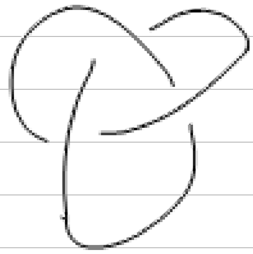
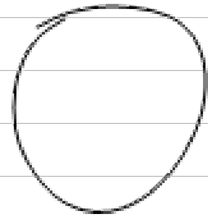
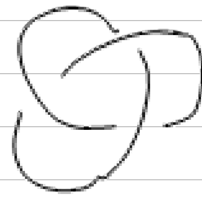
MAT 1350F Topics in Knot Theory
With Dror Bar-Natan
MWF 1-2PM, at http://drorbn.net/20-1350



$\stackrel{?}{=}$



$\stackrel{?}{=}$



Number Theory:

$59^4+158^4=133^4+134^4$

Who cares?

What's the genus of Σ?

Similarly for knot theory!

Possible Topics (not in order).

Introduction to knots. Knot colouring, prime knots, alternating knots.
The fundamental group, quandles.
The Alexander polynomial.
The Jones polynomial (incl. Thistlethwaite).
Khovanov homology.
Braids, combing, Markov traces, Hecke algebras, HOMFLY-PT.
Finite type invariants.
Virtual knots.
Quantum groups and RT.
A word about 3-manifolds.
Hyperbolic invariants.

Topology! Combinatorics! Algebra! Homological Algebra! Representation Theory! Algebraic Topology! Differential Geometry! Quantum Field Theory! Computer Algebra! ....

Prerequisites: Core Algebra and Topology, or the will and the ability to catch up.

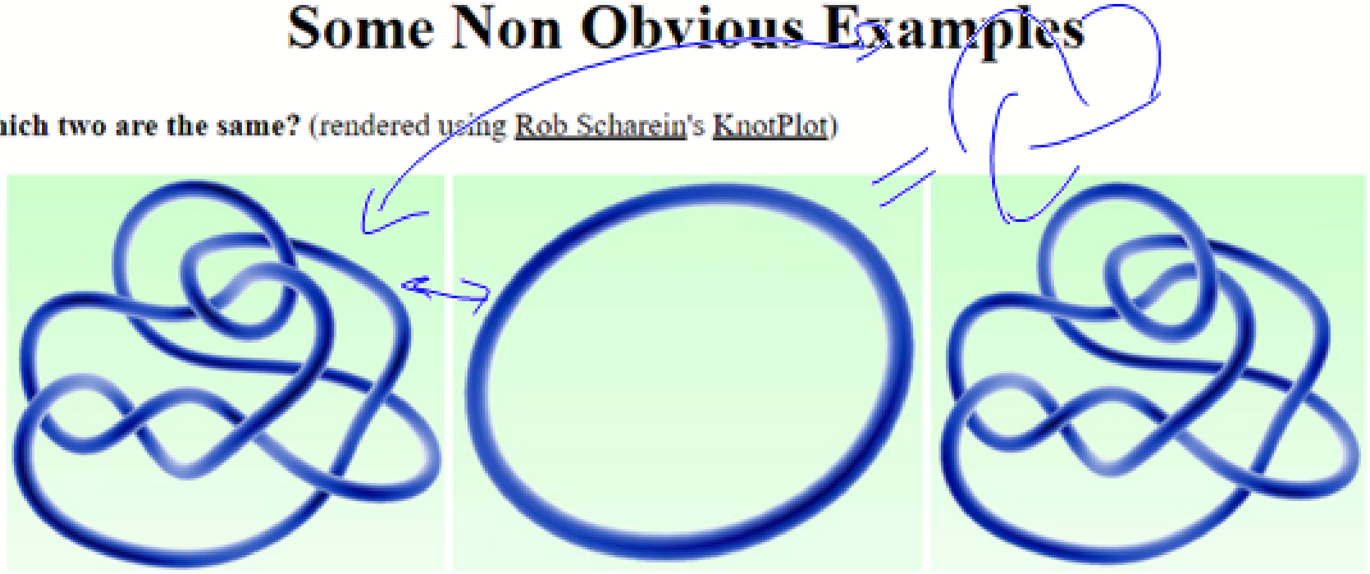
Please, if possible, video on! All classes will be recorded (instructor's side only)

Class of Wednesday September 9: Introduction to Knots, Knot Colourings, the Jones Polynomial.

