Hour 6, Monday September 21: Finish Jones and planar algebras, start Khovanov homology

Last time: over $\mathbb{Z}[A^{\pm 1}]$

KB: Planar Alg of Tangles / R2R3 $\longrightarrow$ $TL := \{$  $\} / \bigcirc = d = -A^2 - A^{-2}$

$\operatorname{Rank}_{\mathbb{Z}[A,A^{-1}]} TL_{2n} = C_n = \frac{1}{n+1}\binom{2n}{n} \sim 4^n / n^{3/2}\sqrt{\pi}$ $n! = \frac{n^n}{e^n}\sqrt{2\pi n}$

C_n = How many histories lead to the score $n-n$ in a soccer game in which team B never leads.

$C_0 = \{(0,0)\} = 1$

$C_1 = \left\{\begin{matrix} 0\,0 \\ 1\,0 \\ 1\,1 \end{matrix}\right\} = 1$ $C_2 = \left\{\begin{matrix} 0\,0 & 0\,0 \\ 1\,0 & 1\,0 \\ 2\,0 & 1\,1 \\ 2\,1 & 2\,1 \\ 2\,2 & 2\,2 \end{matrix}\right\} = 2$

$C_3 = 5$ $C_4 = 14$

$C =$

	-1	-4				0
	-1	-3	-5	-5	0	14
3	-1	-2	-2	0	5	14
2	-1	-1	0	2	5	9
1	-1	0	1	2	3	4
0		1	1	1	1	1	1
		0	1	2	3		

$C = A - B$

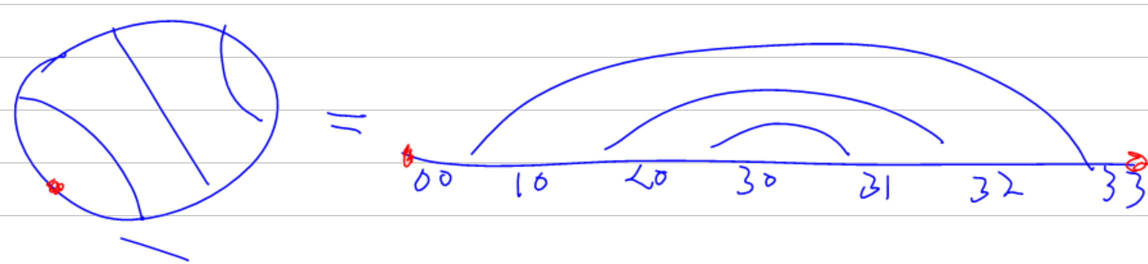
A:

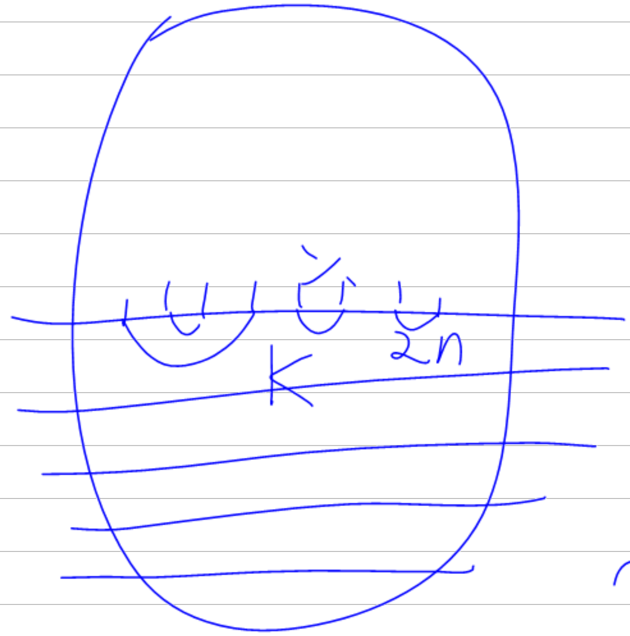
1			
1	3		
1	2	3	
1	1	1	1

B:

