Hour 9, Monday September 28: Khovanov homology following http://drorbn.net/mo13 (Power outage at 2PM?)

The Jones polynomial: $\bigcirc^k \mapsto (q+q^{-1})^k$

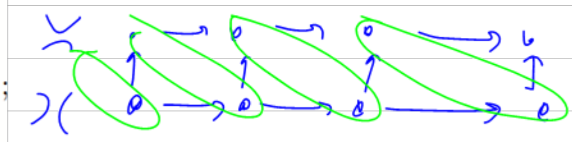
$J : \times \mapsto q)(- q^2 \asymp, \quad J : \times \mapsto -q^{-2} \asymp + q^{-1})($

Khovanov: $K(L)$ is a chain complex of graded $\mathbb{Z}$-modules;

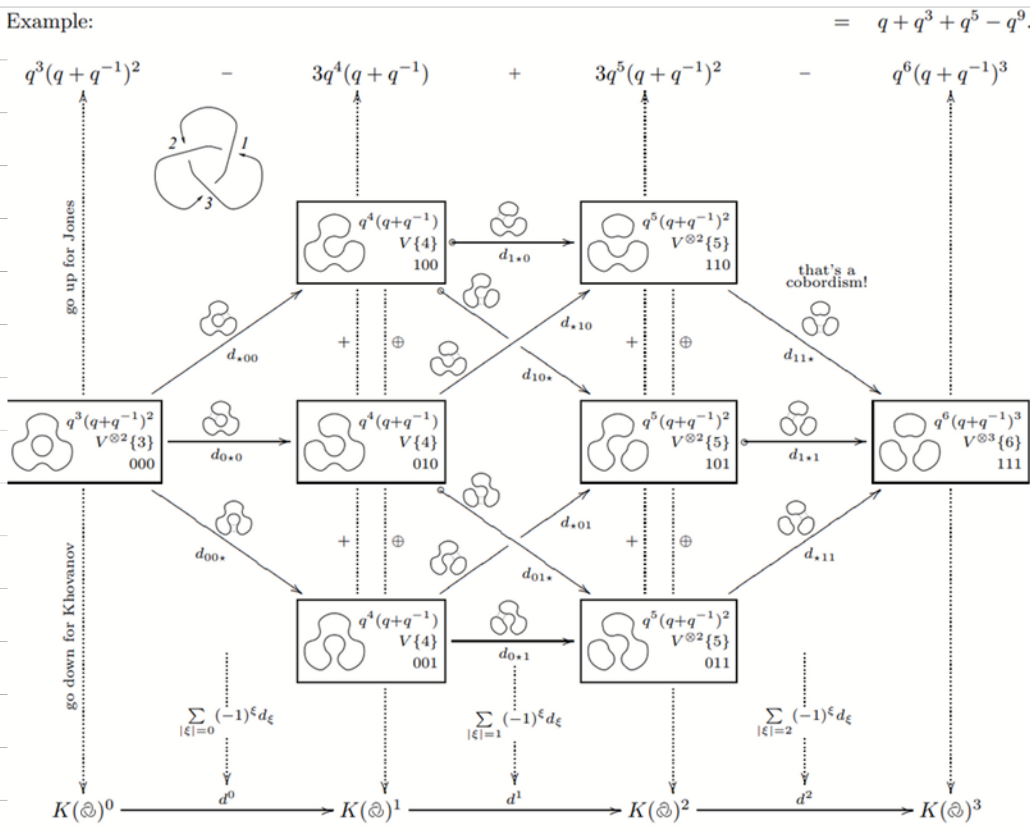
$V = \operatorname{span}\langle v_+, v_-\rangle; \quad \deg v_\pm = \pm 1; \quad q\dim V = q + q^{-1};$

$K(\bigcirc^k) = V^{\otimes k}; \quad K(\times) = \text{Flatten}\left(0 \to \underset{\text{height } 0}{K()(\{1\}} \to \underset{\text{height } 1}{K(\asymp)\{2\}} \to 0\right);$

$\to 0 \longrightarrow \underset{\text{ht } 0}{V^{\otimes k}} \longrightarrow 0 \ldots$



$K(\times) = \text{Flatten}\left(0 \to \underset{\text{height } -1}{K(\asymp)\{-2\}} \to \underset{\text{height } 0}{K()(\{-1\}} \to 0\right);$



(here $(-1)^\xi := (-1)^{\sum_{i<j}\xi_i}$ if $\xi_j = \star$) $= K(\text{trefoil})$.

