Hour 12, Monday October 5: The basics of finite type invariants.

$V : \left\{ \begin{array}{l} \text{oriented knots} \\ \text{in oriented } \mathbb{R}^3 \end{array} \right\} \longrightarrow A$ (an Abelian group)

$$V^{(m)}(\underbrace{\times \cdots \times}_{m}) := V^{(m-1)}(\times \cdots \times) - V^{(m-1)}(\times \cdots \times)$$

"V of type m" means $V^{(m+1)} = 0 = V(\underbrace{\times \cdots \times}_{> m})$

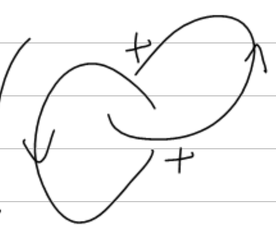
Example 0 $V \overset{(0)}{\equiv} C$ of type 0:

$$V^{(1)} = V^{(0)}(\times) - V^{(0)}(\times) = C - C = 0$$

Example 1 If L is a 2-component link,

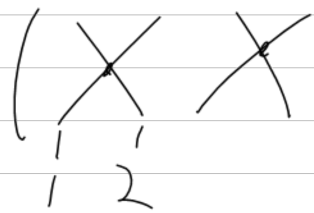
$$lk(L) = \frac{1}{2} \sum_{x:\ \text{xings between the two components}} (-1)^x$$

linking number of K

$lk($  $) = \frac{1}{2}(1+1) = 1$

invt under R1 (doesn't count), R2, R3 (already checked)

$lk(\times) = 0$ $lk(\times) = 1$

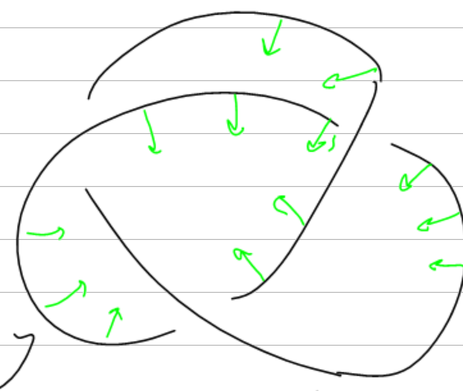
$lk($  $) = $ 0 − 0 / 1 − 1 $= 0$

$\Rightarrow lk$ is of type 1.

If L is a link w/ n components, labeled $1 \ldots n$, let l_{ij} be the linking number of comp $\# i$ w/ comp $\# j$:

$$l_{ij} = \frac{1}{2} \sum_{\substack{x:\ \text{xing} \\ \text{between } i \\ \& \ j}} (-1)^{x}$$

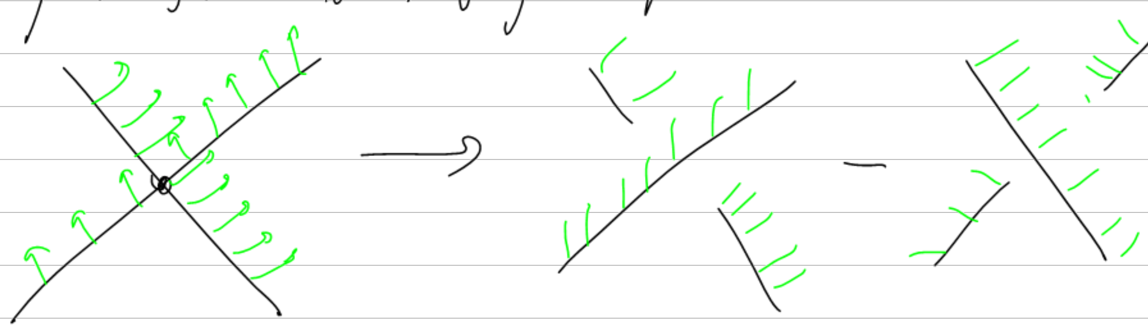
l_{ij} is of type 1.



Example 3 Framed Knot:

A knot w/ an up-to-homotopy choice of a $\neq 0$ normal at every pt.

A framed knot differs from an unframed one by just one integer parameter.



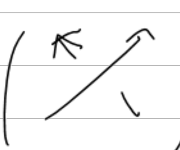
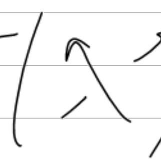
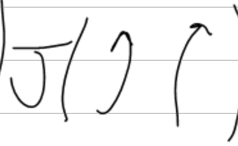
Def If K is a framed knot,

$$sl(K) = lk(K, K^+)$$

sl ← self linking; K^+ ← the push of K in the dir of the framing.

