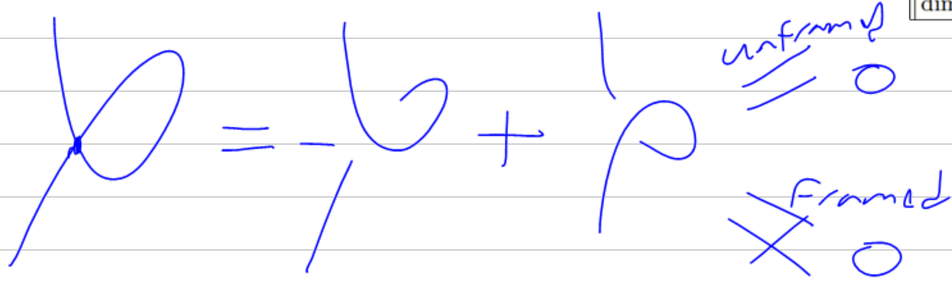
Hour 15, Wednesday October 14: more on \calA

HW4 due by midnight. By then I hope HW5 will be on web and HW3 will be marked.

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}^r_m$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	184	316	548
$\dim \mathcal{P}_m$	0	1	1	1	2	3	5	8	12	18	27	39	55



$C : \{\text{links}\} \longrightarrow \mathbb{Z}[z]$

$C_0\left(\begin{matrix}k\text{ comp}\\ \text{link}\end{matrix}\right) = \delta_{k1}$

$C(\underbrace{\bigcirc \cdots \bigcirc}_{k}) = \delta_{k1}$

$C(\times) = C(\nearrow\!\!\!\!\nwarrow) - C(\nwarrow\!\!\!\!\nearrow) = z\,C(\uparrow\,\uparrow)$

$$C(K) = \sum_{m=0}^{\infty} C_m(K)\cdot z^m$$

is of type m.

$W_{C_m} : \mathcal{A}_m \longrightarrow \mathbb{Z}$

m-singular knot whose underlying c.d. is D.

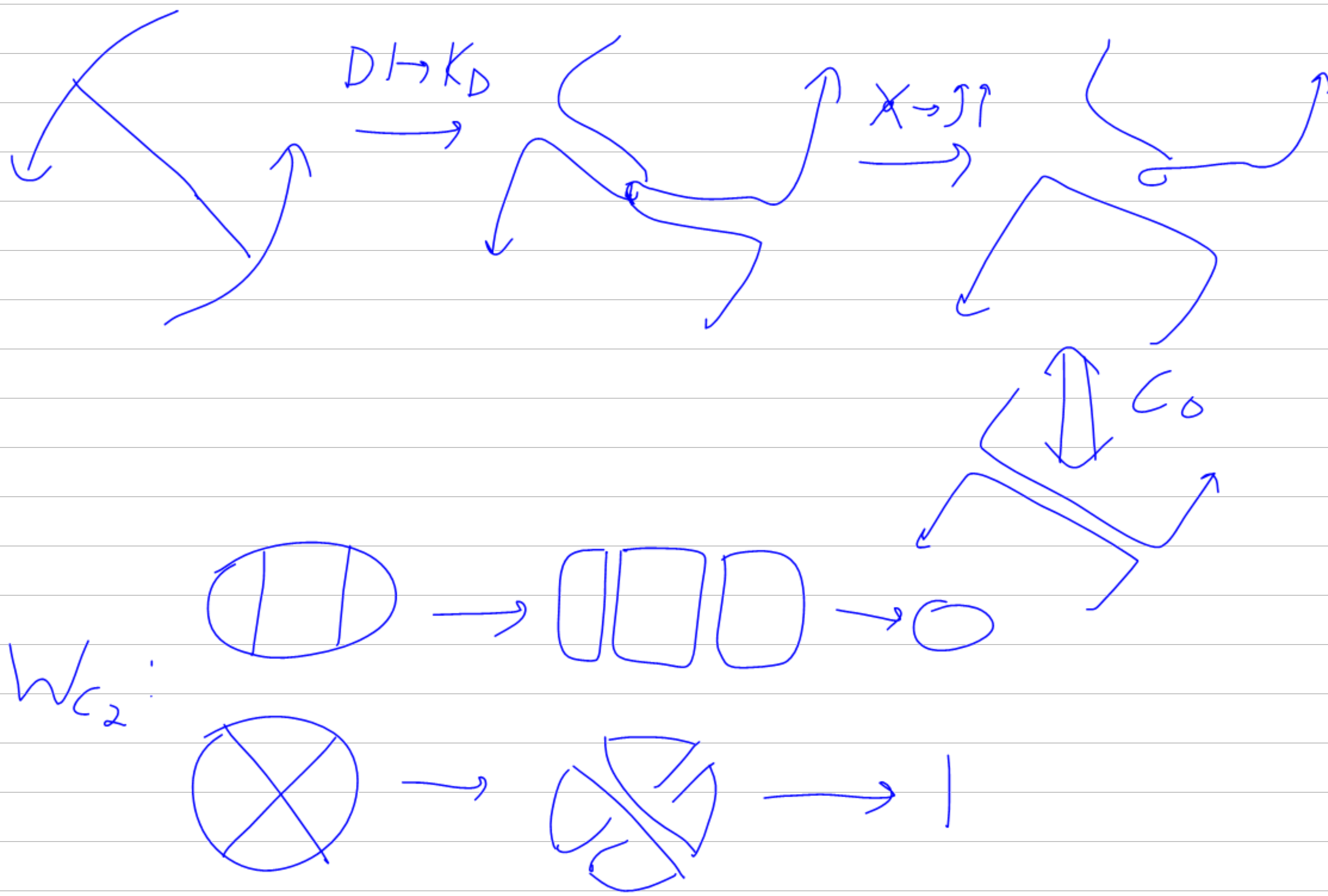
$W_{C_m}\Big(\underbrace{\bigcirc}_{m\text{ chords}} = D\Big) = C_m(K_D)$