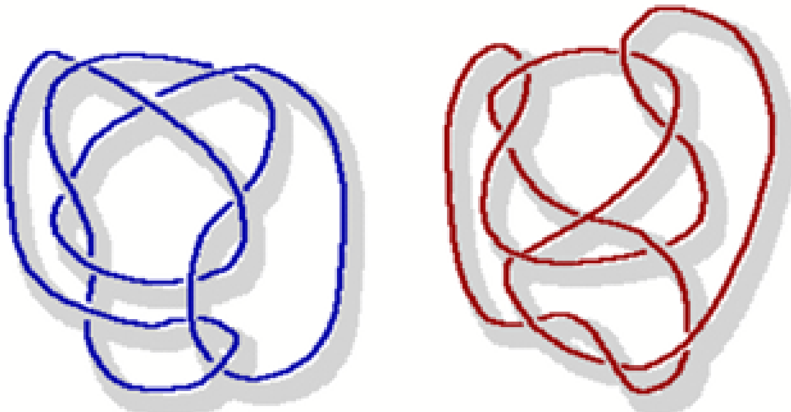
http://www.math.toronto.edu/~drorbn/classes/0304/KnotTheory/NonObvious/NonObvious.html:

Some Non Obvious Examples

Which two are the same? (rendered using Rob Scharein's KnotPlot)



The Perko Pair: (are these the same?) (taken from http://www.math.cuhk.edu.hk/publect/lecture4/perko.html)

