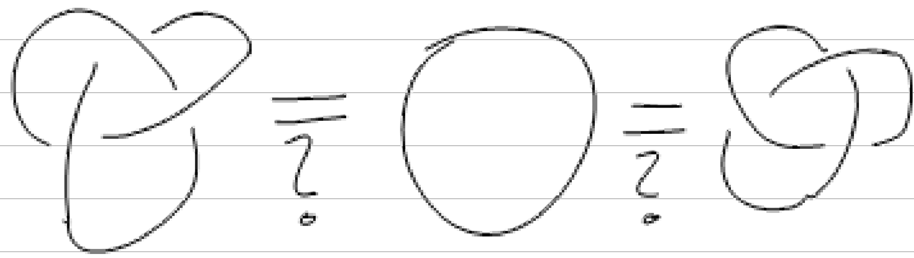
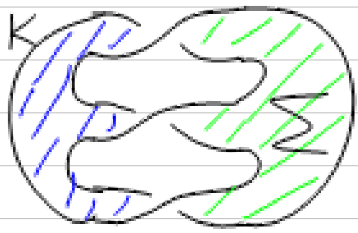
MAT 1350F Topics in Knot Theory
With Dror Bar-Natan
MWF 1-2PM, at http://drorbn.net/20-1350





what's the genus of Σ?

Number Theory:

$59^4+158^4=133^4+134^4$

Who cares?

Similarly for knot theory!

Possible Topics (not in order).

Introduction to knots. Knot colouring, prime knots, alternating knots.
The fundamental group, quandles.
The Alexander polynomial.
The Jones polynomial (incl. Thistlethwaite).
Khovanov homology.
Braids, combing, Markov traces, Hecke algebras, HOMFLY-PT.
Finite type invariants.
Virtual knots.
Quantum groups and RT.
A word about 3-manifolds.
Hyperbolic invariants.

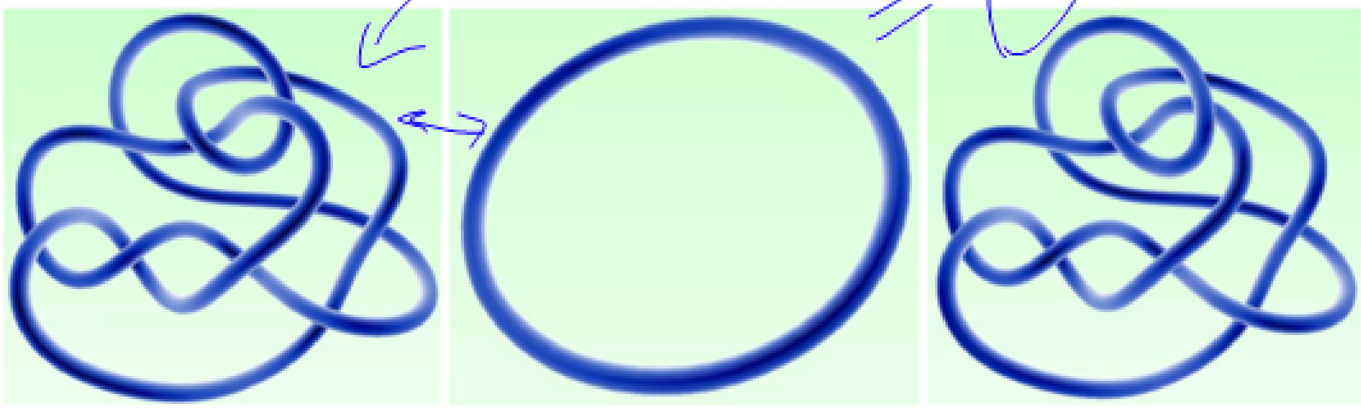
Topology! Combinatorics! Algebra! Homological Algebra! Representation Theory! Algebraic Topology! Differential Geometry! Quantum Field Theory! Computer Algebra! ....

Prerequisites: Core Algebra and Topology, or the will and the ability to catch up.

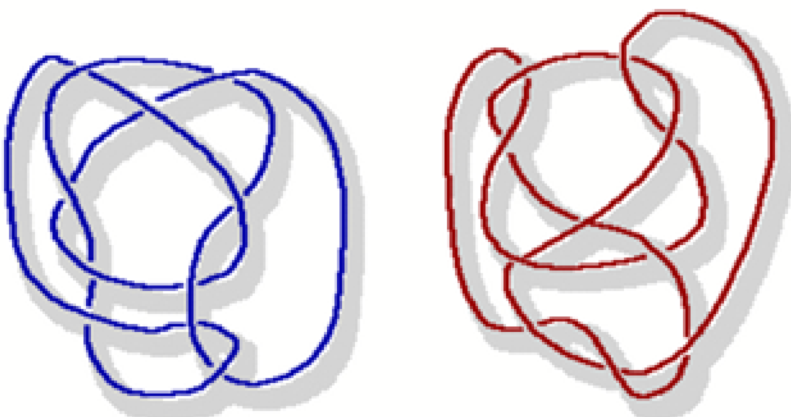
Please, if possible, video on!
All classes will be recorded
(instructor's side only)

Some Non Obvious Examples

Which two are the same? (rendered using Rob Scharein's KnotPlot)