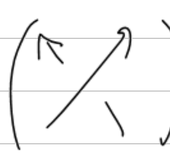
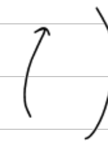
Exercise sl is of type 1.

Exercise What's the relationship between $sl(K)$ & $w(K)$

$q^{-1} J($  $) - q J($  $) = (q^{1/2} - q^{-1/2}) J($  $)$

The Conway polynomial:

$$C : \{\text{knots \& links}\} \longrightarrow \mathbb{Z}[z]$$

$C($  $) - C($  $) = z \cdot C($  $)$

$= C($  $)$

$$C(O^k) = \begin{cases} 1 & k=0 \\ 0 & \text{otherwise.} \end{cases}$$

Thm This gives a well-defined invts.

Thm Write $C(L)(z) = \sum_{m=0}^{\infty} V_m(L) z^m$

$V_m(L)$ ← invts in $\mathbb{Z}$.

then V_m is of type m.

$$C(\underbrace{\times \cdots \times}_{m+1}) = z\,C(\nwarrow\!\!\!\nearrow \times \cdots \times)$$

$$\vdots$$

$$= z^{m+1} C(\nwarrow\!\!\!\nearrow \cdots \nwarrow\!\!\!\nearrow)$$

So $V_m(\underbrace{\times \cdots \times}_{m+1})$ = coeff of z^m in $C(\underbrace{\times \cdots \times}_{m+1}) = 0$

<u>Example 5</u>

$$q^{-1} J(\text{positive crossing}) - q\,J(\text{negative crossing}) = (q^{1/2} - q^{-1/2})\,J(\nwarrow\!\!\!\nearrow)$$

$$J(L)(q = e^x) = \sum_{m=0}^{\infty} V_m(L)\,x^m$$

($V_m(L)$ is of type m.)

$$(\underline{\underline{1}} - x + \cdots)\,J(\text{positive crossing}) - (\underline{\underline{1}} + x + \cdots)\,J(\text{negative crossing})$$

$$= (x + \cdots)\,J(\nwarrow\!\!\!\nearrow)$$

$$J(\times) = J(\text{positive crossing}) - J(\text{negative crossing}) = \begin{matrix}(x + \cdots)(J(\text{positive crossing}))\\(x + \cdots)\,J(\text{negative crossing})\\(x + \cdots)\,J(\nwarrow\!\!\!\nearrow)\end{matrix} = x\,(\text{Junk})$$

$$J(\underbrace{\times \cdots \times}_{m+1}) = x^{m+1}(\text{Junk})^{m+1} \qquad \square$$

$$q = (1 + x + \cdots$$

$$q^{-1} = (1 - x + \cdots)$$

Hour 13, Wednesday October 7: The basics of finite type invariants.

HW 2 will not be marked.
HW3 due at midnight (extensions will be considered)
HW 4 on web soon.

$$V \in \mathcal{V}_m \iff V^{(m+1)} = 0 \iff V(\underbrace{\times \cdots \times}_{>m}) \equiv 0$$

$\mathcal{V}_m$ ↑ v.s. of type m invts

$\mathcal{A}$ v.s. or $\mathbb{Z}$-module $\supset \mathcal{V}_{m-1}$

$V^{(m+1)} \equiv 0 \Longrightarrow V^{(m)}$ is "constant"
"independent of the embedding into the ambient space $\mathbb{R}^3$"