Need: Such that:

$(\bigcirc\bigcirc \longrightarrow \bigcirc) \longrightarrow (V \otimes V \xrightarrow{m} V)$ 1. $\deg m = \deg \Delta = -1$.

$(\bigcirc \longrightarrow \bigcirc\bigcirc) \longrightarrow (V \xrightarrow{\Delta} V \otimes V)$ 2.

Aside: Hopf Algebras: $(a+b)\cdot c = ac + bc$

Algebra: A a v.s. w/ $m : V \otimes V \longrightarrow V$

$$\begin{tikzcd}[column sep=large] a\otimes b\otimes c & (ab)\otimes c \\ V\otimes V\otimes V \arrow[r, "m\otimes 1"] \arrow[d, "1\otimes m"'] \arrow[dr, phantom, "\circlearrowright"] & V\otimes V \arrow[d, "m"] \\ V\otimes V \arrow[r, "m"] & V \\ a\otimes (bc) \arrow[r] & a(bc) = (ab)\cdot c \end{tikzcd}$$

"Associativity"

$$\begin{tikzcd} a\otimes b \arrow[rr] & & b\otimes a \\ V\otimes V \arrow[rr, "\sigma"] \arrow[dr, "m"'] & \circlearrowright & V\otimes V \arrow[dl, "m"] \\ & V & \end{tikzcd}$$

"Commutative law"

Co-algebra C w/ $\Delta : C \to C\otimes C$

s.t.

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowright"] & C\otimes C \arrow[d, "\Delta\otimes 1"] \\ C\otimes C \arrow[r, "1\otimes\Delta"] & C\otimes C\otimes C \end{tikzcd}$$

"Co-associativity"

Co-commutative: if

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C\otimes C \arrow[rr, "\sigma"] & & C\otimes C \end{tikzcd}$$

commutes.

<u>Claim</u> If A is a f.d. algebra, then $C = A^*$ is a co-algebra. $m : A\otimes A \to A$

Indeed $\Delta = m^* : A^* \to A^*\otimes A^*$

$C \to C\otimes C$

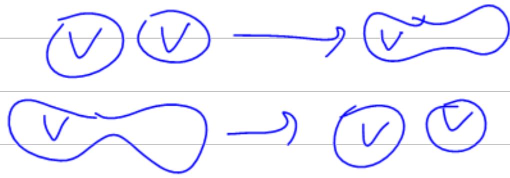
<u>Hopf Algebra</u> A v.s. which is both an algebra & a co-alg at the same time (has m, Δ; satisfying both assoc. & co-assoc) [Forgetting axioms related to units & "inverses"]

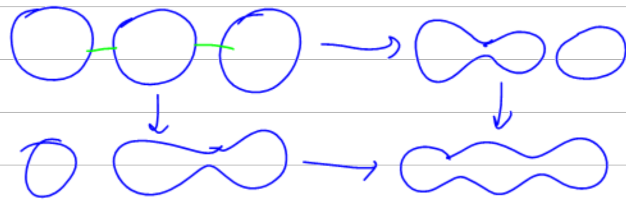
and s.t.

$(a\otimes b)(c\otimes d) = (ac\otimes bd)$

$\Delta : A \to A\otimes A$ is a morphism of algebras.