3			
2	3		
1	1	1	1

$$C_{n,n} = A_{n,n} - B_{n,n} = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$$

$TL_{2n} =$ 



C_n – Laurant polys in $A^{\pm 1}$

Complexity $=$ (# xings) $C_n \cdot$ (complexity of ops in $\mathbb{Z}[A^{\pm 1}]$)

$\sim 4^n \sim 4^{\sqrt{\text{xings}}}$

Khovanov homology (1999)

A graded chain complex for each knot diagrams, whose q Euler characteristic is the Jones poly, whose homology is invariant (stronger than Jones) + ... more. Finite.

$$0 \to \cdots \to C^{r-1} \xrightarrow{d^{r-1}} C^r \xrightarrow{d^r} C^{r+1} \longrightarrow \cdots \to 0$$

F.d. V.S.

$d^r \circ d^{r-1} = 0$

$d^2 = d \circ d = 0$

r-bndries ; r-cycles

$$B^r = \operatorname{im} d^{r-1} \subset \ker d^r = Z^r$$

$$H^r = Z^r / B^r \quad \text{"r-th homology"}$$

$$\chi = \sum (-1)^r \dim C^r$$

rank-nullity: $\operatorname{rank}(d^r) + \operatorname{nullity}(d^r) = \dim C^r$

$$\begin{array}{c} \operatorname{rank}(d^r) \\ \| \\ \dim B^{r+1} \end{array} + \dim Z^r = \dim C^r$$

$$\chi = \sum (-1)^r \left(\dim Z^r + \dim B^{r+1} \right)$$

$$= \sum (-1)^r \left(\dim Z^r - \dim B^r \right) = \sum (-1)^r \dim H^r$$

Hour 7, Wednesday September 23: Khovanov homology following http://drorbn.net/mo13
HW2 on web, HW1 due at 11:59PM.

Sums and products of complexes.
Graded vector spaces, q-dim, sums, prodcuts, shifts.
Graded complexes and graded Euler characteristic.
Then http://drorbn.net/mo13 ....

$A, B \qquad A\oplus B \qquad \dim(A\oplus B)=\dim(A)+\dim(B)$

$A, B \qquad A\otimes B$

$\{a_i\}_1^n \ \{b_j\}_1^m \qquad \langle a_i\otimes b_j\rangle_{i=1\ j=1}^{n\ \ m} \qquad \dim(A\otimes B)=\dim(A)\cdot\dim(B)$

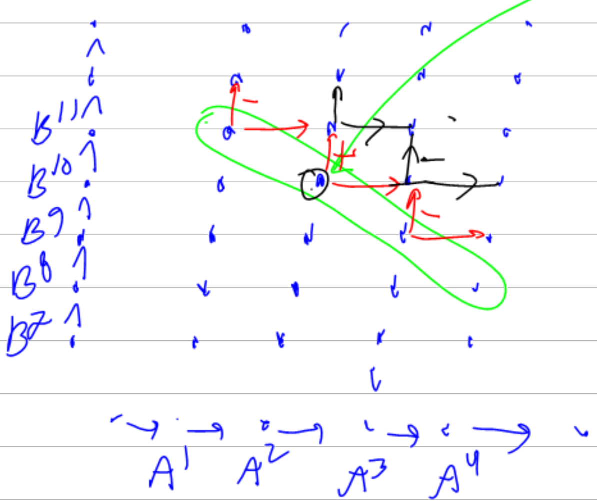
$(A^r, d_A) \quad (B^r, d_B) \qquad (A^r\oplus B^r, (d_A, d_B))$

$(A\oplus B)^r := A^r\oplus B^r \qquad \chi(A^\cdot\oplus B^\cdot)=\chi\ldots+\chi\ldots$

$(A^\cdot\otimes B^\cdot, d_A+d_B) \qquad \chi(A^*\otimes B^*)=\chi(A)^*\times\chi(B^*)$

$$(A\otimes B)^{r} \overset{ht}{=} \bigoplus_{r_1+r_2=r} A^{r_1}\otimes B^{r_2}$$