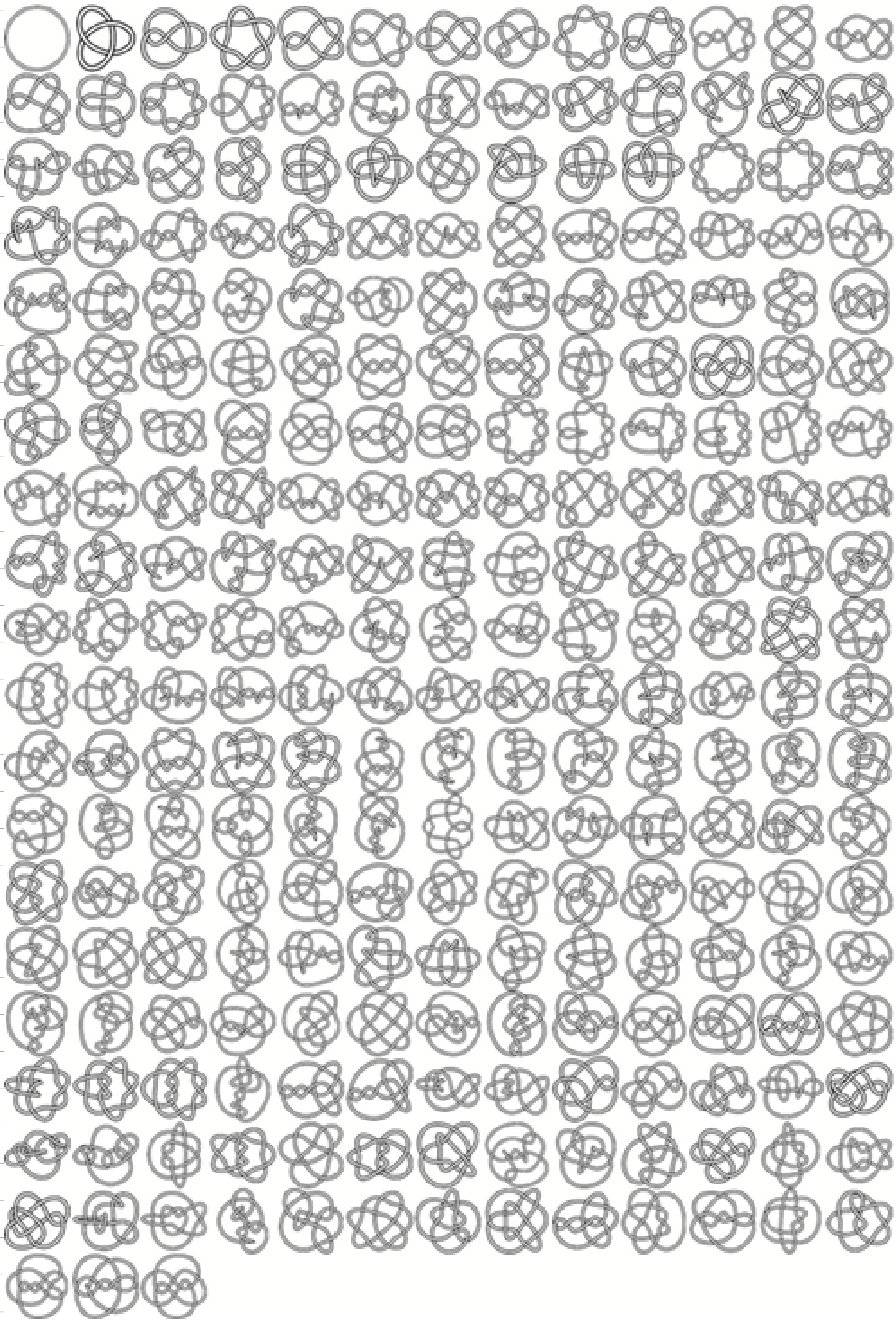
The Perko Pair: (are these the same?) (taken from http://www.math.cuhk.edu.hk/publect/lecture4/perko.html)



Is this the unknot? (From a book by A.B. Sossinsky. Thanks, Ian Agol)

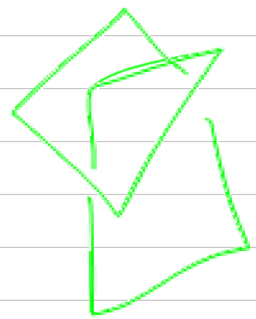
Figure 3.5. Wolfgang Haken's "Gordian knot."

The Rolfsen Knot Table Mosaic



wrong

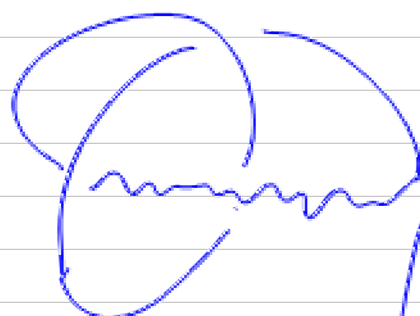
Def A knot is a ~~cont~~ smooth PL 1-1 map $\gamma : S^1 \to \mathbb{R}^3$ modulo homotopies (~~cont~~ smooth PL) of such things

