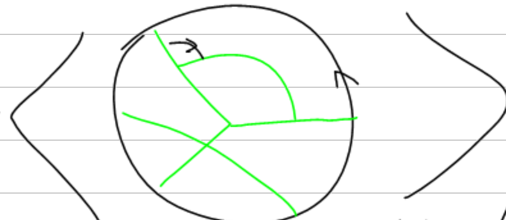
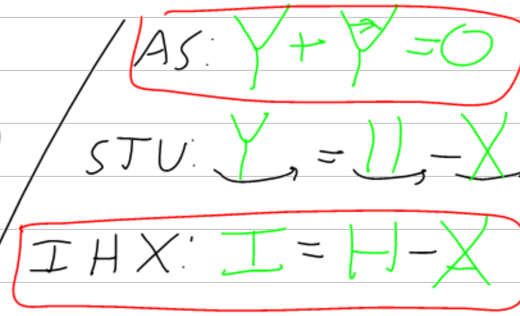
Hour 17, Monday October 19: \calA^t and Lie Algebras

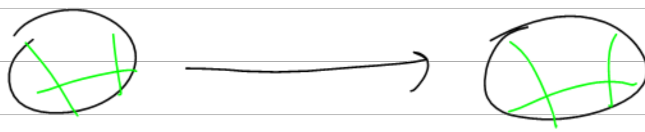
Thm $\mathcal{D}/4T = \mathcal{A} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathcal{A}^t := \langle$  $\rangle$ / 

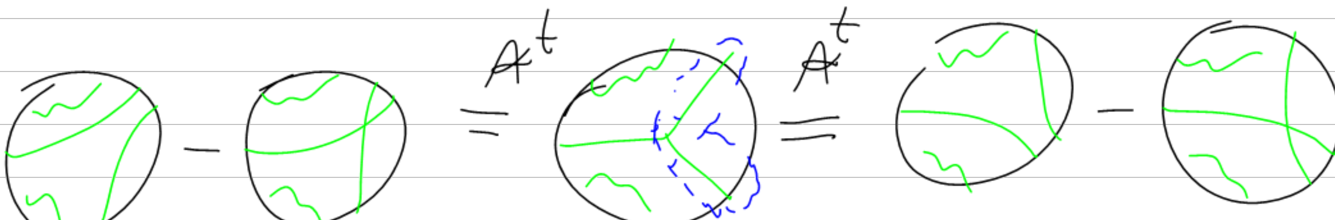
* connected, oriented vertices, $\deg = \frac{1}{2}(\text{verts})$

$\theta =$  $\dim \mathcal{A}_m = \dim \mathcal{A}^r_m + \dim \mathcal{A}_{m-1}$

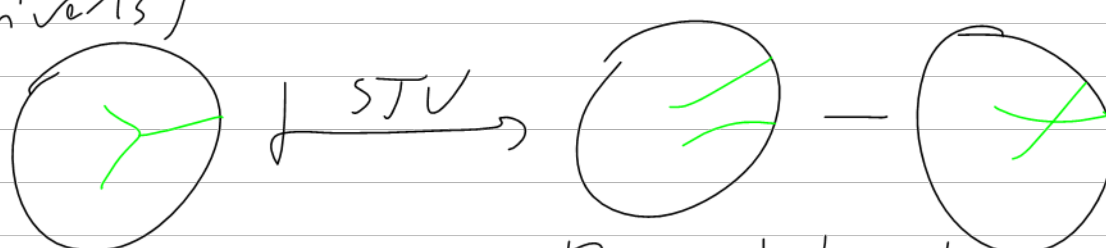
Polys in $\theta, x, y \ldots$ — Polys w/ no θ — Poly of deg 1 less in $\theta, x, y, \ldots$