$V \in \mathcal{V}_m \Longrightarrow$

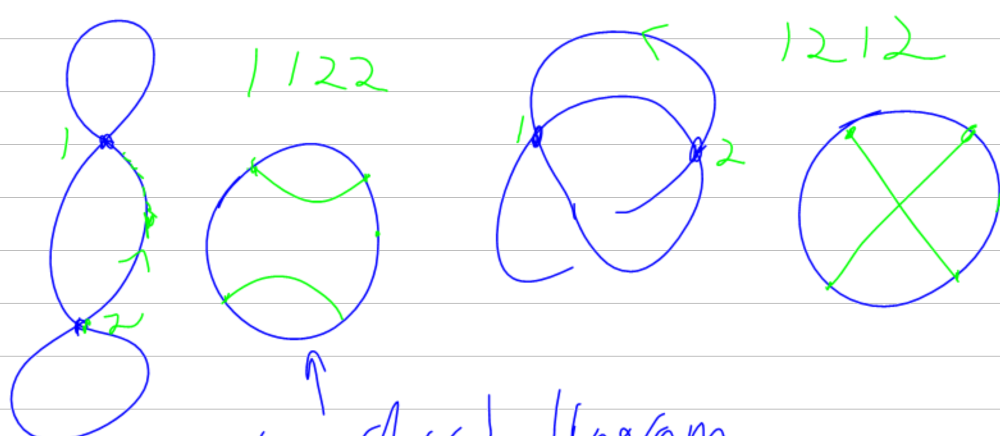
$$V(\underbrace{\times \cdots \times}_{m} \diagdown\!\!\diagup) - V(\times \cdots \times \diagup\!\!\diagdown) = V(\underbrace{\times \cdots \times}_{m} \times) = 0$$

So

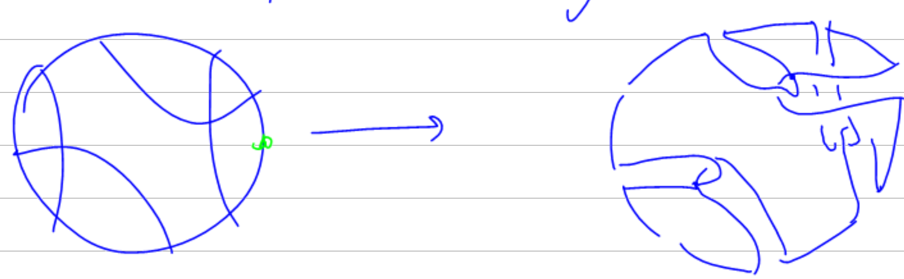
$$V^{(m)}: \{m\text{-singular knots}\} \big/ \left(\diagdown\!\!\diagup = \diagup\!\!\diagdown\right) \longrightarrow \mathcal{A}$$

$\xrightarrow[\text{ex.}]{1\text{-}1?}$ {m chord diagrams}

m = 2



m-chord diagram



$W_V = V^{(m)} : \{ m\text{-chord diagrams} \} \longrightarrow A$

"weight system of V"

$\updownarrow$

$W_V : \mathcal{D}_m \to A \qquad \mathcal{D}_m = A\langle m\text{-chord diagrams} \rangle$

Cor $\mathcal{V}_m$ is an A-module of finite rank.

Pf $\mathcal{V}_{m-1} \longrightarrow \mathcal{V}_m \xrightarrow{V \mapsto W_V} \mathcal{D}_m^*$

$\mathcal{V}_{m-1}$: finite rank by induction

onto?