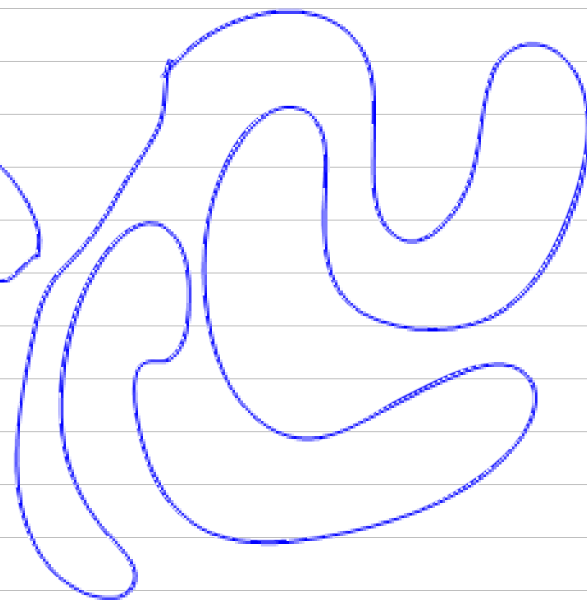


$\varphi : \mathbb{R}^3 \to \mathbb{R}^3$

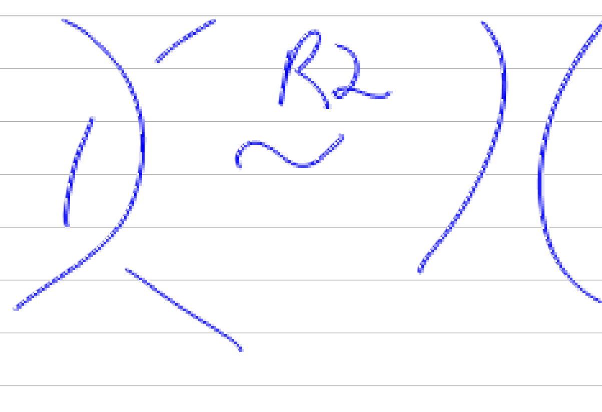
A combinatorial def of knots:

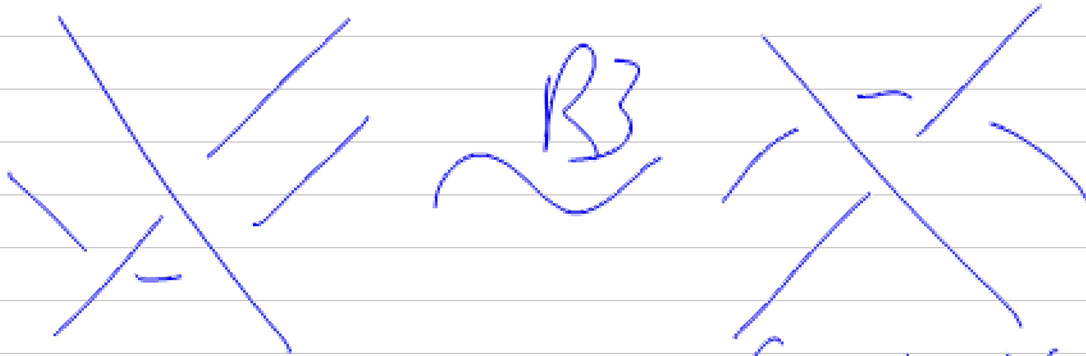
A knot diagram is



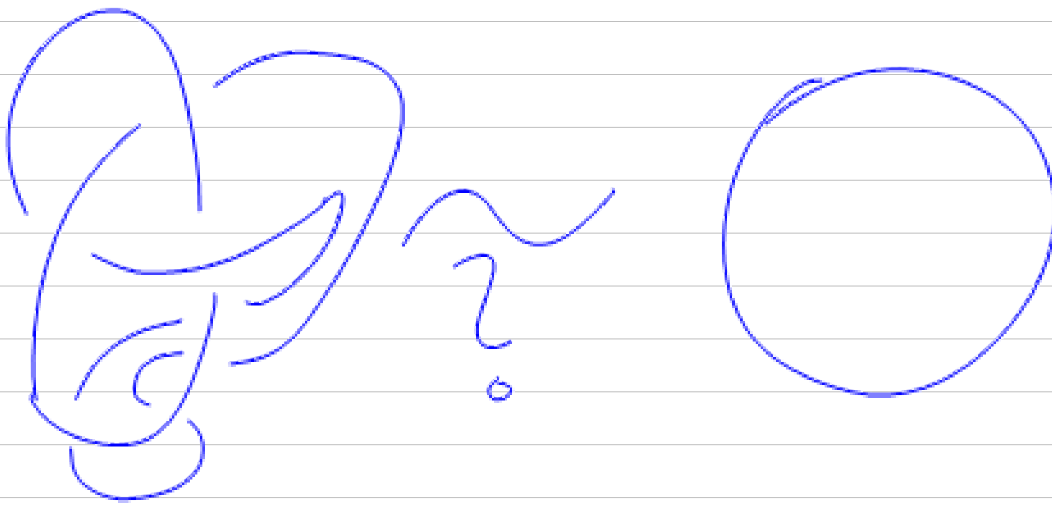


Reidemeister moves



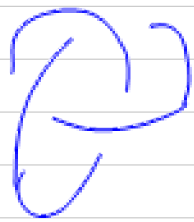
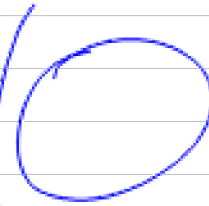


Thm $\{\text{Knots}\} = \dfrac{\{\text{knot diagrams}\}}{\text{Reid. moves}}$



Find "Knot invariants"