$\mathcal{A} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathcal{A}^t$

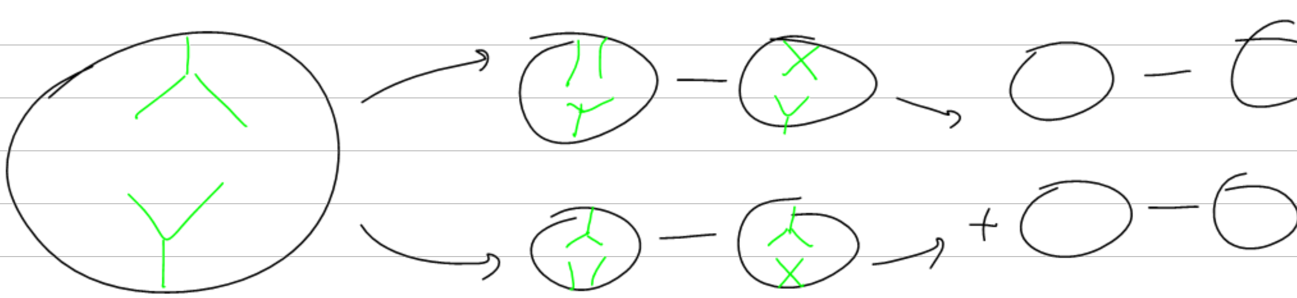
ϕ: 

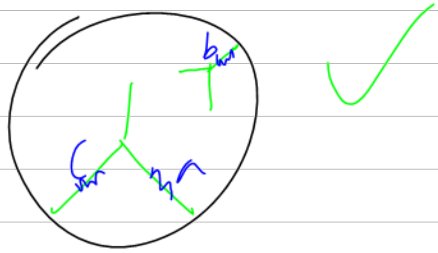


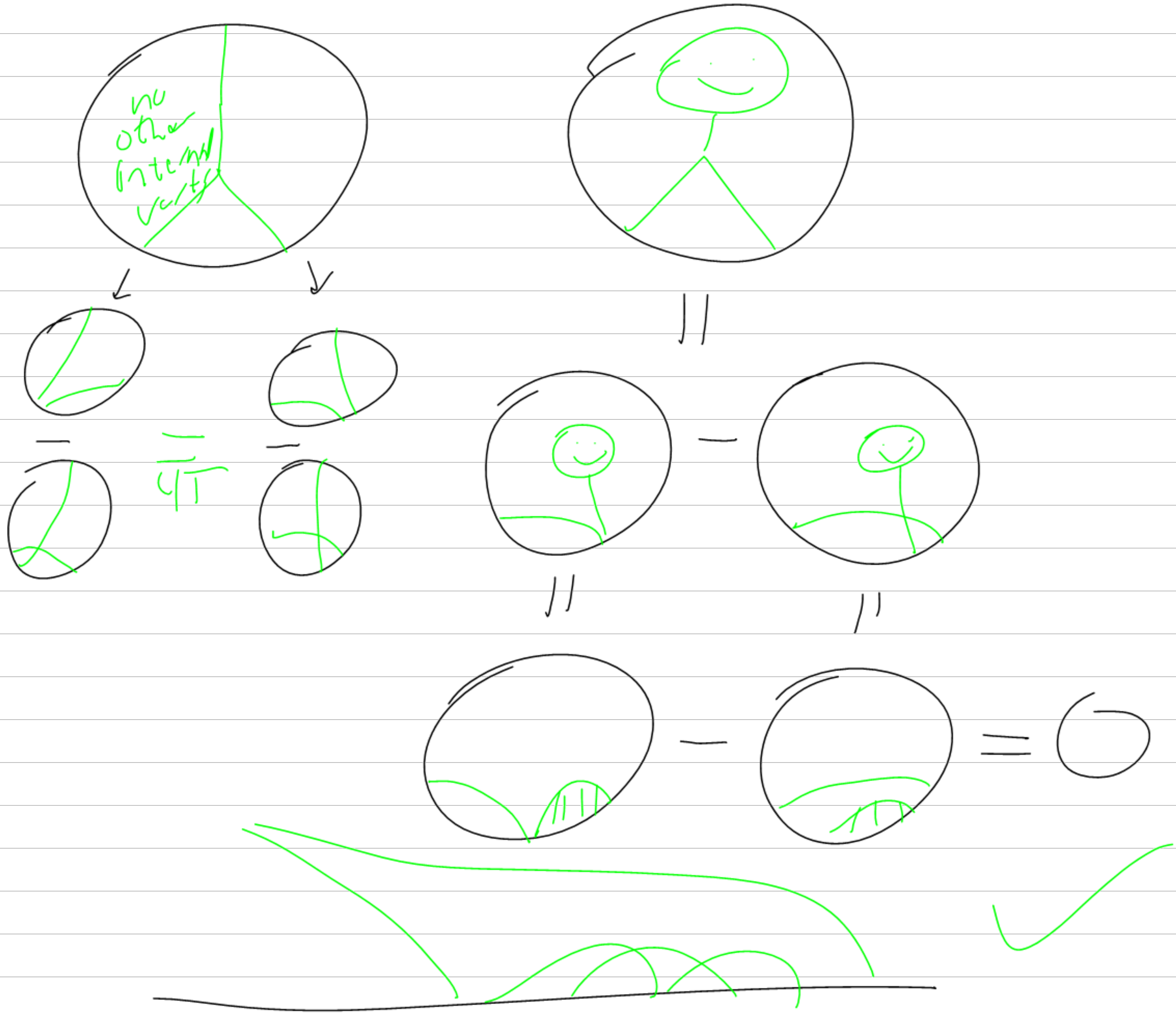
ψ(diags w/ no green verts) = itself. So ϕ is well def.