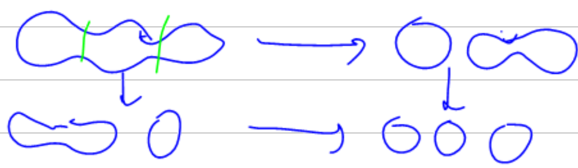
$m: V\otimes V \to V$ s.t. 1. $\deg m = \deg \Delta = -1$
$\Delta: V \to V\otimes V$ 2. m is commutative.
3. Δ is co-commutative
4. m is associative
5. Δ co-associative



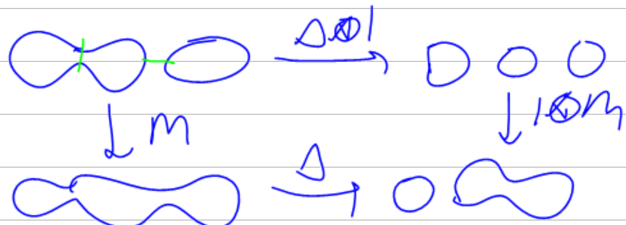


$\to$

$$\begin{tikzcd} V\otimes V\otimes V \arrow[r, "m\otimes 1"] \arrow[d, "1\otimes m"'] & V\otimes V \arrow[d, "m"] \\ V\otimes V \arrow[r, "m"'] & V \end{tikzcd}$$



$$\begin{tikzcd} V \arrow[r] \arrow[d] & V\otimes V \arrow[d] \\ V\otimes V \arrow[r] & V\otimes V\otimes V \end{tikzcd}$$



6. $\Delta\otimes 1 \,/\!/\, 1\otimes m = m \,/\!/\, \Delta$

2 $\to$ 2+7 1 $\to$

$$\begin{aligned} m:\ & V_+\otimes V_+ \longrightarrow V_+ \\ & V_+\otimes V_- \longrightarrow V_- \\ & V_-\otimes V_- \longrightarrow 0 \end{aligned}$$

$$\begin{aligned} \Delta:\ & V_+ \longrightarrow (V_+\otimes V_- + V_-\otimes V_+) \\ & V_- \longrightarrow (V_-\otimes V_-) \end{aligned}$$

W1: $n>0$, $a_i>0$, $\sum a_i = b$ known to W1 & W2
$\prod a_i = a$ age of W1,
W2: If I knew n & a I'd know a_i
W1: No you couldn't
W2: I know a; what's b?

Hour 10, Wednesday September 30: Khovanov homology following http://drorbn.net/mo13

HW3 on web!
HW2 due tonight!

The Jones polynomial: $\bigcirc^k \mapsto (q+q^{-1})^k$

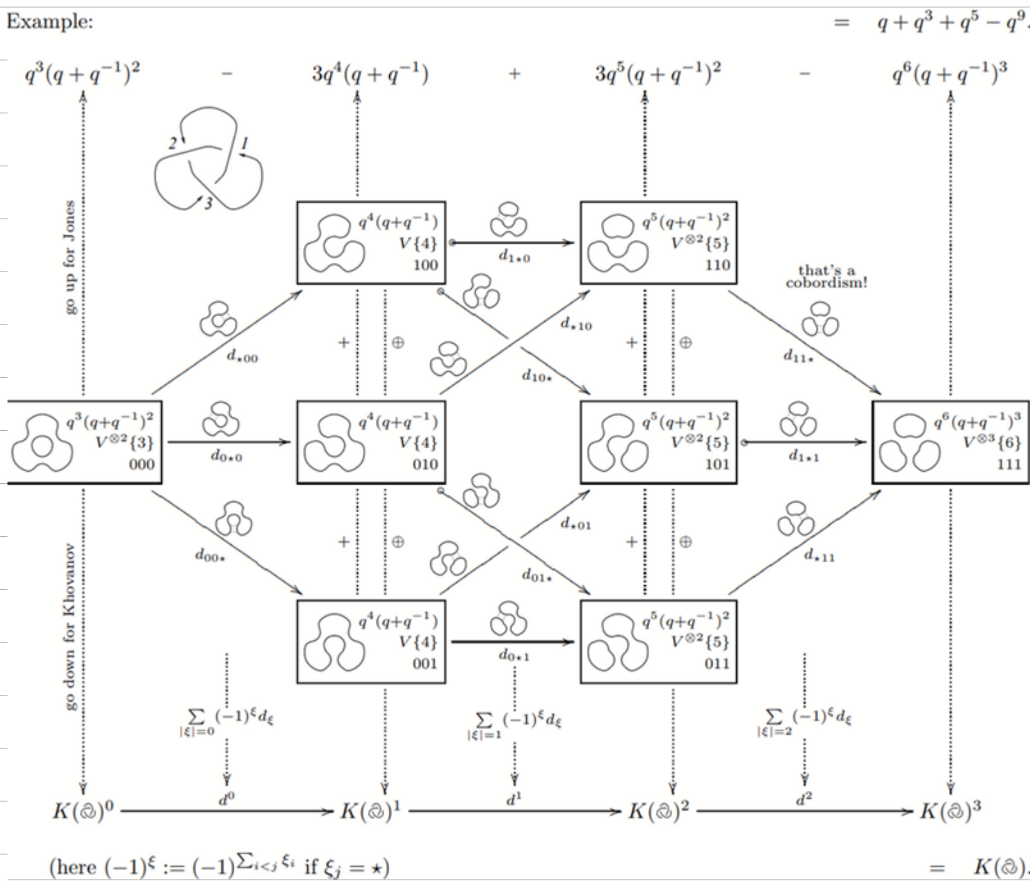
$J : \overcrossing \mapsto q)(-q^2\asymp, \quad J : \undercrossing \mapsto -q^{-2}\asymp + q^{-1})($