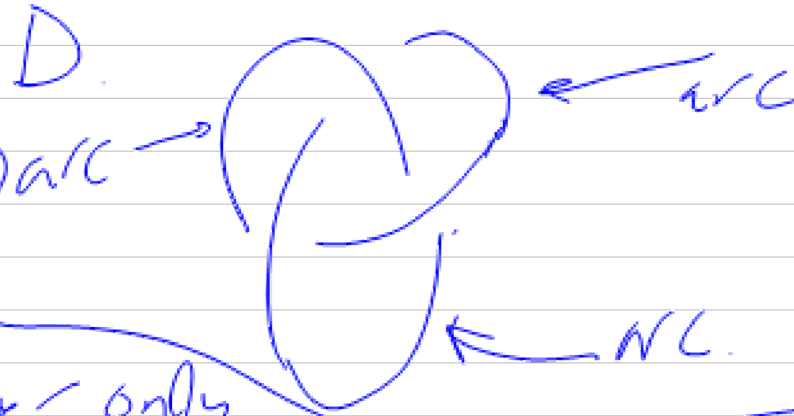
A function on the set of knot diagrams with values in $\mathbb{Z}$, polys, matrices. and that does not change if you perform R-moves.

$\lambda($  $) = 9$ $\lambda($  $) = 3$

Example "3-colouring" λ:

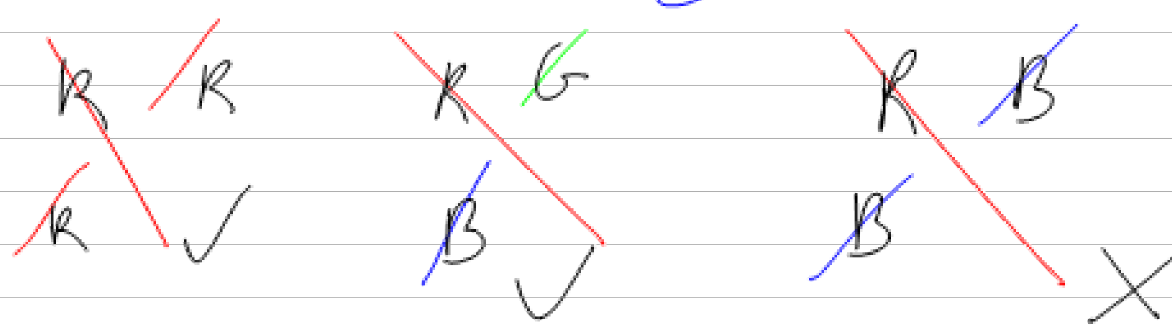
$\lambda(D) = \#$ of legit 3-cols of D

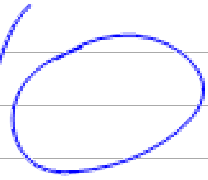
3-colouring: An assignment of colours {R, G, B} to the arcs of D

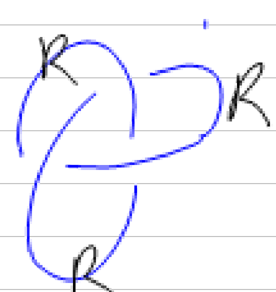


Legit:

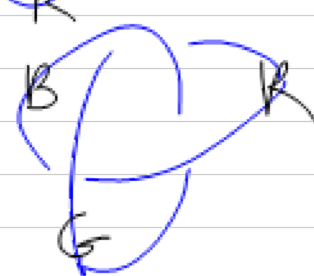
either - only one colour appears or all 3.

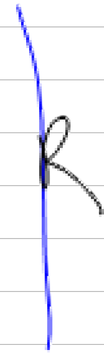


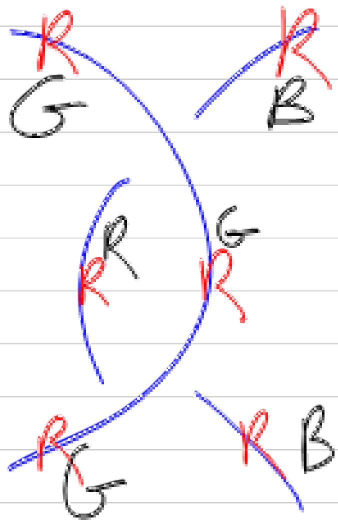
$\lambda($  $) = 3$

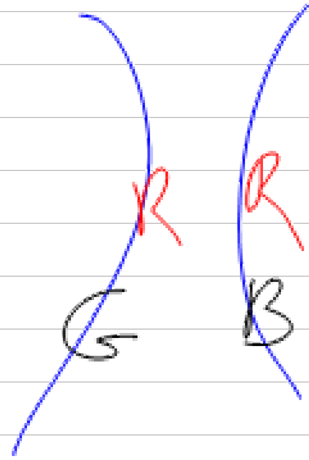


$\lambda($  $) = 3 + 6 = 9$





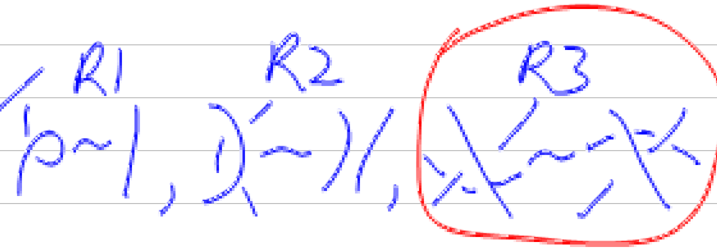




R3 exercise

Class of Friday September 11: The Jones Polynomial.