$\mathcal{D}_m^*$: finite rank

$(\mathcal{V}_0 = \{\text{const}\})$

$V_1 \quad V_2$

$W_{V_1} = W_{V_2}$

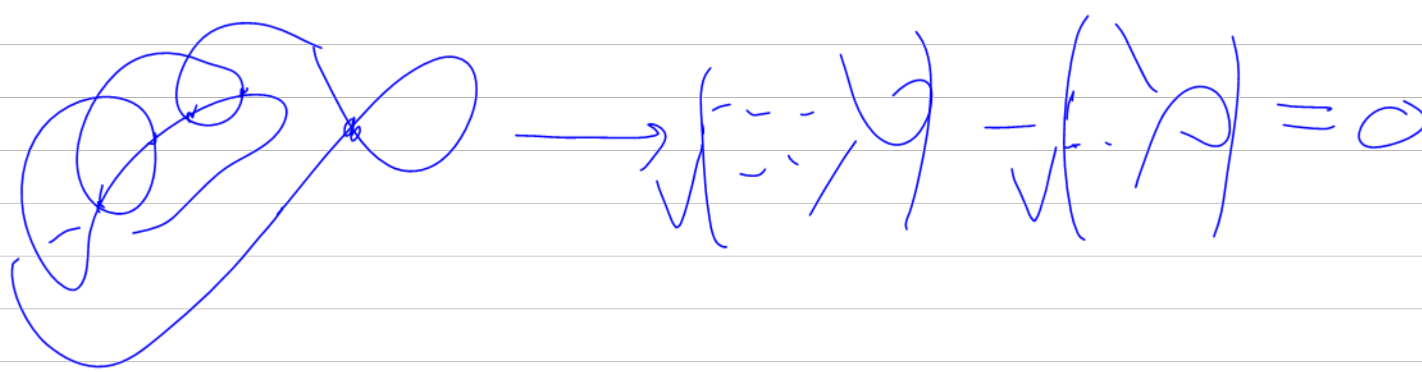
$V_1^{(m)} = V_2^{(m)}$

$(V_1 - V_2)^{(m)} = 0$

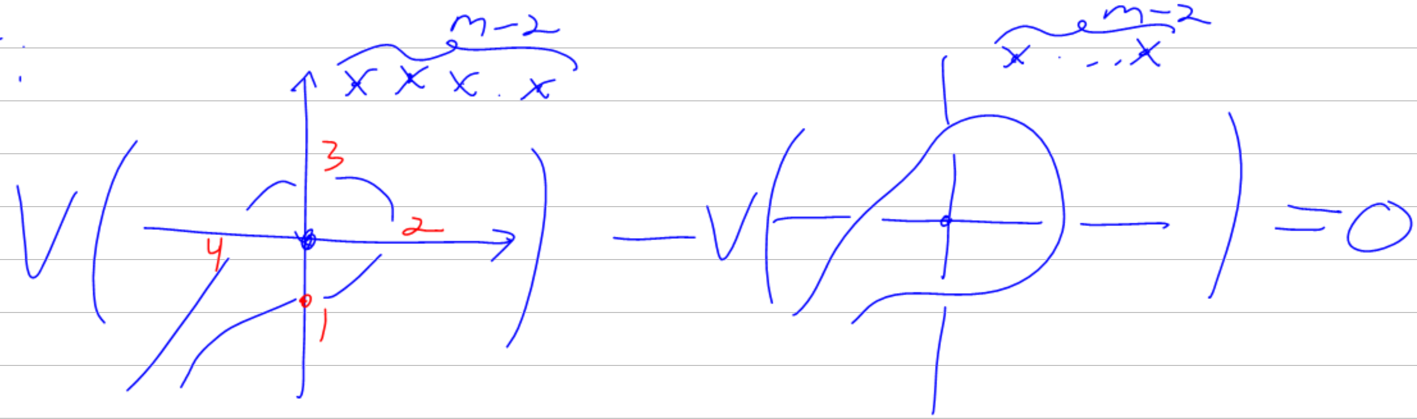
$V_1 - V_2 \in \mathcal{V}_{m-1}$

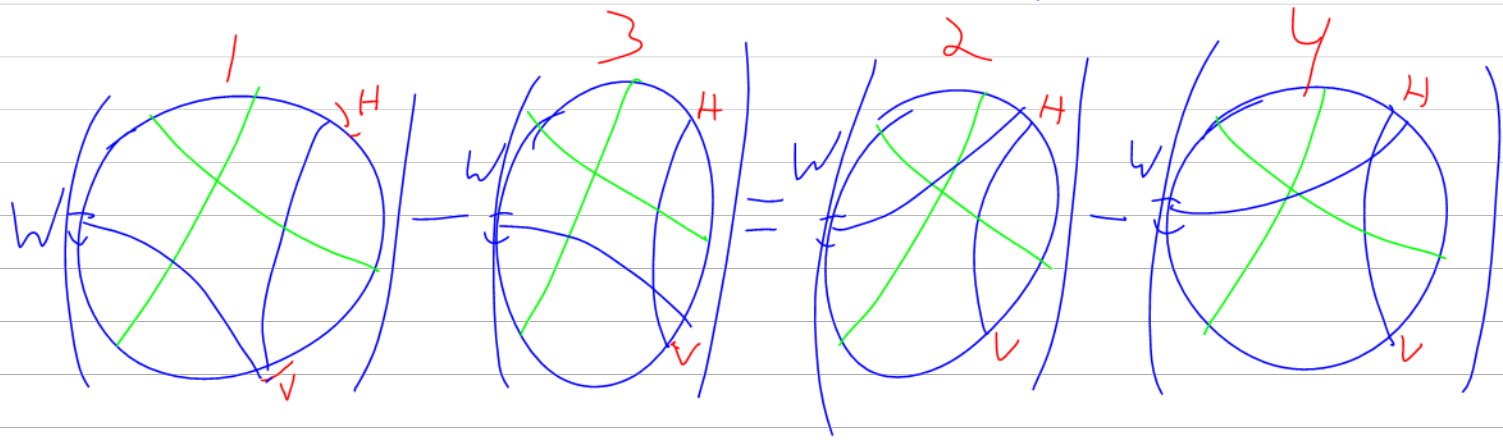
W_V satisfies two relation:

FI: Framing Independence



4T:





<u>Thm</u> "The Fundamental Thm of F.T. Invariants": Over $\mathbb{Q}$, this is all.