otherwise ψ: 

Then iterate...



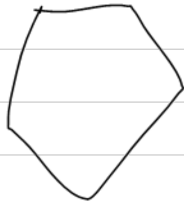


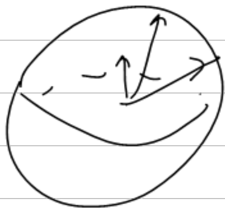


Def A Lie Alg is a V.S. L along with a bilinear map $[\,,\,]: L^{\otimes 2} \to L$ s.t.

1. $[x,y]+[y,x]=0$
2. $0=[x,[y,z]]+[y,[z,x]]+[z,[x,y]]$

↑ Jacobi identity

 $\longrightarrow D_{10}$

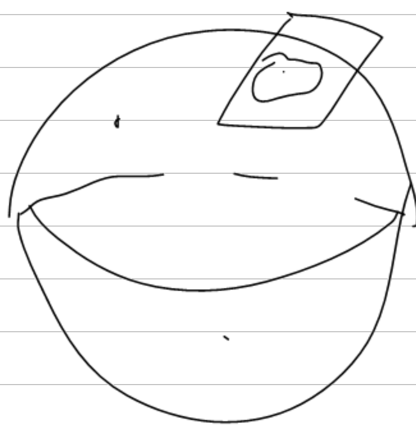
 $S^2 \longrightarrow O(3) = \begin{pmatrix} | & | & | \end{pmatrix} = A$

$\{A : A^T A = I\}$

$$SO(n) = \left\{ A \in M_{n\times n}(\mathbb{R}) : \begin{matrix} A^T A = I \\ \det A = 1 \end{matrix} \right\}$$

$$GL(n) = \left\{ \begin{matrix} \text{all invertible} \\ n\times n \text{ matrices} \end{matrix} \right\}$$

$$SL(n) = \{ A \in GL(n) : \det(A) = 1 \}$$



Enough to understand

$T_1 G = L$

↑ Lie algebra of the group.

A metric on L is a non-degenerate symmetric, invariant, bilinear form on L:

$$\langle\ ,\ \rangle : L \otimes L \longrightarrow \mathbb{Q}$$

s.t.

1. $\langle x, y\rangle = \langle y, x\rangle$
2. $\langle [z,x], y\rangle + \langle x, [z,y]\rangle = 0$
3. non-deg: if $x \neq 0$ then $\exists\, y$ s.t. $\langle x, y\rangle \neq 0$.