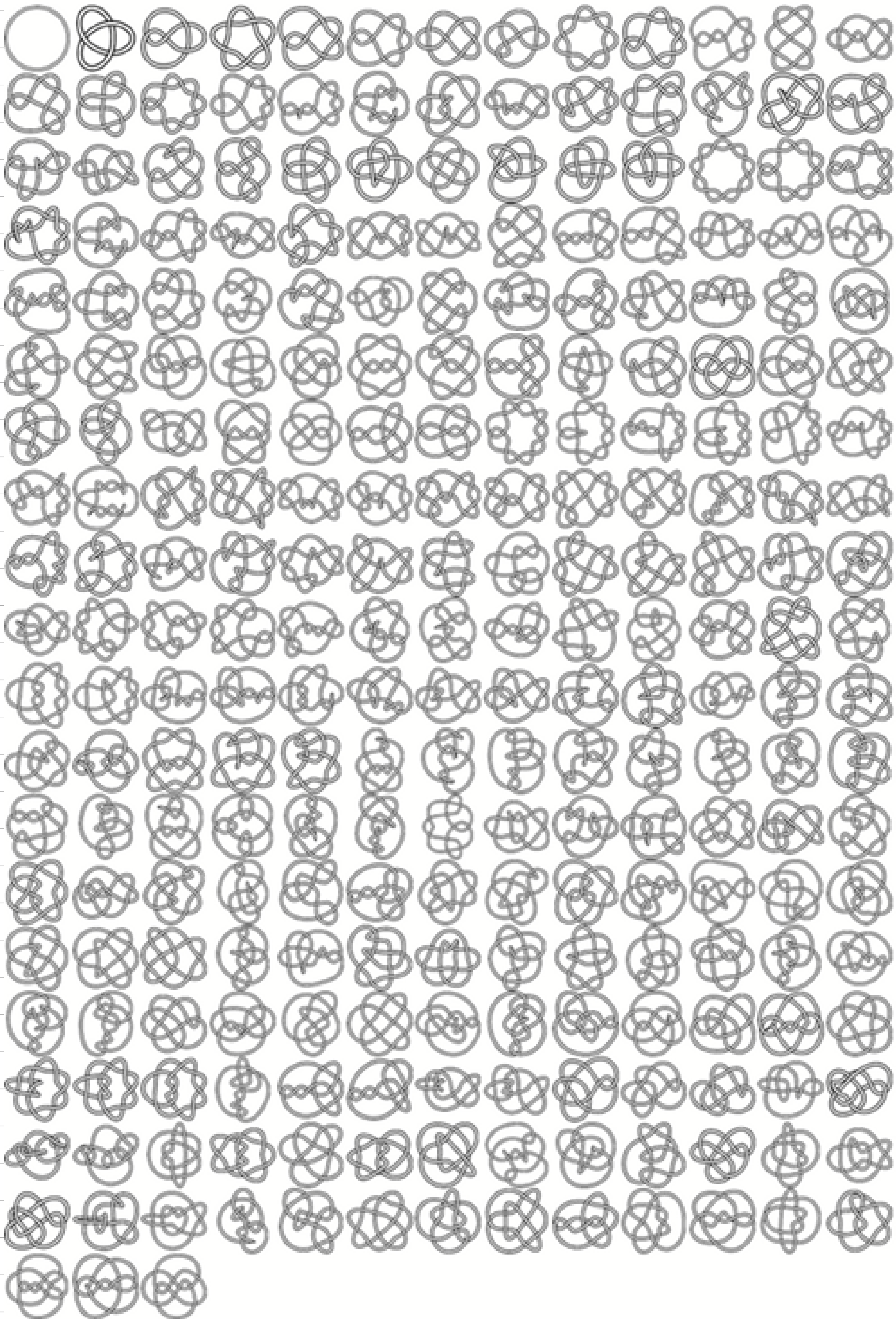


Is this the unknot? (From a book by A.B. Sossinsky. Thanks, Ian Agol)

Figure 3.5. Wolfgang Haken's "Gordian knot."

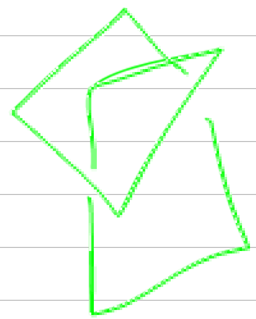
http://katlas.org/wiki/The_Rolfsen_Knot_Table_Mosaic

The Rolfsen Knot Table Mosaic



wrong

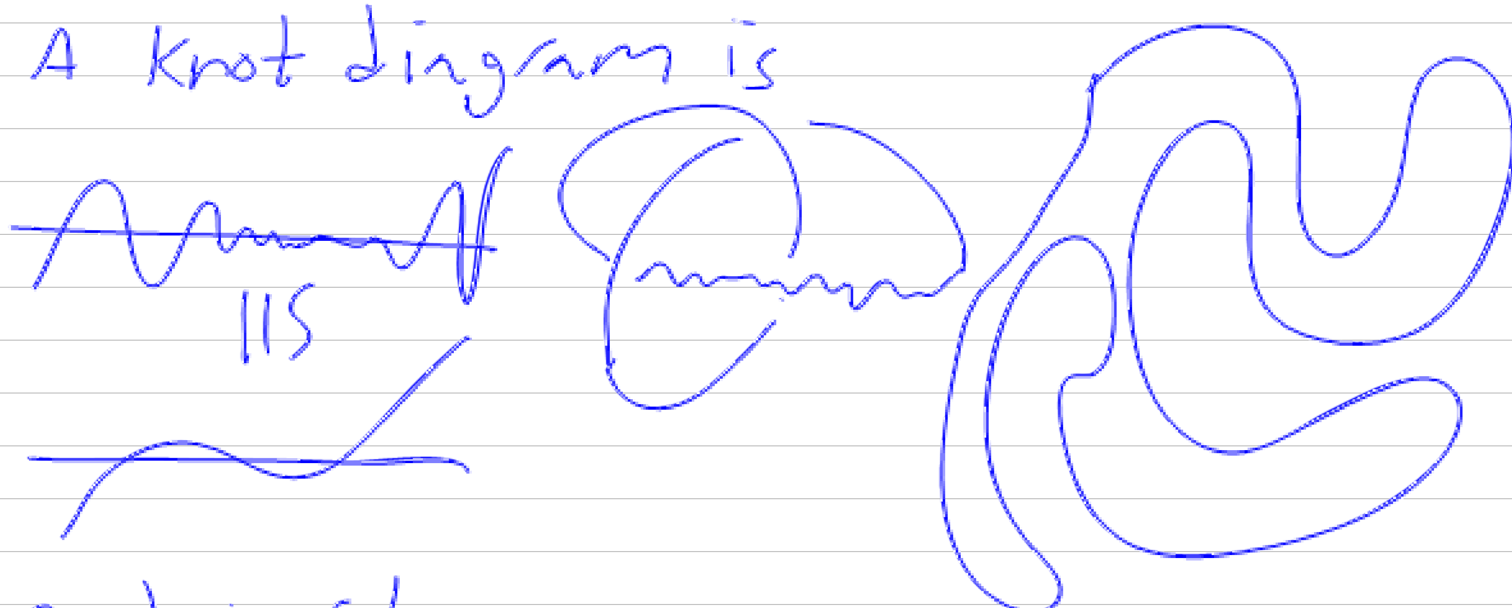
Def A knot is a ~~cont~~ smooth PL 1-1 map $\gamma: S^1 \to \mathbb{R}^3$ modulo ~~cont~~ smooth PL homotopies of such things