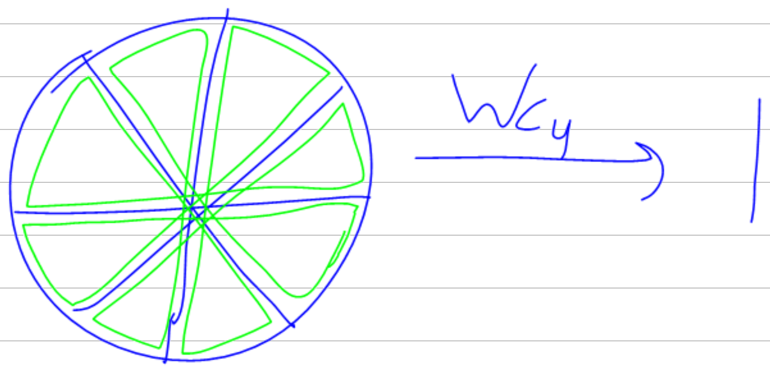
$= \mathrm{Coeff}_{z^m}\left(z^m C\left(K_D / \times \longrightarrow \uparrow\,\uparrow\right)\right)$

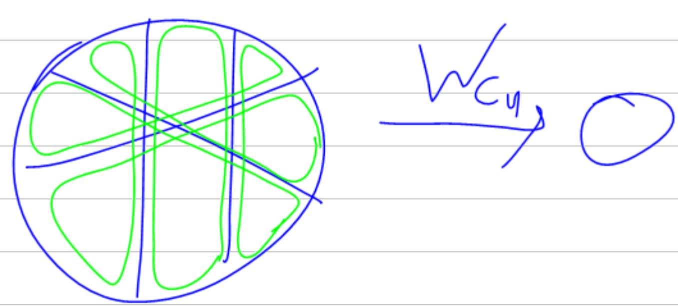
$= \mathrm{Coeff}_{z^0}\left(C\left(K_D / \times \longrightarrow \uparrow\,\uparrow\right)\right)$

$= C_0\left(K_D / \times \longrightarrow \uparrow\,\uparrow\right)$

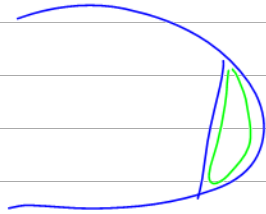
$$W_{C_n}(D) = \begin{cases} 1 & \text{if } K_D / X \longrightarrow \text{ is connected} \\ 0 & \text{otherwise} \end{cases}$$