$A^2\otimes B^{10}\subset(A\otimes B)^{12}$



$$\underbrace{d_{A\otimes B}}_{\text{odd}}(a\otimes b)=(\overbrace{d_A}^{\text{odd}}a\otimes b)+(-1)^{ht(a)}(a\otimes \underbrace{d_B}_{\text{odd}} b)$$

$$\chi(A^*\otimes B^*)=\chi(A^\cdot)\cdot\chi(B^\cdot)$$

$$\sum(-1)^r \overbrace{\sum_{r_1+r_2=r} A^{r_1}\otimes B^{r_2}}^{n}_{\dim} = \sum_{r_1, r_2}(-1)^{r_1}(-1)^{r_2}(\dim A^{r_1})\cdot(\dim A^{r_2})$$

Graded v.s: $V = \bigoplus_{n=n_1}^{n_2} \underbrace{V_n}_{\text{elements of deg } n}$

$$q\text{-dim } V = \sum_n (\dim V_n) \cdot q^n$$

$$(V_1 \oplus V_2)_n = V_{1n} \oplus V_{2n}$$

$$(V_1 \otimes V_2)_r = \sum_{r_1+r_2=r} (V_1)_{r_1} \otimes (V_2)_{r_2}$$

$$q\dim(V_1 \otimes V_2) = (q\dim V_1)(q\dim V_2)$$

$$\cdots \to A^{r-1} \longrightarrow A^r \xrightarrow{d} A^{r+1} \cdots$$

$$a \longrightarrow da$$

$$\deg a = \deg(da) \qquad \deg d = 0$$

$$\cdots A_7^{r-1} \longrightarrow A_7^r \longrightarrow A_7^{r+1} \cdots$$

$$\cdots A_6^{r-1} \longrightarrow A_6^r \longrightarrow A_6^{r+1} \cdots$$

$$A_5 \to A_5 \longrightarrow A_5 \cdots$$

$$\chi_q(A^*) = \sum_r (-1)^r q\dim A^r$$

$$= \sum (-1)^r q\dim H^r$$

$$\chi_q(A \otimes B) = \chi_q(A) \cdot \chi_q(B) \cdots$$

The Jones polynomial: $\bigcirc^k \mapsto (q+q^{-1})^k$

$J : \times \mapsto q\,)(- q^2 \asymp, \quad J : \times \mapsto -q^{-2} \asymp + q^{-1}\,)(,$