Today: The most disturbing open problem about Khovanov homology.

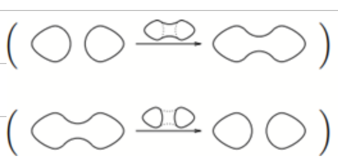
Khovanov: $K(L)$ is a chain complex of graded $\mathbb{Z}$-modules;
$V = \mathrm{span}\langle v_+, v_-\rangle$; $\deg v_\pm = \pm 1$; $q\dim V = q + q^{-1}$;

$$K(\bigcirc^k) = V^{\otimes k}; \quad K(\overcrossing) = \text{Flatten}\left(0 \to \underset{\text{height } 0}{K()()\{1\}} \to \underset{\text{height } 1}{K(\asymp)\{2\}} \to 0\right);$$

$$K(\undercrossing) = \text{Flatten}\left(0 \to \underset{\text{height } -1}{K(\asymp)\{-2\}} \to \underset{\text{height } 0}{K()()\{-1\}} \to 0\right);$$



Need:



$\longrightarrow (V \otimes V \xrightarrow{m} V)$

$\longrightarrow (V \xrightarrow{\Delta} V \otimes V)$

$$m : \begin{cases} v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto 0 \end{cases}$$

$$\Delta : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$$

Such that:
1. $\deg m = \deg \Delta = -1$
2. m is commutative and associative
3. Δ is co-commutative and co-associative
4. A funny "Frobenius compatibility" of m & Δ holds.

1. $v_+(v_-v_+) \overset{?}{=} (v_+v_-)v_+$ ✓

2. $v_+ \xrightarrow{\Delta} v_+v_- + v_-v_+ \xrightarrow{1\otimes\Delta} v_+v_-v_- + v_-(v_+v_- + v_-v_+)$ ✓
$\xrightarrow{\Delta\otimes 1}$ − − − − − −

<u>Claim</u> $(V, m) \cong \mathbb{Z}[x]/x^2=0$ $\quad 1 \to v_+$, $x \to v_-$

<u>Thm</u> Suppose $\mathcal{C}' \subset \mathcal{C}$ is an inclusion $\text{)}\sim\text{)(}$ of complexes

$$\begin{array}{ccccccccc}
\mathcal{C}' & & C'^{r-1} & \to & C'^{r} & \to & C'^{r+1} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C} & \cdots \to & C^{r-1} & \to & C^{r} & \to & C^{r+1} & \to & \cdots \\
\mathcal{C}/\mathcal{C}' & & & \to & C^r/C'^r & \to & & &
\end{array}$$

