{\left(\text{framing-dependent counter-term}\right)} \in \mathcal{A}(\circlearrowleft)$$

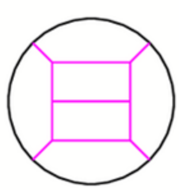
D. Thurston

IHX, STU, AS

N := # of stars

c := # of chopsticks

e := # of edges of D

$\mathcal{A}(\circlearrowleft)$:= Span $\left\langle\right.$  $\left.\right\rangle$ / oriented vertices, AS: Y + Y = 0 & more relations

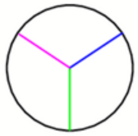
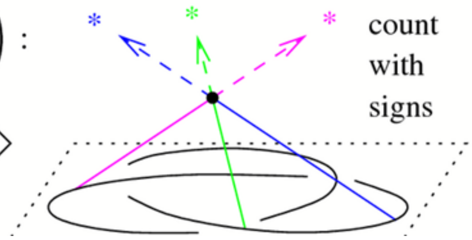
dbnsymb

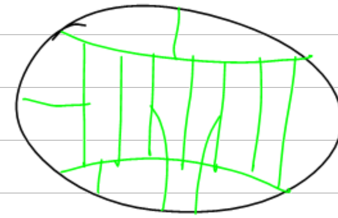
Hour 22 on Friday October 30: Cosmic coincidences and the IHX relation

$Z : \mathcal{K} \to \mathcal{A} = \mathcal{D} \Big/ \begin{matrix}AS\\ IHX\\ STU\end{matrix}$

$\langle D, K\rangle_{\sqcap} := \begin{pmatrix}\text{The signed Stonehenge}\\ \text{pairing of } D \text{ and } K\end{pmatrix}$: count with signs

$\mathcal{D}$

$D=$  $K=$  $\Rightarrow$ 



The generating function of all cosmic coincidences:

D. Thurston

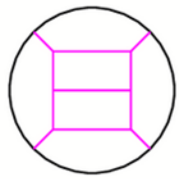
$$Z(K) := \cancel{\lim_{N\to\infty}} \sum_{\text{3-valent } D} \frac{\langle D, K\rangle_{\sqcap} D}{2^{c} \cancel{c! \binom{N}{e}}} \cdot \cancel{\begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix}} \in \mathcal{A}(\circlearrowleft)$$