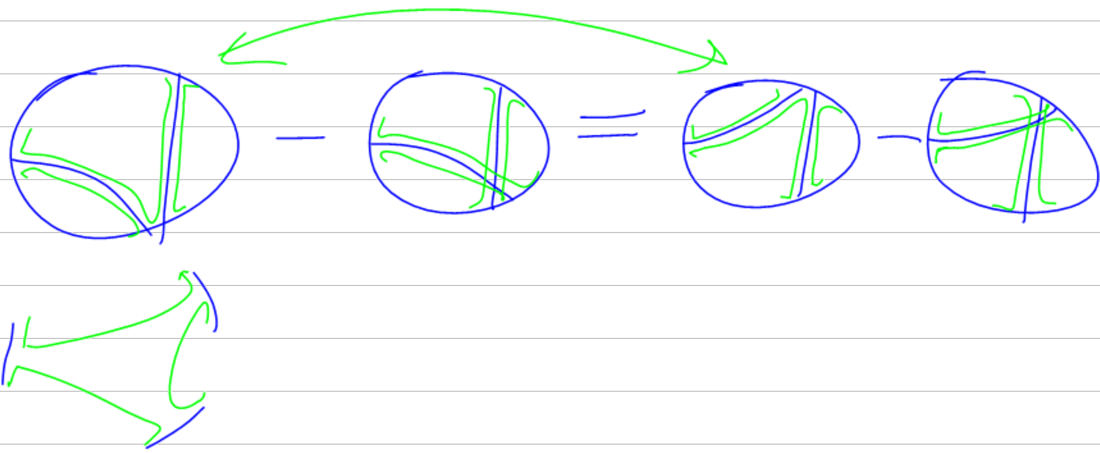




FI

 $\longrightarrow 0$

4T



<u>Prop</u> If $V_1 \in \mathcal{V}_{m_1}$ & $V_2 \in \mathcal{V}_{m_2} \Rightarrow V_1 \cdot V_2 \in \mathcal{V}_{m_1+m_2}$.

<u>Prop</u> $\exists! \ \square : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

$$W_{V_1 \cdot V_2} = \square /\!/ W_{V_1} \cdot W_{V_2}$$

$$\partial\partial\partial\partial\partial\partial\partial\partial(x^3 \cdot x^4) \qquad (fg)' = f'g + fg'$$

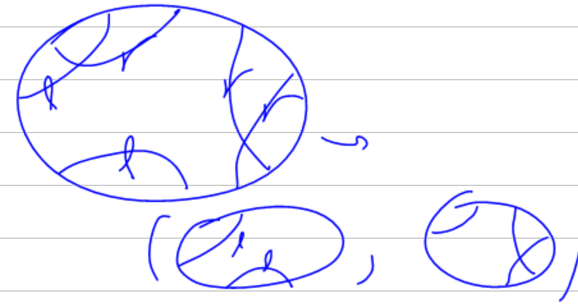
$\deg f = 2$ $\quad$ $\deg g = 3$

$$\partial\partial\partial\partial\partial(f \cdot g)$$

= sum of all ways of splitting 5 ∂ into 2 to left & 3 to rgt.

V_1 of type 2

V_2 of type 3



$W_{V_1 \cdot V_2}$ () $= \sum W_{V_1}(D_L) \cdot W_{V_2}(D_R)$

$2+3$ $\quad$ $\| \; D$

splitting of D into "left" D_l & "right" D_r

$$\square : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$

$$W_{V_1 \cdot V_2}(D) = \square \,//\, W_{V_1} \cdot W_{V_2}$$

<u>claims</u> 1. This is right.