If $W \in \mathcal{D}_m^*$ satisfies FT & 4T, then $\exists V \in \mathcal{V}_m$ s.t. $W_V = W$.

over $\mathbb{Q}$ $\quad \mathcal{V}_m / \mathcal{V}_{m-1} \cong \left(\mathcal{D}_m / {FI \atop 4T} \right)^*$

No class on Monday!

The Fundamental Thm of FTI (Kontsevich+....): $\forall W \in \mathcal{W}_m := (\mathcal{D}_m/FI, 4T)^*$ $\exists V \in \mathcal{V}_m$ s.t. $W = W_V$ $(\sim V^{(m)})$. $\mathcal{A}^r_m$

$\mathcal{A}_m := \mathcal{D}_m / 4T$

m	0	1	2	3	4	5	6	7	8	9	10	11	12
dim $\mathcal{A}^r_m$	1	0	1	1	3	4	9	14	27	44	80	132	232
dim $\mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	184	316	548

Def $\mathcal{A}^{(r)} = \bigoplus_{m=0}^{\infty} \mathcal{A}^{(r)}_m$ $\quad$ $\hat{\mathcal{A}}^{(r)} := \prod_{m=0}^{\infty} \mathcal{A}^{(r)}_m$

$(\mathcal{A}_m)^* \sim$ invts of type m

$(\mathcal{A})^* \sim$ invts of all type invts $\sim (\mathcal{K})^*$ → Knots.

$\mathcal{K}$ has a commutative product

$\mathcal{K}(\bigcirc) \cong \mathcal{K}(\uparrow)$:

long knots

clearly has a product

$-\boxed{K_1}- \# -\boxed{K_2}- = -\boxed{K_1}-\boxed{K_2}\rightarrow$?= $-\boxed{K_2}-\boxed{K_1}\rightarrow$

Commutative!

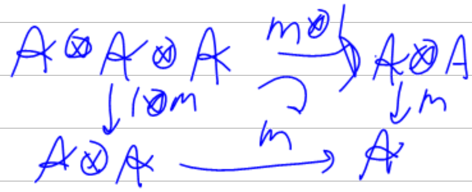
$\mathcal{K}^*$ also a commutative algebra

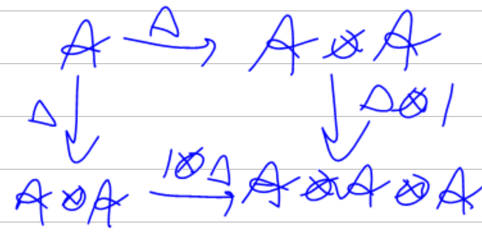
$\mathcal{A}_0 = \{\bigcirc\}$

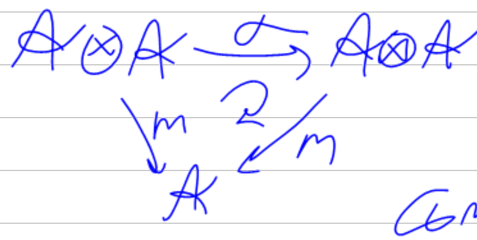
Thm $\mathcal{A}$ is a connected graded commutative & co-commutative bi-algebra:

$\dim \mathcal{A}_0 = 1$ $\quad \mathcal{A} = \bigoplus_{m=0} \mathcal{A}_m \quad m \geq 0$

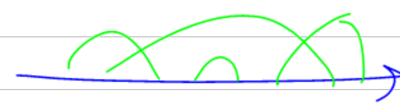
$m: \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ $\qquad \Delta: \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$