1. Knots = Knot diagrams / (R1, R2, R3)



2. $\lambda(D) := \left|\left\{\begin{array}{l}\text{colorings of the arcs of } D \\ \text{in } R, G, B, \text{ s.t. no crossing} \\ \text{displays exactly 2 colours}\end{array}\right\}\right|$

(D: a knot diagram)

$\lambda(\text{trefoil}) = 9 \quad \lambda(\bigcirc) = 3$

1. Is λ always a power of 3?
2. 100 xings ~ 100 arcs, ~3^{100} colorings?
Can you compute λ more efficiently?

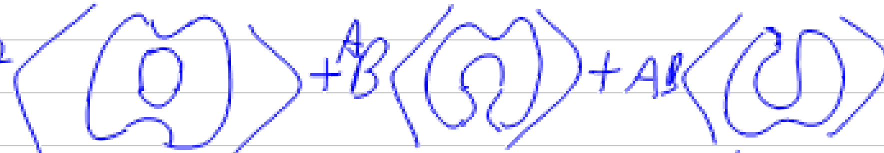
Vaughan Jones (Dec 31, 1952 - Sep 6, 2020):

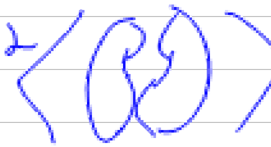
The Kauffman Bracket $\langle D \rangle$

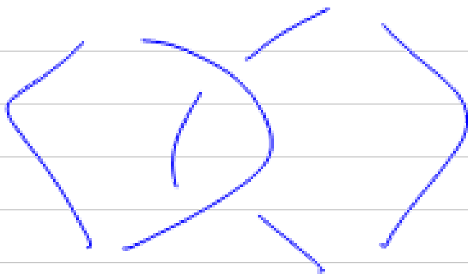
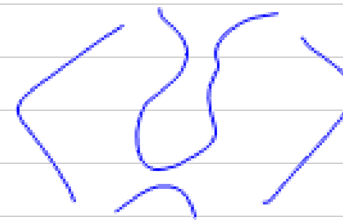
$\langle O \sqcup D \rangle = d \langle D \rangle$ $\quad$ $\langle$  $\rangle = A \cdot d + B d^2$

$\langle \emptyset \rangle = 1$

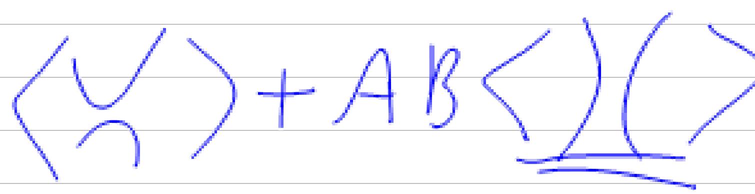
$\langle$  $\rangle =$

$A^2 \langle \cdot \rangle + AB \langle \cdot \rangle + AB \langle \cdot \rangle$


$= A^2 d^2 + 2AB \cdot d + B^2 d^2 \quad + B^2 \langle$  $\rangle$

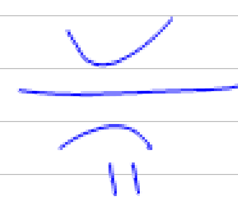
$\langle$  $\rangle = \qquad + B^2 \langle$  $\rangle$

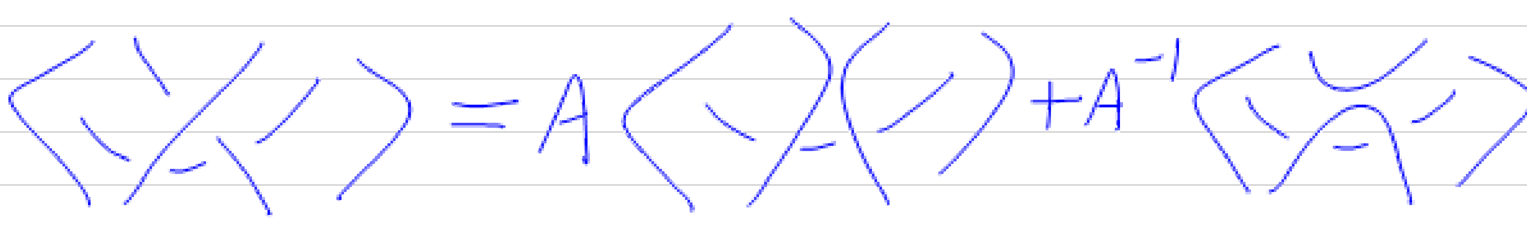
$A^2 \cdot \langle \cdot \rangle + AB \langle \cdot \rangle + AB \langle \cdot \rangle$