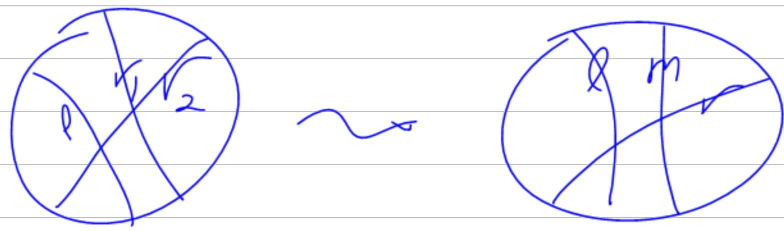
2. $\square : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ descends mod 4T: $\square : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$.

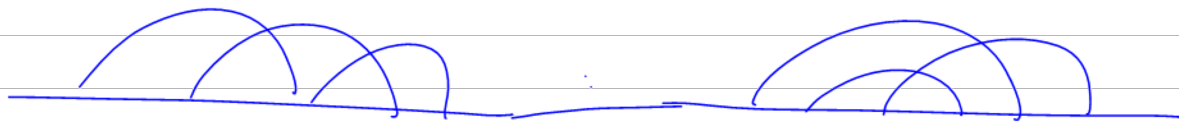
<u>Thm</u> $(\mathcal{A}, m, \square, \bar{\epsilon}, \bar{\eta})$ is

co-unit $\quad$ unit

a connected graded commutative co-commutative bi-algebra.

$$\Box /\!/ 1 \otimes \Box = \Box /\!/ \Box \otimes 1$$





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<u>Thm</u> $(\mathcal{A}, m, \Box, \eta, \epsilon)$ is a connected graded commutative co-commutative bialgebra; $\dim \mathcal{A}_m < \infty$.

$$\mathcal{A} = \left\{ \bigotimes \right\} / 4T \xleftarrow[\sim]{Z} \mathcal{K}$$

1·1·1·1 3·1
1·1·2
2·2

Unit: $1 \in A$: $1a = a \cdot 1 = a$

alt: $\eta : \mathbb{Q} \to A$ "$\alpha \mapsto \alpha \cdot 1$"

$$\begin{tikzcd} \alpha \otimes a & \mathbb{Q} \otimes A \arrow[r, "\eta \otimes I"] \arrow[dr] & A \otimes A \arrow[d, "m"] & \alpha 1 \otimes a \\ & \alpha a & A & \alpha 1 a \end{tikzcd}$$

Co-unit: $\epsilon : A \to \mathbb{Q}$ (in a co-algebra)

$$\begin{tikzcd} & A \arrow[dl, "\Delta"'] \arrow[dr, "\sim"] & \\ A \otimes A \arrow[rr, "\epsilon \otimes I"] & & \mathbb{Q} \otimes A \end{tikzcd}$$

circle, not zero

In $\mathcal{A}$, $\eta : \mathbb{Q} \to \mathcal{A}$ is $\alpha \mapsto \alpha \cdot \bigcirc$

ϵ: $\epsilon(D) = \begin{cases} 1 & \text{if } D = \bigcirc \\ 0 & \text{if } \deg D \geq 1 \end{cases}$

Milnor-Moore Thm (60s) (co-commutative version)

If A is . . . . . ., then A is a graded polynomial algebra . . . .

$$\mathbb{Q}[P_1\ P_2\ P_3 \ldots\ldots] \quad \deg P_i \geq 1$$

$$\mathbb{Q}[x, y] \qquad \begin{matrix}\deg x = 1 \\ \deg y = 2\end{matrix} \qquad \deg xy^2 = 5.$$

$$\Box P_i = 1 \otimes P_i + P_i \otimes 1 \qquad m, \epsilon$$

$$\Box(xy^2) = \Box(x) \cdot (\Box(y))^2$$

$$= (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y)^2 = \ldots$$

If P is a graded vector space,

$$S(P) = \left\langle \underset{P_i \in P}{P_1 \otimes P_2 \otimes \ldots \otimes P_n} \right\rangle \Big/ \ \begin{matrix} \forall \sigma \in S_n \\ P_1 \otimes \ldots \otimes P_n = \\ P_{\sigma 1} \otimes \ldots \otimes P_{\sigma n} \end{matrix}$$

If $p \in P$ decree that $\Box(p) = p \otimes 1 + 1 \otimes p$.