A. If $H(\mathcal{C}')=0$ then $H(\mathcal{C}/\mathcal{C}') = H(\mathcal{C})$
"$\mathcal{C}'$ is acyclic"

B If $H(\mathcal{C}/\mathcal{C}')=0$ then $H(\mathcal{C}) = H(\mathcal{C}')$

<u>Pf</u> Whenever you have a short exact seq. of cmplxs:

$$0 \to \mathcal{C}' \to \mathcal{C} \to \mathcal{C}/\mathcal{C}' \to 0$$

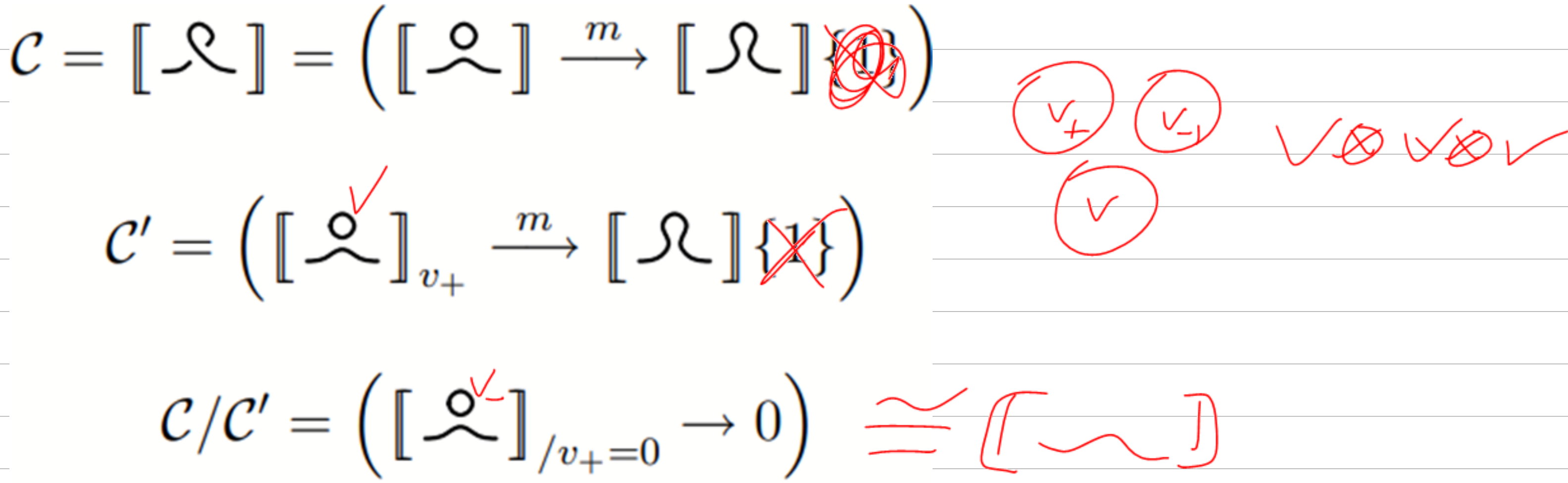
there is a corresponding long exact seq of H's:

$$\to H^{r-1}(\mathcal{C}/\mathcal{C}') \to$$
$$\to H^r(\mathcal{C}') \to H^r(\mathcal{C}) \to H^r(\mathcal{C}/\mathcal{C}') \to$$
$$\to H^{r+1}(\mathcal{C}') \to$$

A

B

Invariance under R1:



Invariance under R2:

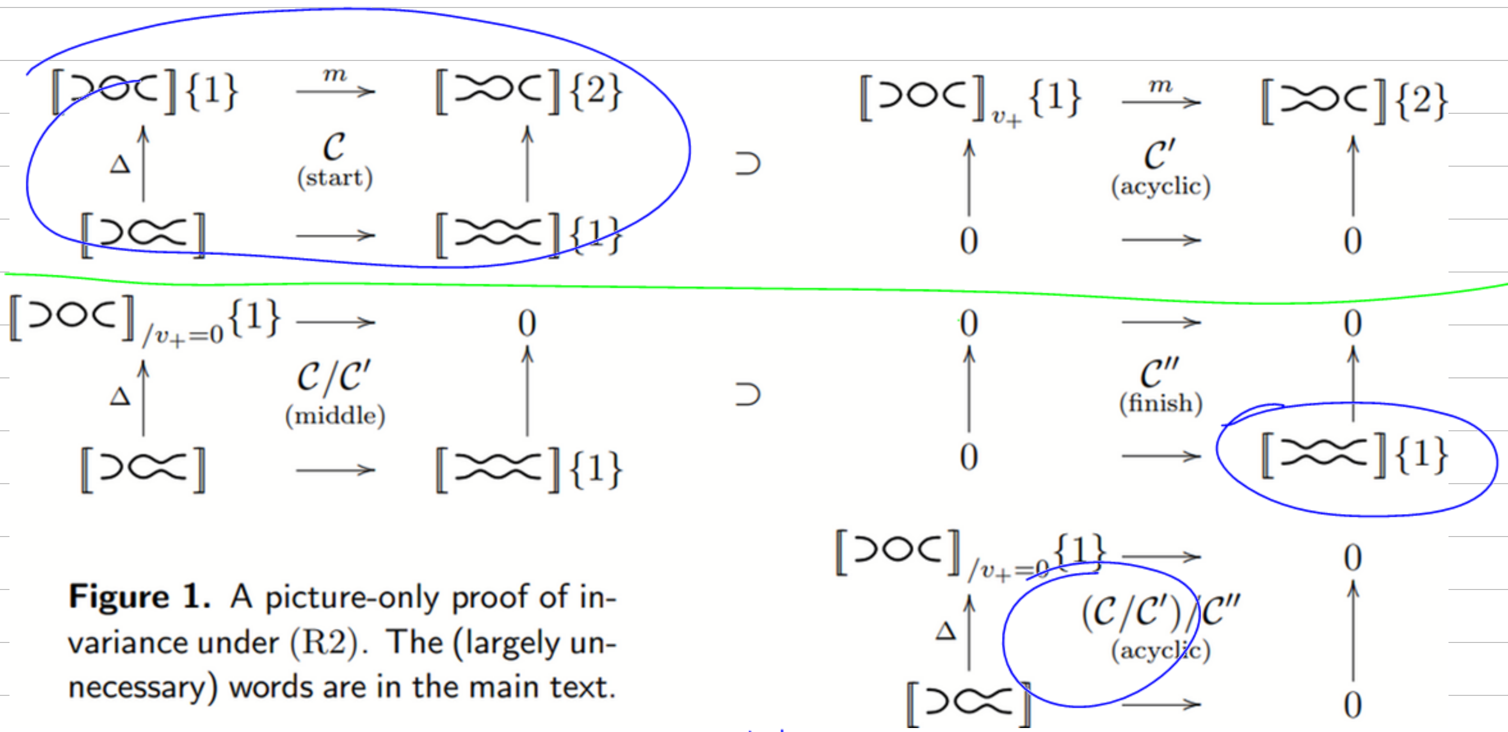


Figure 1. A picture-only proof of invariance under (R2). The (largely unnecessary) words are in the main text.

$\Delta: V_- \to V_- \otimes V_-$ $\xrightarrow[\text{on left}]{\text{mod by } V_+=0}$ $\to V_- \otimes V_-$

$V_+ \to V_- V_+ + V_+ V_-$ $\to V_- \otimes V_+$

Invariance under R3?
http://www.math.toronto.edu/~drorbn/papers/Categorification/Categorification.pdf or wait a bit.