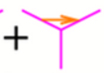
N := # of stars

c := # of chopsticks

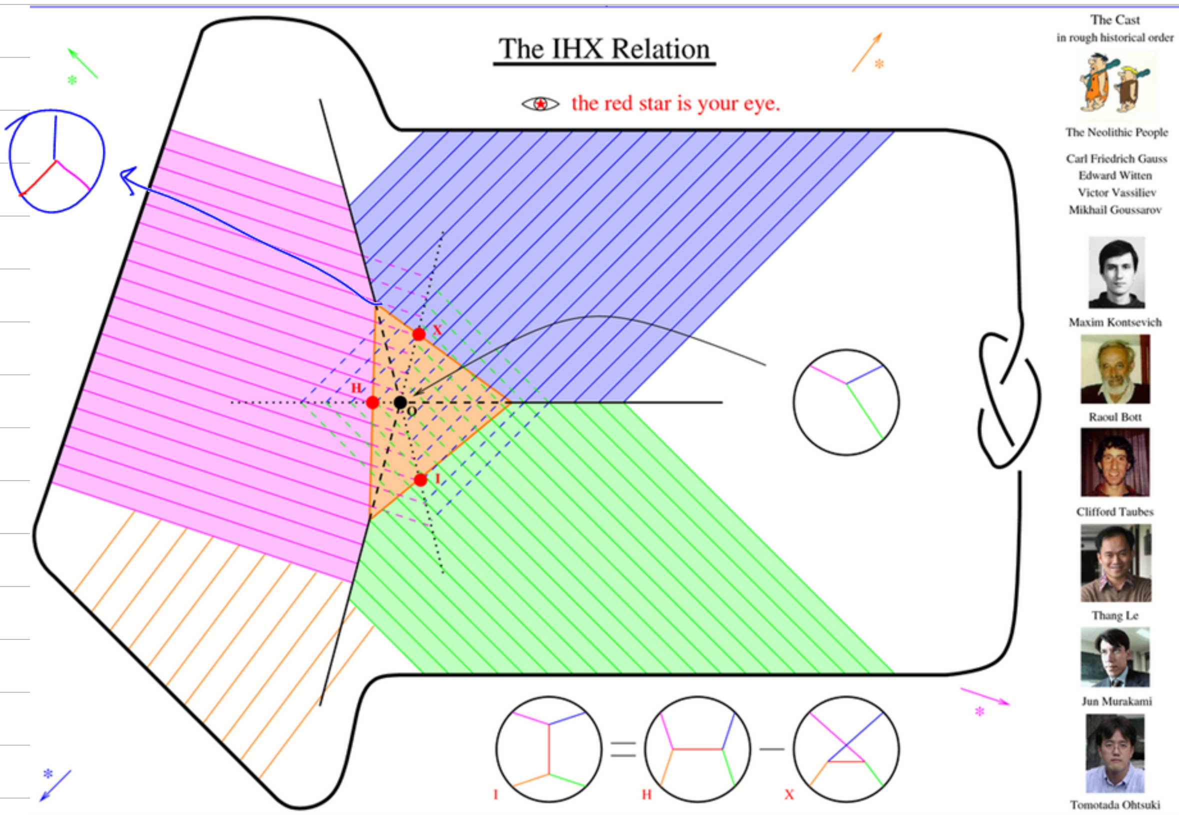
$\parallel$

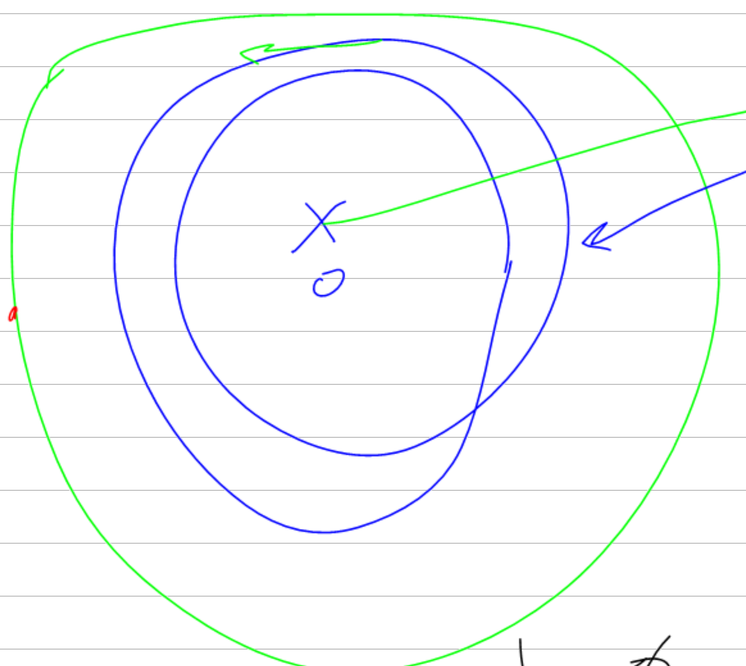
e := # of edges of D

$\mathcal{A}(\circlearrowleft)$:=Span $\left\langle\right.$  $\left.\right\rangle \Big/$ oriented vertices

AS:  +  =0

& more relations





γ

$$w(\gamma) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z}$$

$$\phi : M^n \longrightarrow N^n \ni w$$

$p \qquad \int_N w = 1$

$$\deg \phi = \phi^{-1}(p) = \int_M \phi^*(w)$$

$\langle \Theta, K \rangle_\pi \qquad \phi : \underbrace{S^1\times S^1\times S^1\times \mathbb{R}^3}_{M} \longrightarrow \underset{w\,\times\, w\,\times\, w}{S^2\times S^2\times S^2}$

$$\int_M \phi^*(w\wedge w\wedge w)$$

When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,

(see below)

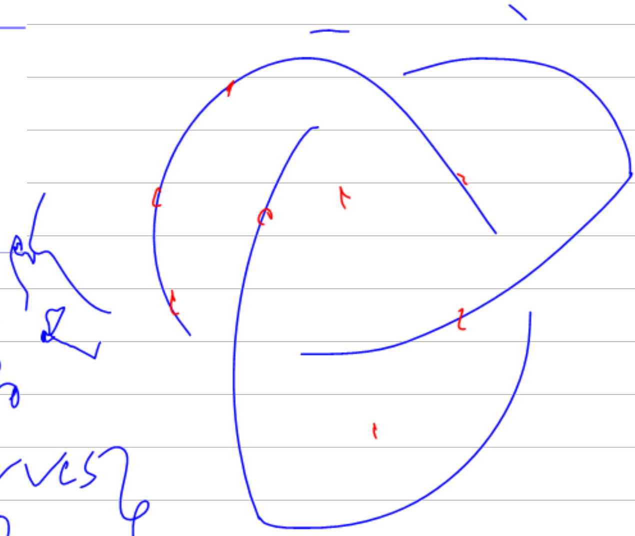
An intersection line cuts through the knot – Solution: Impose STU,

(similar argument)

The Gauss curve slides over a star – Solution: Multiply by a framing–dependent counter–term.

(not shown here)

Theorem. Modulo Relations, $Z(K)$ is a knot invariant!



$$\{\text{curves in } \mathbb{R}^3\} \xrightarrow[\text{curve}]{\text{Gauss}} \{\text{curves in } S^2\}$$

Chern-Simons-Witten theory and Feynman diagrams.

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right] = \sum_{m=0}^{\infty} \sum_{\text{Feynman diagram}} \iiint_{\mathbb{R}^{N(D)}} e$$

Witten

$$\longrightarrow \sum_{D:\text{ Feynman diagram}} W_{\mathfrak{g}}(D) \sum\!\!\!\!\!\!\int \mathcal{E}(D) \longrightarrow \sum_{D:\text{ Feynman diagram}} D \sum\!\!\!\!\!\!\int \mathcal{E}(D)$$

Feynman

$$\int_{\mathbb{R}^I \to \mathbb{R}} e^{Q+P} = \sum_{x_1 \ldots x_{17}=1}^{n} e$$

$\mathbb{R}^{\mathbb{R}}$ $\mathbb{R}^{\infty}$