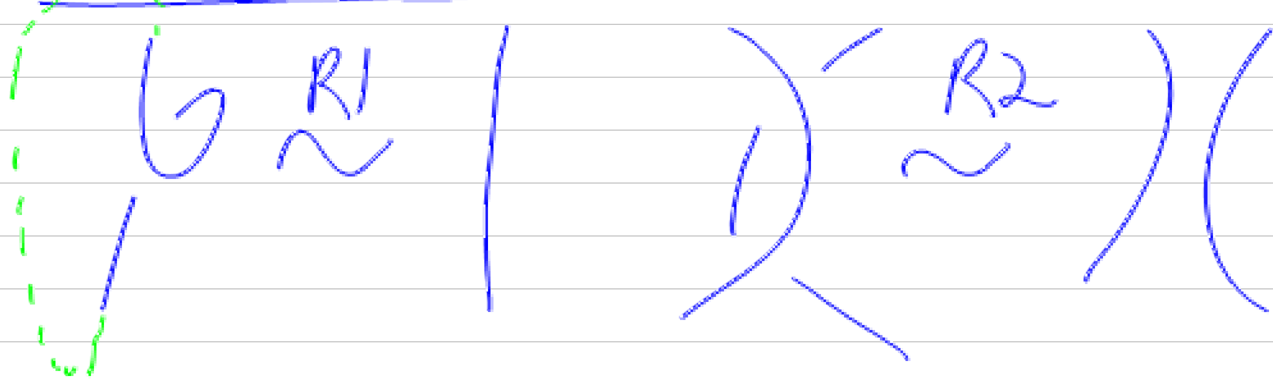
$\varphi: \mathbb{R}^3 \to \mathbb{R}^3$

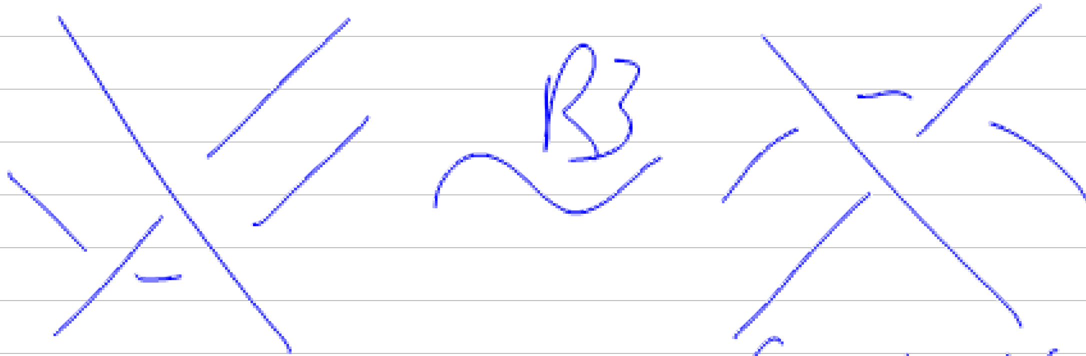
A combinatorial def of knots:

A knot diagram is

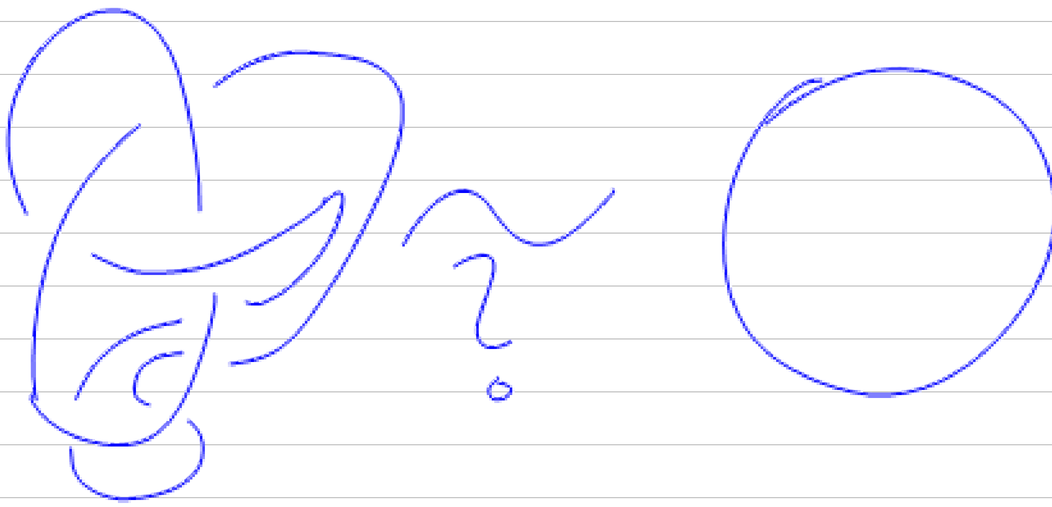


Reidemeister moves



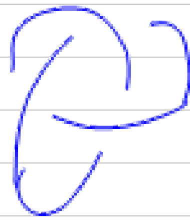
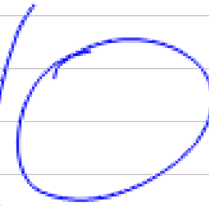


$$\underline{\text{Thm}} \quad \{\text{Knots}\} = \frac{\{\text{knot diagrams}\}}{\text{Reid. moves}}$$



Find "Knot invariants"

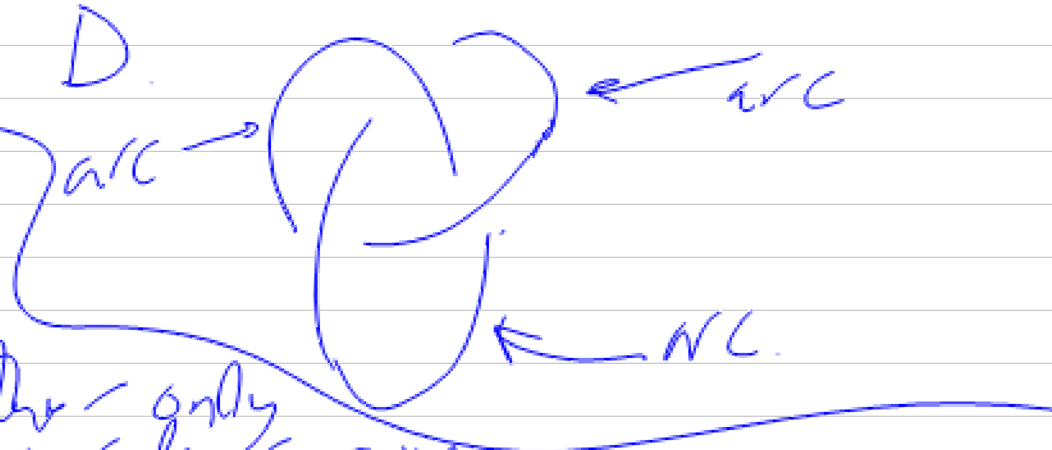
A function on the set of knot diagrams with values in $\mathbb{Z}$, Polys, matrices..
and that does not change if you perform R-moves.