Examples

$$\mathfrak{gl}(n) = \{A : A \in M_{n\times n}\}$$

$$[A,B] = AB - BA.$$

$$[A,[B,C]] = [A, BC - CB] =$$

$$= \cancel{ABC} - \cancel{ACB} - \cancel{BCA} + \cancel{CBA}$$

$$= \cancel{BCA} - BAC - CAB + \cancel{ACB}$$

$$= CAB - \cancel{CBA} - \cancel{ABC} + BAC$$

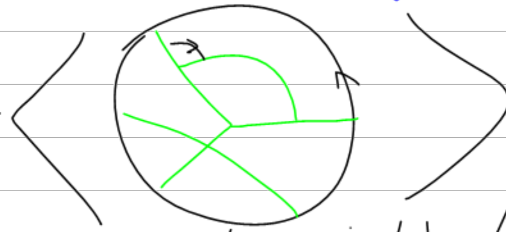
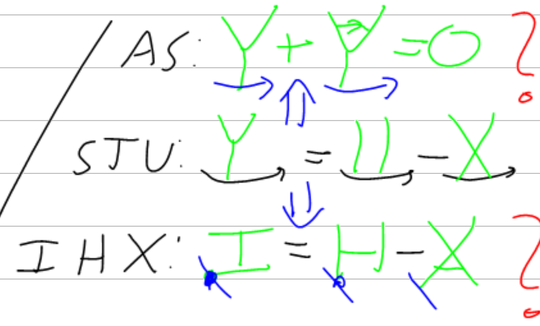
$$\langle A, B\rangle := \operatorname{tr} AB = \operatorname{tr} BA = \langle B, A\rangle$$

$$\langle [C,A], B\rangle + \langle A, [C,B]\rangle \stackrel{?}{=} 0$$

$$\operatorname{tr}\left(\cancel{CAB}-\cancel{ACB}\right)+\operatorname{tr}\left(\cancel{ACB}-\cancel{ABC}\right)$$

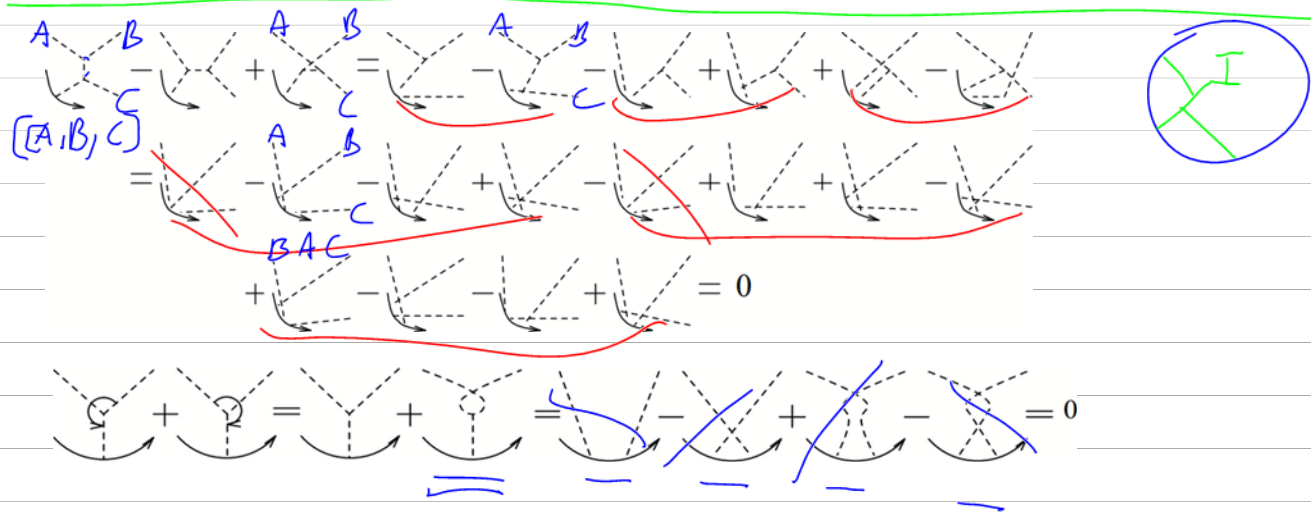
$$\forall A\neq 0\ \exists B \text{ s.t. } \langle A,B\rangle \neq 0$$

Hour 18, Wednesday October 21: Lie Algebraic Weight Systems, gl(N).
By midnight: HW5 due, HW6 assigned.

Thm $\mathcal{A} \cong \mathcal{A}^t := \Big\langle$  $\Big\rangle \Big/$ 

$\mathcal{A} = \mathcal{D}/4T$

* connected, oriented vertices, $\deg = \frac{1}{2}(v's)$



Reminders

$\mathfrak{g}$: a metrized Lie algebra

$\langle\ ,\ \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{Q}$ sym, non-deg, inv.

$$\langle [Z,X], Y\rangle + \langle X, [Z,Y]\rangle = 0$$

R: representation of $\mathfrak{g}$

$$\rho : \mathfrak{g} \xrightarrow[\text{alg}]{\text{morphism of Lie}} \operatorname{End}(R) \sim \mathfrak{gl}_{\dim(R)}$$

v.s.

$$\rho : \mathfrak{g} \to M_{n\times n}(\mathbb{Q})$$

$$\rho([X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$$

Constructio/
Thm Given a f.d. Lie alg $\mathfrak{g}$,
and a f.d. representation thereof R
$$\exists\ W_{\mathfrak{g},R} : \mathcal{A} \longrightarrow \mathbb{Q}$$
[A F.T. invariant of knots for each m.]