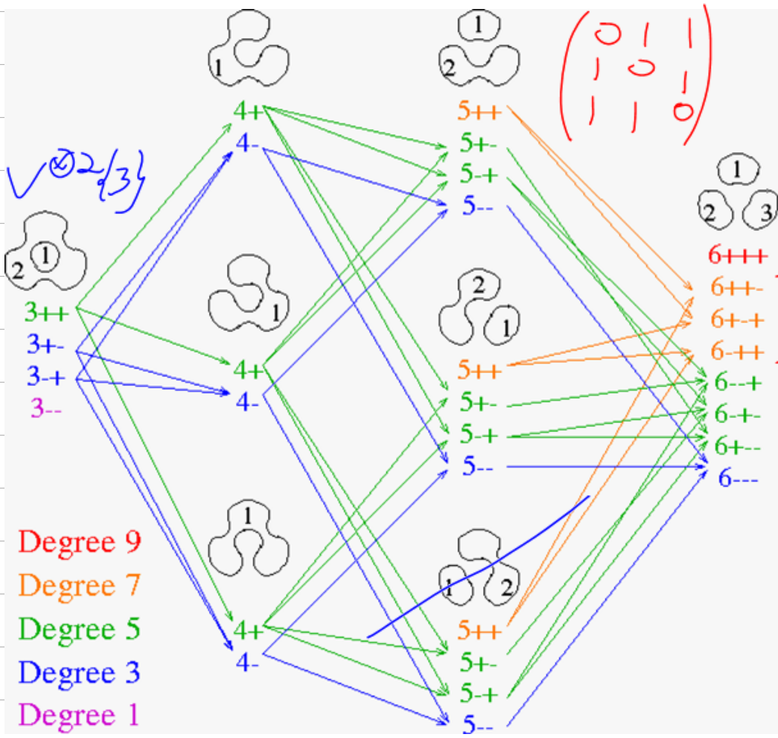
Thm $Kh(K) = V \Rightarrow K = \bigcirc$

Pf ~~wrongest~~ ever (morally)

Way too complicated.

Hour 11, Friday October 2: Last class on "plain" Khovanov homology.

Open problems:
1. 3D interpretation?
2. Why $Kh(K)=Kh(O) \Rightarrow K=O$?




```
In[1]:= << KnotTheory`
        Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
        Read more at http://katlas.org/wiki/KnotTheory.
In[2]:= Kh[PD@Mirror[Knot[3, 1]]][q, t]
        KnotTheory: Loading precomputed data in PD4Knots`.
        KnotTheory: The Khovanov homology program JavaKh-v2 is an update of
           Jeremy Green's program JavaKh-v1, written by Scott Morrison in
           2008 at Microsoft Station Q.
Out[2]= q + q^3 + q^5 t^2 + q^9 t^3
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$Kh = \sum t^r q \dim H^r$

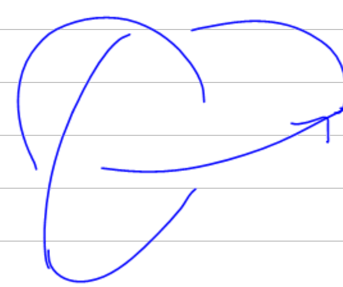
So coeff of $t^r q^d$ is $\dim (H^r)_d$

$Kh/_{t\to -1} = J$

Parts of homological alg make sense even if ker & im don't 0 | Polynomials on {knots}

sl_2 V_2

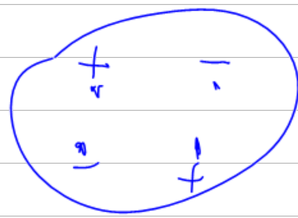
Let V (for Vassiliev) be an inv. of oriented knots in oriented $\mathbb{R}^3/S^3$. V can be extended to 1-singular knots via:



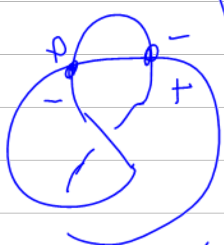
$V^{(1)}$ is "first derivative" of V

$$V^{(m)}(\underbrace{\times\times\cdots\times}_{m}) = V^{(m-1)}(\times\ \times\cdots\times) - V^{(m-1)}(\times\ \times\cdots\times)$$

"m-singular knots"

 + analysis

$$V(\times\times) = V(+\ +) - V(+\ -) - V(-\ +) + V(-\ -)$$



$$V^{(m)}(\underbrace{\times\cdots\times}_{m}) = \sum_{s\in\{-1,+1\}^m} \left(\prod s_i\right) V(K_s)$$

<u>Def</u> V is poly of deg m,

if $V^{(m+1)} \equiv 0$