$$S(\text{v.s.} \langle P_1 \ldots P_n \ldots \rangle) = \mathbb{Q}[P_1 \ldots P_n].$$

MM: $$A \underset{\text{as a bialgebra}}{\cong} S(P(A))$$

$P(A) : \{ p \in A : \Box p = p \otimes 1 + 1 \otimes p \}$

$\mathbb{Q}[x, y]$ $\quad \deg(x) = 1$, $\deg(y) = 2$

$y + 0.01 \cdot x^2$

$\mathcal{A} = \langle \ominus, \otimes - ⦶, \ldots \rangle$

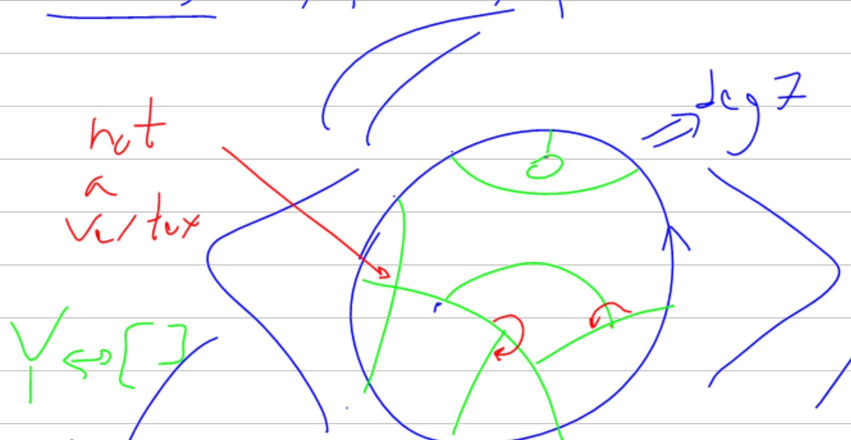
$\mathcal{A}^r = \mathcal{A} / \langle \ominus \rangle \qquad) = 0$

$\mathcal{A} \xrightarrow{proj} \mathcal{A}^r \xrightarrow{HWI} \mathcal{A}$

$\mathcal{A}^r = \mathbb{Q}\langle \otimes - ⦶, \ldots \rangle$

$\mathcal{A} \otimes \begin{pmatrix} \text{Lie Algebras} \\ \text{\& reps} \end{pmatrix} \longrightarrow \mathbb{Q}$

<u>Thm</u> $\mathcal{A} \cong \mathcal{A}^t \longrightarrow$ trivalent temporary



AS: Y + Y = 0

STU: Y = U − X

IHX: I = H − X

oriented vertices / connected / degree $= \frac{1}{2}$(total # of verts.)

mult.

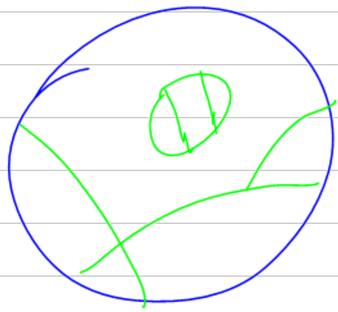
$$L \quad [\,,\,] : L \otimes L \to L \qquad [A,B] = AB - BA$$

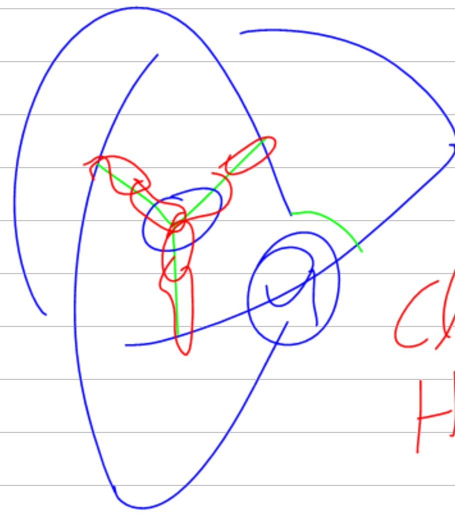
$$[a,b] = -[b,a]$$

$$0 = [a,[b,c]] + [b,[c,a]] + [c,[a,b]]$$

"Jacobi-identity"

Not connected:





Claspers
Habiro.