Construction:

$$\mathfrak{g} = \langle X_a \rangle_{a=1}^{\dim \mathfrak{g}} \qquad R = \langle e_\alpha \rangle_{\alpha=1}^{\dim R}$$

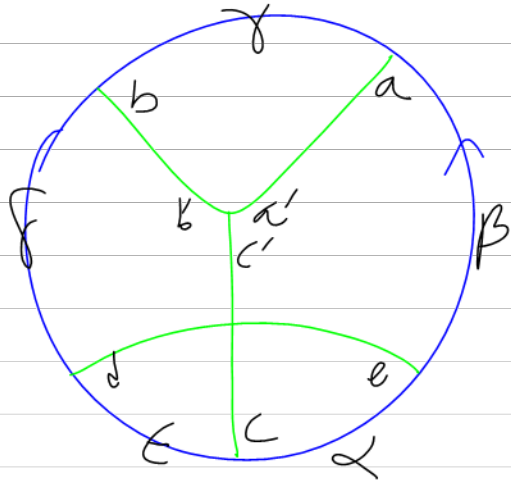
$$[X_a, X_b] = \sum f^c_{\underbrace{ab}_{AS}} X_c = f^c_{ab} X_c$$

"the structure constants"

$$\langle X_a, X_b \rangle = t_{\underbrace{ab}_{sym}} \qquad t_{ab}\, t^{\overbrace{bc}^{sym}} = \delta^c_a$$

$$f_{\underbrace{abc}_{\text{totally AS}}} = \langle [X_a, X_b], X_c \rangle = f^d_{ab}\, t_{dc}$$

$$X_a e_\alpha = \rho(X_a) e_\alpha = r^\beta_{a\alpha} e_\beta$$



$$\longrightarrow \sum_{\substack{a'b'c'\\ abcde \\ \alpha\beta\gamma\delta\epsilon}} F_{a'b'c'}\, t^{aa'}\, t^{bb'}\, t^{cc'}\, t^{ed} \cdot r_{a\beta}^{\gamma}\, r_{b\gamma}^{\delta}\, r_{d\delta}^{\epsilon}\, r_{c\epsilon}^{\alpha}\, r_{e\alpha}^{\beta}$$

$\cancel{M}\cancel{M}M$ $\quad \bar{M}\cancel{M}$ $\quad \bar{M}\cancel{M}$ $\quad \bar{M}\bar{M}$ $\quad \bar{M}\bar{M}$

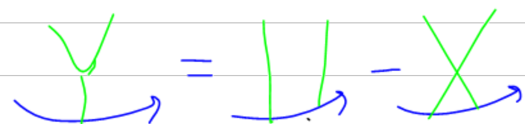
$MN\bar{N}$ $\quad \in \mathbb{Q}$

$F_{\dots} \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}^* \qquad t^{\cdot\cdot} \in \mathfrak{g} \otimes \mathfrak{g} \qquad r \in \mathfrak{g}^* \otimes R^* \otimes R$

AS STU IHX

$\widetilde{AS}$

AS.



Jacobi id.

$\rho([X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$

"representations represent" □

F_{abc}

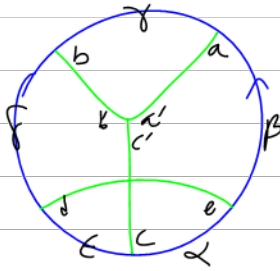
t^{ab} $\qquad \longrightarrow \qquad \int\int\int\int$

$r_{a\alpha}^{\beta}$

Hour 19, Friday October 23: gl(N) and maybe also sl(2).