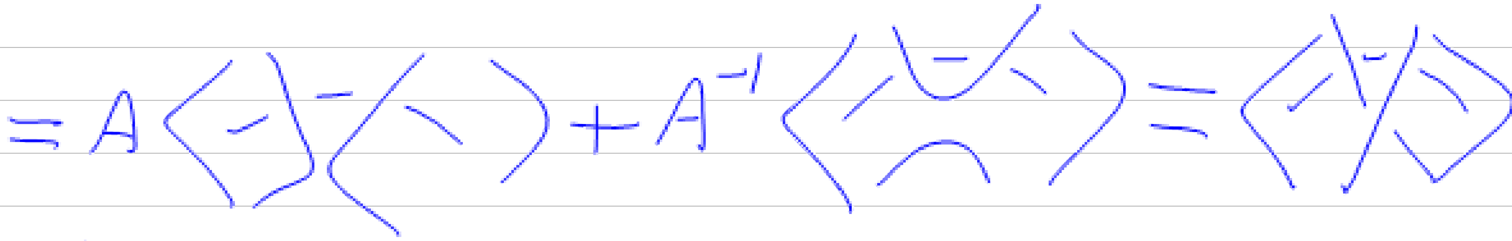

$= (A^2 + B^2 + ABd) \langle \cdot \rangle + AB \langle \cdot \rangle$


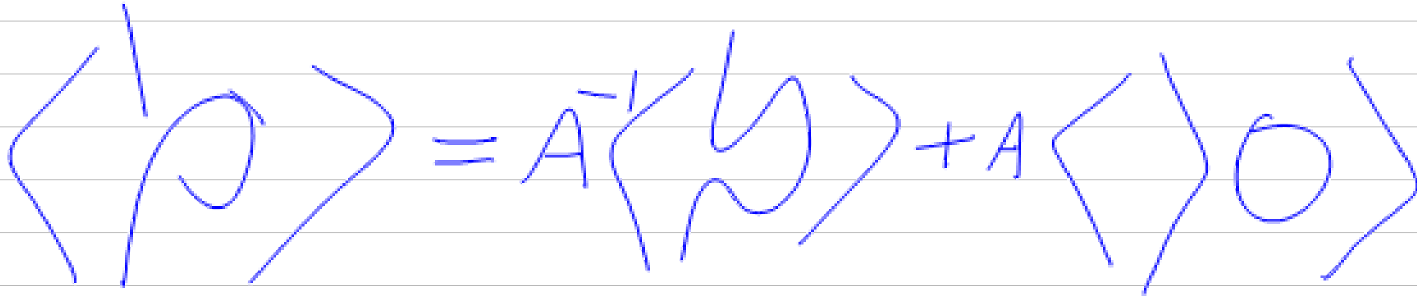
$AB = 1 \qquad A^2 + B^2 + ABd = 0$

$B = A^{-1} \qquad d = -A^2 - A^{-2}$



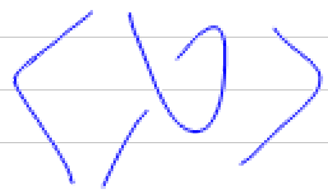
$\langle \cdot \rangle = A \langle \cdot \rangle + A^{-1} \langle \cdot \rangle$






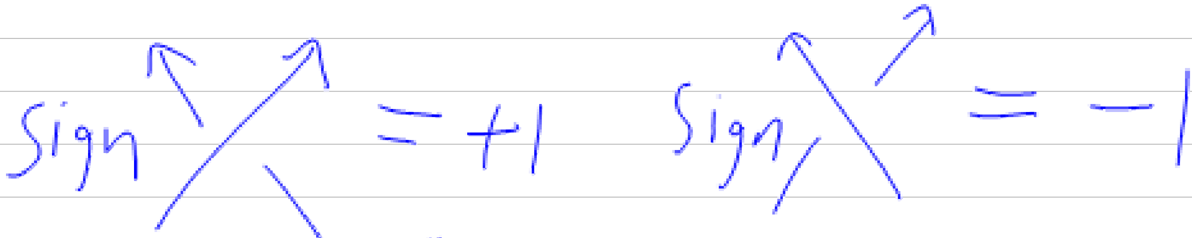
$$= \langle | \rangle \cdot \left(A^{-1} + A\left(-A^{2} - A^{-2}\right)\right)$$

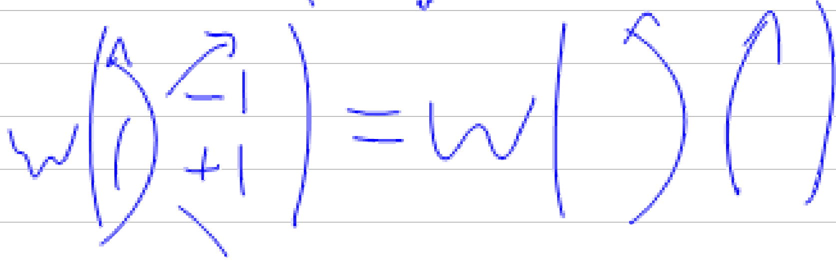
$$= \langle | \rangle \cdot \left(-A^{3}\right)$$

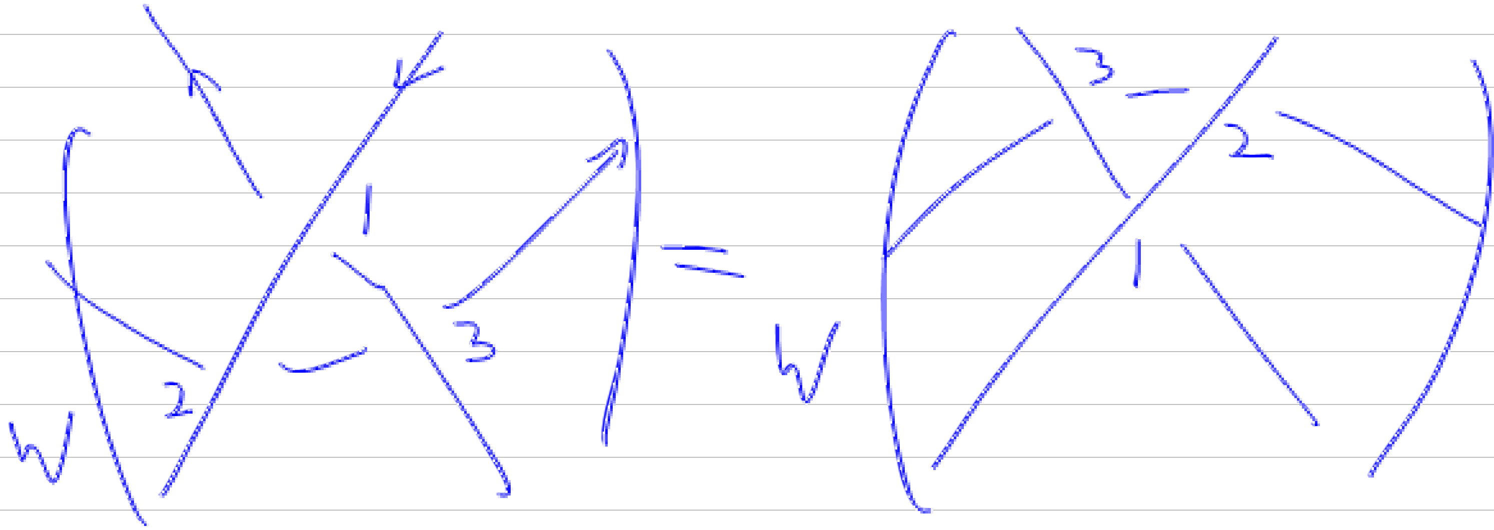
 $= \ldots = \langle | \rangle \cdot \left(-A^{-3}\right)$