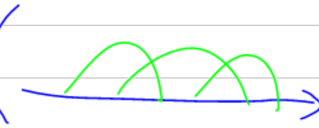




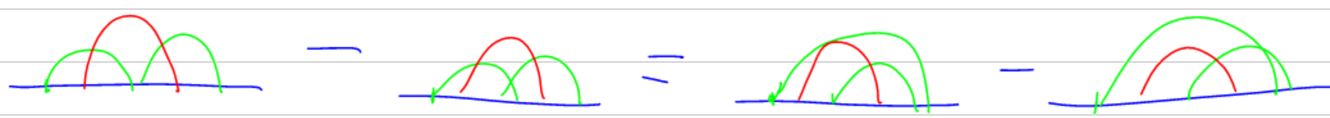


Compatibility

$K(\bigcirc) \cong K(\uparrow)$ $\qquad$ FI:  $= 0$

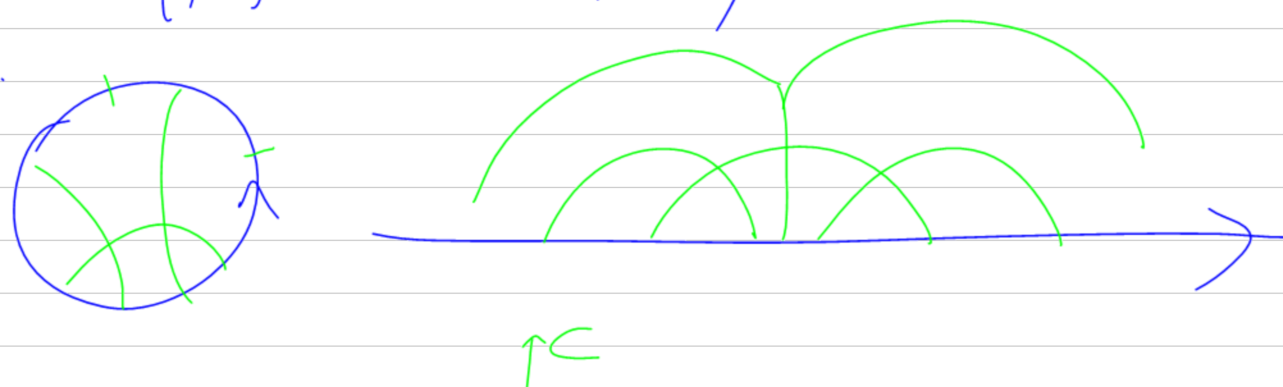
Thm $\mathcal{A}(\bigcirc) \cong \mathcal{A}(\uparrow) = \langle$  $\rangle / 4T$

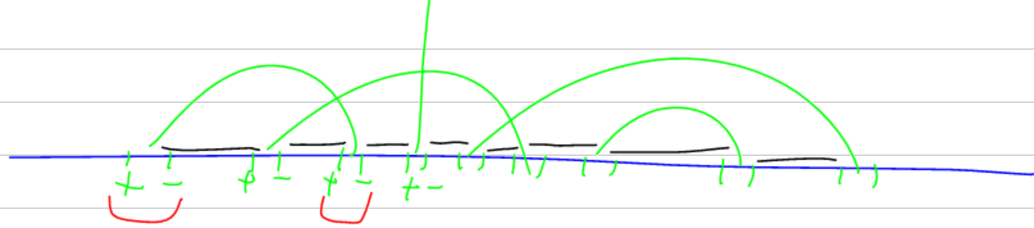
4T



Pf $\mathcal{A}(\uparrow) \longrightarrow \mathcal{A}(\bigcirc)$ obvious.

$\Leftarrow$:





S = sum of all ways of connecting C to the hooks, with signs $\in \mathcal{A}$

= sum over chords (sum of hooks next to its ends) + 2 ~~for the connections~~ $= 0$

= sum over edges (~~sum over & hooks on~~ on includge) + cont from hooks at ends

clearly associative.

Prop 1 If $V_1 \in \mathcal{V}_{m_1}$ & $V_2 \in \mathcal{V}_{m_2}$ then

$$V_1 \cdot V_2 \in \mathcal{V}_{m_1+m_2}$$

$\xrightarrow{f} \xrightarrow{g}$

$g \circ f$

$f /\!/ g := g \circ f$

Prop 2 $\exists! \, \square : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

$$W_{V_1 \cdot V_2} = \square /\!/ W_{V_1} \cdot W_{V_2}$$

$$\mathcal{A} \to \mathcal{A}\otimes\mathcal{A} \xrightarrow[W_{V_2}]{W_{V_1}\otimes} R\otimes R \to K$$