$q+q^{-1} = \bigcirc = -A^2 - A^{-2}$

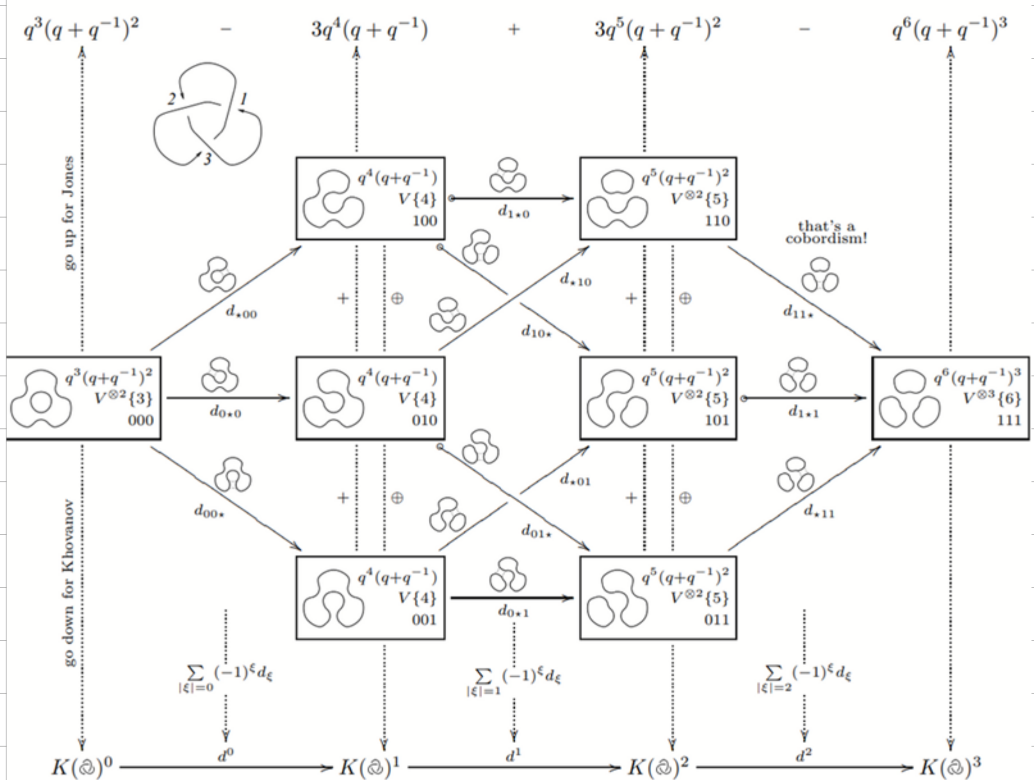
$q = -A^2 \quad A = \sqrt{-q}$

$J(K) = \langle K \rangle (-A^{-3})^{w(K)}$

$w(\times) = +1$

$w(\times) = -1$

Example: $= q + q^3 + q^5 - q^9.$



(here $(-1)^\xi := (-1)^{\sum_{i<j}\xi_i}$ if $\xi_j = \star$) $= K(\circledast).$

V is a graded v.s., $V = \bigoplus_n V_n$

$(V\{m\})_n := V_{n-m}$

$\text{qdim}\, V\{m\} = q^m \cdot \text{qdim}\, V$

Riddle. $\exists$? Cont. $f: \mathbb{R} \to \mathbb{R}$ s.t. $f \circ f = \cos$?

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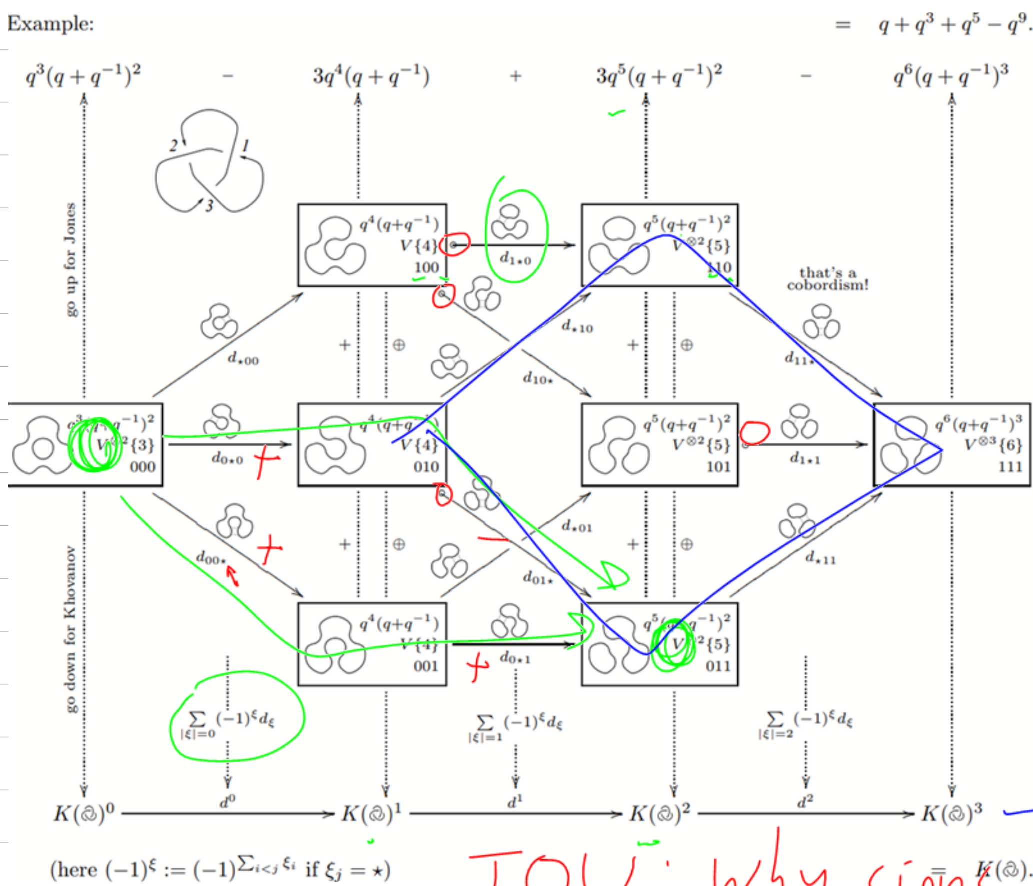
$J: \times \mapsto q)(- q^2 \asymp$, $\quad J: \times \mapsto -q^{-2} \asymp + q^{-1})($,