$$w(D) = \sum_{\substack{\text{crossings} \\ x \text{ in } D}} \operatorname{sign}(x)$$

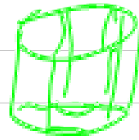
↑ writhe






w
1
2
3
w
3
2
1

Hour 3, Monday Sep 14



$\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$ $\langle \bigcirc^k \rangle = (-A^2 - A^{-2})^k$

invt under R2 & R3 but $\langle \text{ᑭ} \rangle = -A^3 \langle | \rangle$ $\langle \text{ᑫ} \rangle = -A^{-3} \langle | \rangle$

$W(D) = \sum_{\substack{x \text{ xing} \\ \text{in } D}} \text{sign}(x)$ $\text{sign}(\nearrow\!\!\nwarrow) = +1$ $\text{sign}(\nwarrow\!\!\nearrow) = -1$

invt under R2 & R3 but $w(\text{ᑭ}) = w(|) + 1$ $w(\text{ᑫ}) = w(|) - 1$.

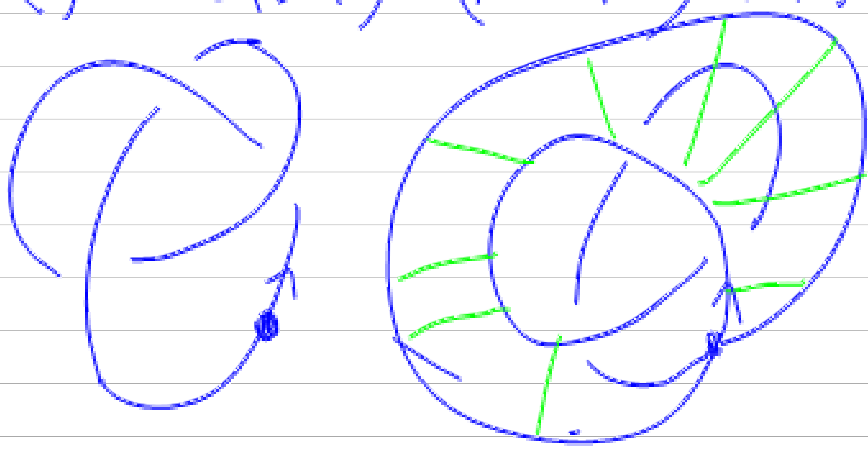
$$J(K) := (-A^3)^{-w(K)} \frac{\langle K \rangle}{d} \Big/ . \; A \to q^{-1/4}$$

The Jones Skein relation:

$$J(\nearrow\!\!\nwarrow) = -q^{3/4} \left(q^{-1/4} J()() + q^{1/4} J(\asymp) \right)$$

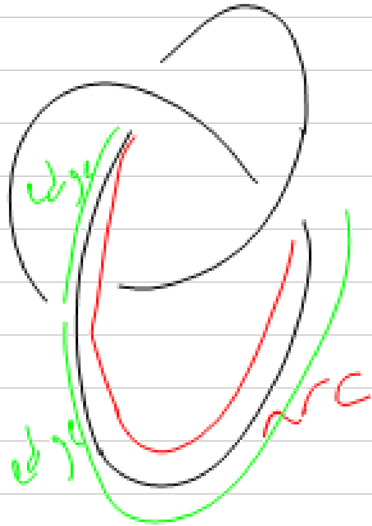
$$J(\nwarrow\!\!\nearrow) = -q^{-3/4} \left(q^{-1/4} \langle \asymp \rangle + q^{1/4} \langle)(\rangle \right)$$

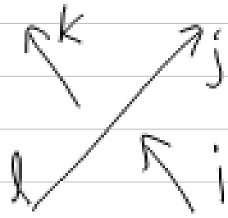
$$\Longrightarrow \quad q^{-1} J(\nearrow\!\!\nwarrow) - q\, J(\nwarrow\!\!\nearrow) = (q^{1/2} - q^{-1/2})\, J(\uparrow\uparrow)$$