$\mathfrak{g} = \langle X_a \rangle_{a=1}^{\dim \mathfrak{g}} \qquad R = \langle \ell_\alpha \rangle_{\alpha=1}^{\dim R}$

$f_{abc} = \langle [X_a, X_b], X_c \rangle \qquad \langle X_a, X_b \rangle = t_{ab} \qquad t_{ab} t^{bc} = \delta_a^c \qquad X_a \ell_\alpha = r_{a\alpha}^\beta \ell_\beta$

$W_{\mathfrak{g},R}:$  $\longrightarrow \sum_{\substack{a'b'c' \\ abcde \\ \alpha\beta\gamma\delta\epsilon}} f_{a'b'c'} t^{aa'} t^{bb'} t^{cc'} t^{ed} \cdot r_{a\beta}^{\gamma} r_{b\gamma}^{\delta} r_{d\delta}^{\epsilon} r_{c\epsilon}^{\alpha} r_{e\alpha}^{\beta} \in \mathbb{Q}$

AS IHX
STU

$\mathfrak{gl}_N$ w/ defining rep.

$X_{ij} = \begin{pmatrix} 0 & & 0 \\ & & 1 \\ 0 & & 0 \end{pmatrix} \qquad a \longleftrightarrow (ij)$

$X_{ij} X_{kl} = \delta_{jk} X_{il}$

$[X_{ij}, X_{kl}] = \delta_{jk} X_{il} - \delta_{il} X_{kj} \qquad (AB)_{ik} = \sum A_{ij} B_{jk}$

$t_{(ij)(kl)} = tr(X_{ij} X_{kl}) = \delta_{jk} \delta_{il}$, where $X_{ij} X_{kl} = \delta_{jk} X_{il}$

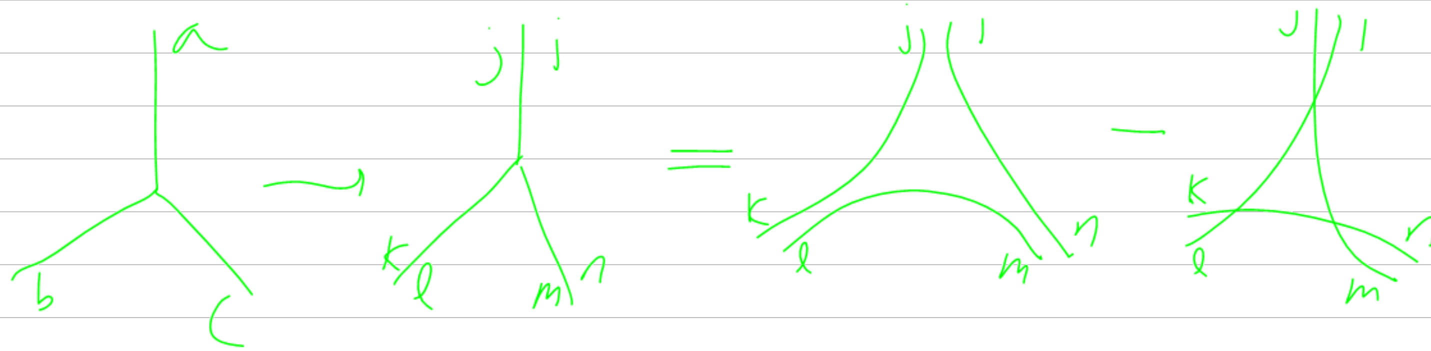
$t^{(ij)(kl)} = \delta_{jk} \delta_{il} \qquad t^{(ij)(kl)} t_{(kl)(mn)} = \sum_{k,l} \delta_{jk} \delta_{li} \delta_{lm} \delta_{kn} = \delta_{jn} \delta_{im}$