$\lambda($  $) = 9$ $\lambda($  $) = 3$

Example "3-colouring" λ:

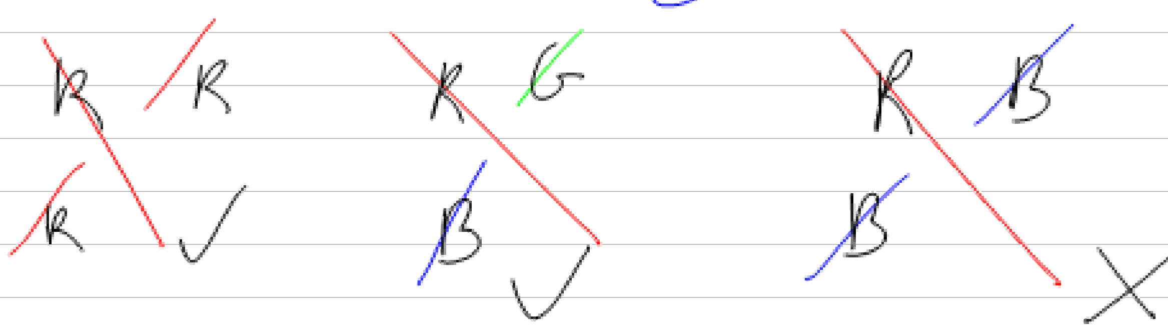
$\lambda(D) = \#$ of legit 3-cols of D

3-colouring: An assignment of colours $\{R, G, B\}$ to the arcs of D

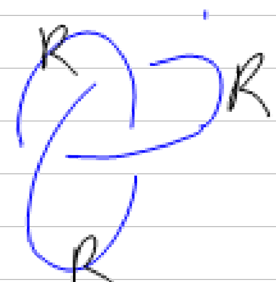


Legit:

either - only one colour appears or all 3.



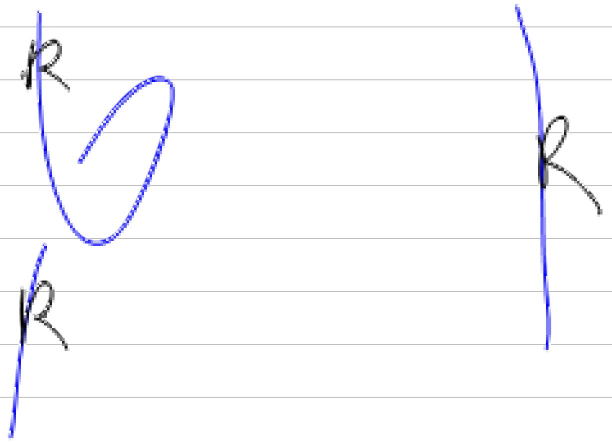
$\lambda(\bigcirc) = 3$



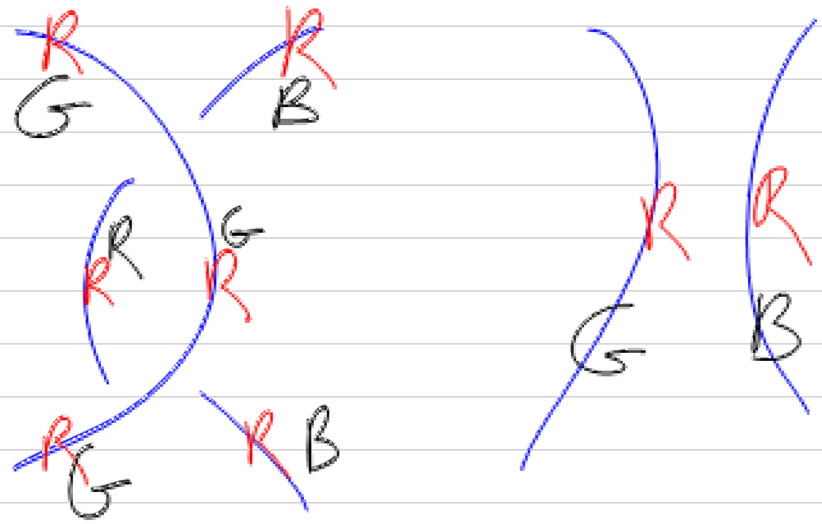
3

$\lambda($  $) = 3 + 6 = 9$

$3 \cdot 2 \cdot 1$



R3
exercise



Class of Friday September 11: The Jones Polynomial.