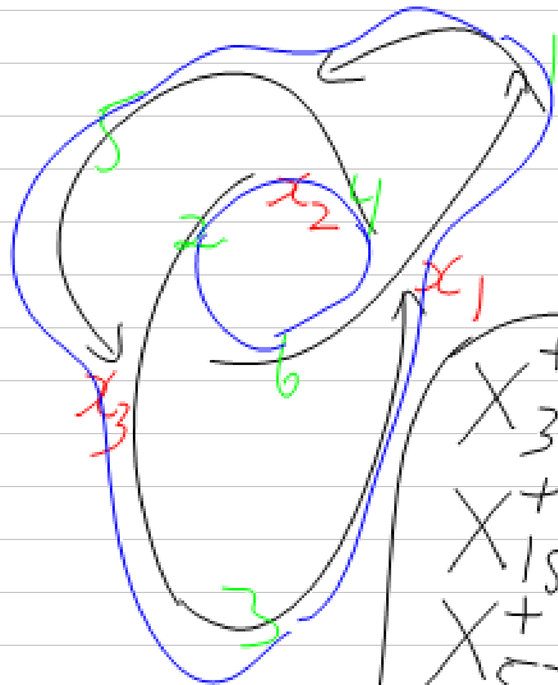
$$J(\bigcirc^k) = (-q^{1/2} - q^{-1/2})^k$$

PD – notation





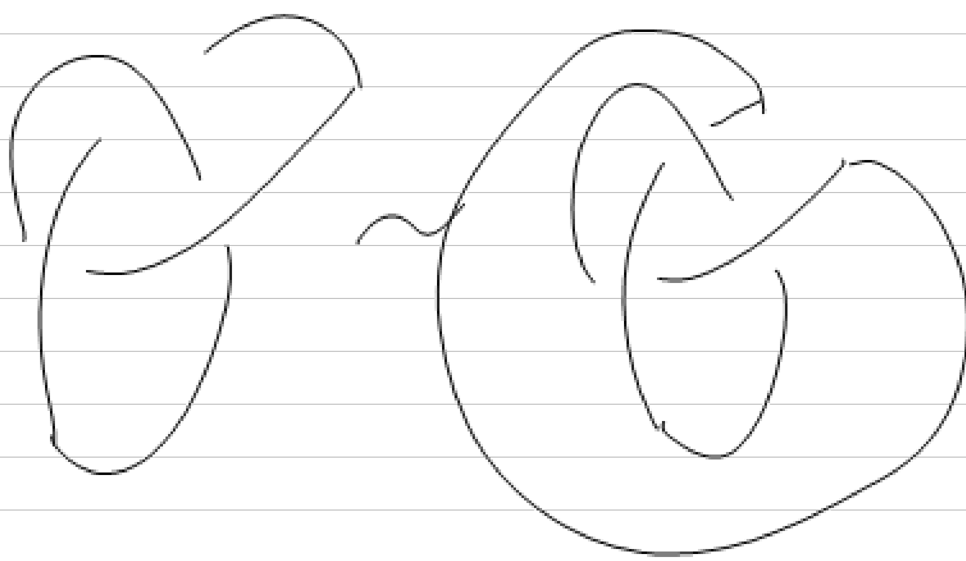
$\longrightarrow X^{s}_{ijkl}$

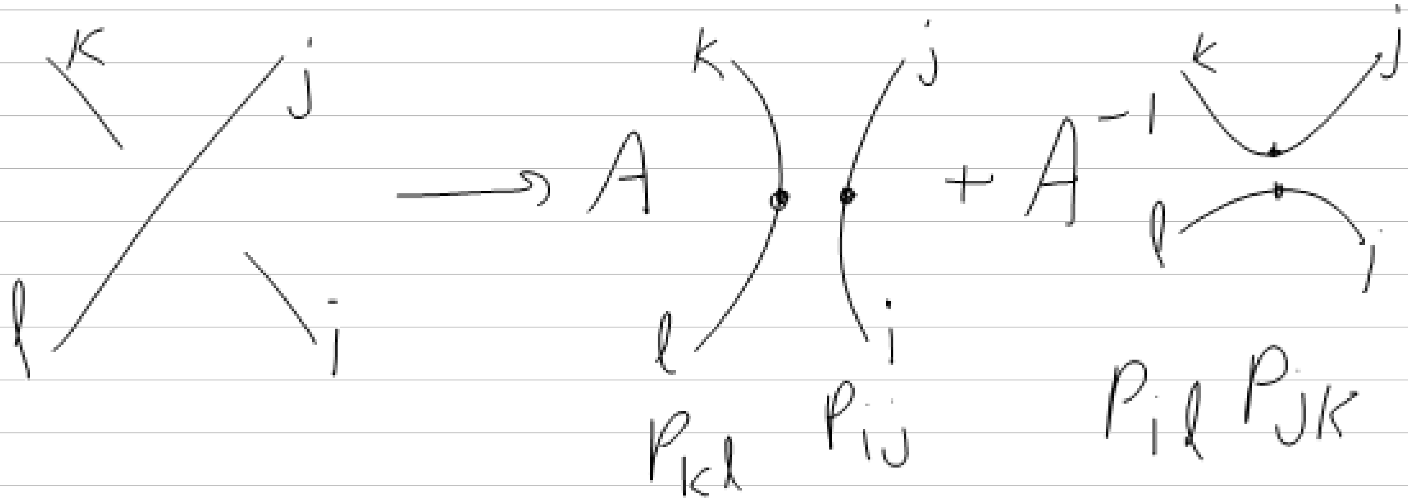


A^3

X^{+}_{3146}
X^{+}_{1524}
X^{+}_{5362}

Exercise: This list of xing info determines D as a diagram on S^2




k
j
l
i
A
+ A⁻¹
ρkl
ρij
ρil ρjk