$V \quad qdim = q + q^{-1}$

$V = \langle v_+, v_- \rangle$

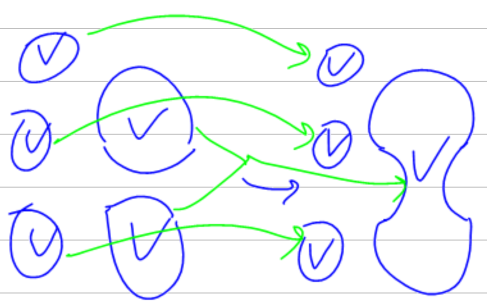
$\deg(v_\pm) = \pm 1$

$V^{\otimes 3}\{6\}$

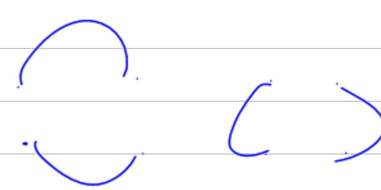
$d^2 = 0$

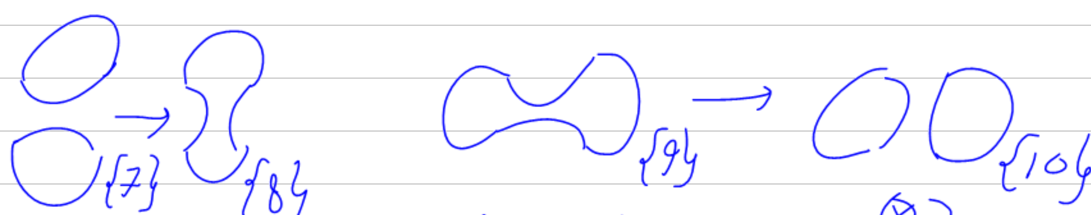
$\xrightarrow{\chi_q} J(\text{trefoil})$

JOU: why signs work and where they came from,



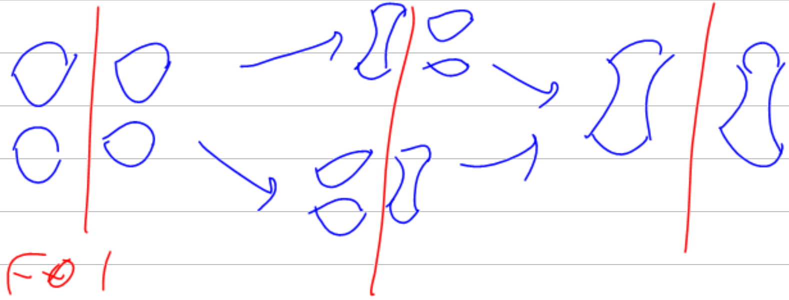
$V^{\otimes 5} \to V^{\otimes 4}$





$m: V^{\otimes 2} \to V$ $\quad \Delta: V \to V^{\otimes 2}$

Cond #1: $\deg(m) = \deg(\Delta) = -1$



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