$\updownarrow$

$$f_{abc} = f_{(ij)(kl)(mn)} = \langle [X_{ij}, X_{kl}], X_{mn} \rangle$$

$$= \langle \delta_{jk} X_{il} - \delta_{il} X_{kj}, X_{mn} \rangle$$

$$= \delta_{jk}\delta_{in}\delta_{lm} - \delta_{il}\delta_{kn}\delta_{jm}$$

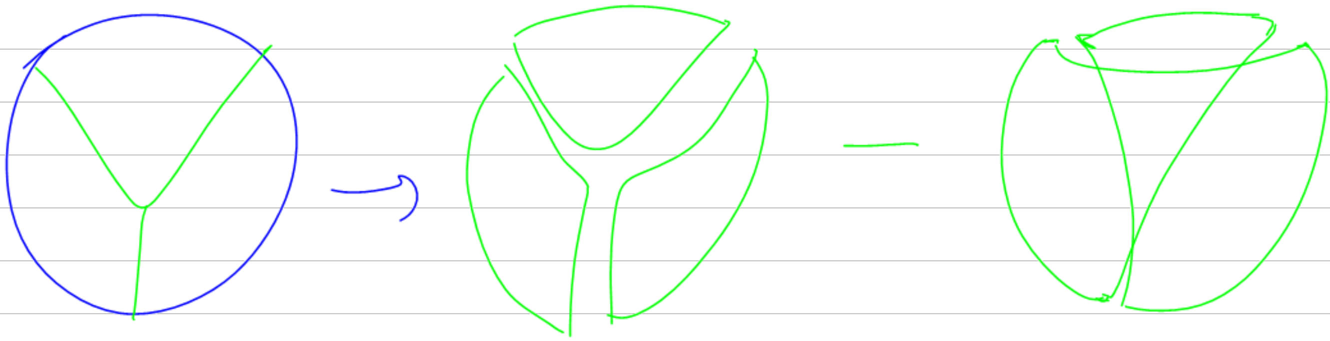


$$\{e_\alpha\} \xrightarrow{R=} \langle e_i \rangle \qquad e_i = i\begin{pmatrix}0\\1\\0\end{pmatrix}$$

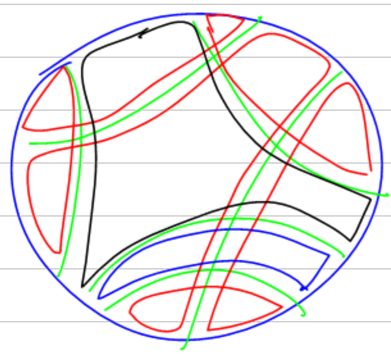
$$X_a e_k = X_{ij} e_k = \delta_{jk} e_i = \sum_l \delta_{kl}\delta_{il} e_l$$

$$V^{l}_{(ij)k} = \delta_{jk}\delta_{il}$$

$$N^2 - N^2 = 0$$



$$= N^3 - N$$



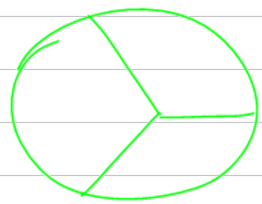
$\to N^3$

$$sl_2 \cong SO_3 = \langle X_1, X_2, X_3 \rangle$$

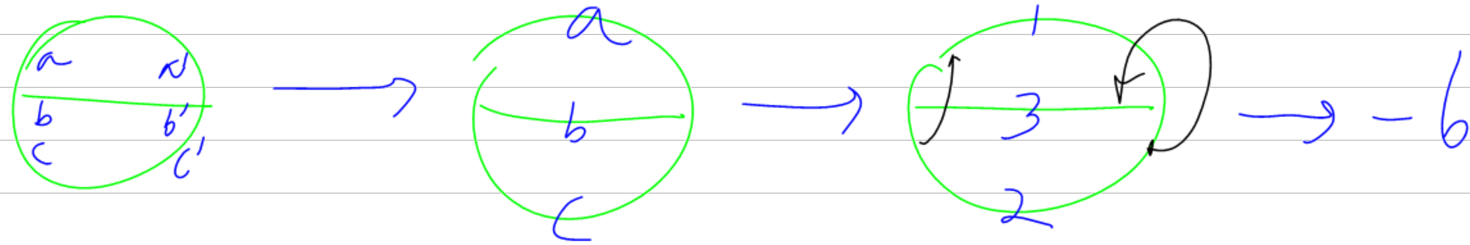
$$\begin{pmatrix} 0&0&0\\0&0&1\\0&-1&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&-1\\0&0&0\\1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&0\\-1&0&0\\0&0&0 \end{pmatrix}$$

$$f_{abc} = \epsilon_{abc} = \begin{cases} (-1)^{abc} & \text{if } [abc] \text{ is a permutation} \\ 0 & \text{otherwise} \end{cases}$$

$$t^{ab} \sim \delta_{ab}$$



$$\begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ [a & b & c] \end{matrix}$$



Thm If D is green-only, and planar, all contributions come w/ same sign

$W_{sl(2)} = W_{so(3)}(D) = \pm \#$ edge 3-colorings of D

$= \pm \#$ Face 4-colorings of D $= \pm \#$ 4 colorings of the map complementary to D

A statement about W equiv to ~~th~~ the 4CT.