1. Knots = Knot diagrams / (R1, R2, R3)

2. $\lambda(D) := \left|\left\{\begin{array}{l}\text{colorings of the arcs of } D\\ \text{in } R, G, B, \text{ s.t. no crossing}\\ \text{displays exactly 2 colours}\end{array}\right\}\right|$

a knot diagram

$\lambda(\;) = 9 \qquad \lambda(\bigcirc) = 3$

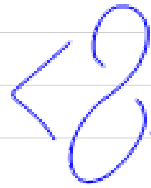
1. Is λ always a power of 3?
2. 100 xings $\sim$ 100 arcs, $\sim 3^{100}$ colorings?
   Can you compute λ more efficiently?

Vaughan Jones (Dec 31, 1952 - Sep 6, 2020):

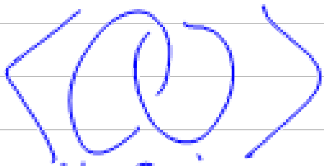
The Kauffman Bracket $\langle D \rangle$

$\langle \; \rangle = A \langle \; \rangle + B \langle \; \rangle$

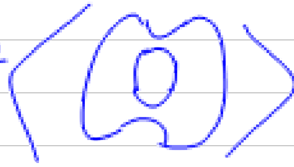
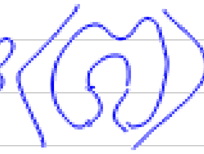
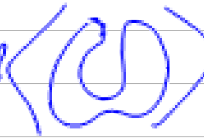
1 0 — 0-smoothing — 1-smoothing

$\langle O \cdot D \rangle = d \langle D \rangle$  $= A \cdot d + B d^2$

$\langle \emptyset \rangle = 1$

 =

Hopf Link

A^2  $+ AB$  $+ AB$ 

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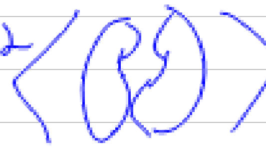
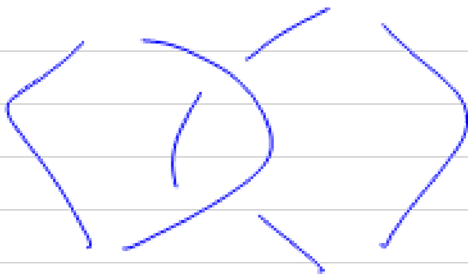
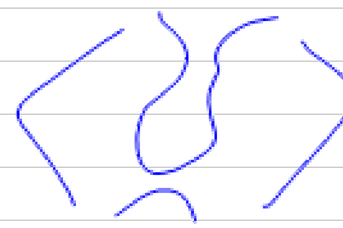
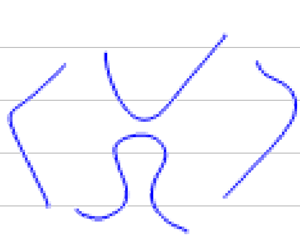
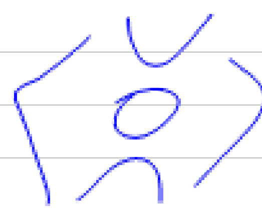
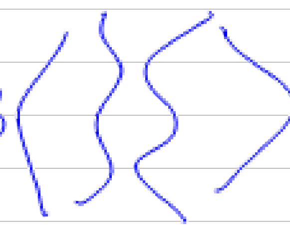
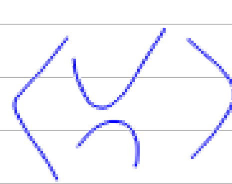
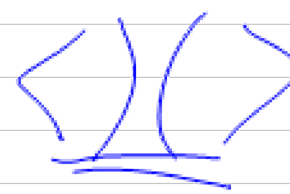
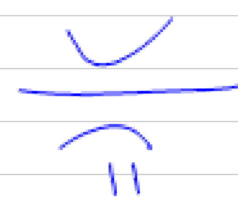
$= A^2 d^2 + 2AB \cdot d + B^2 d^2$ $+ B^2$ 

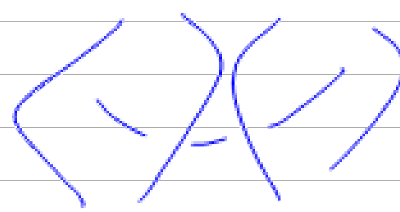
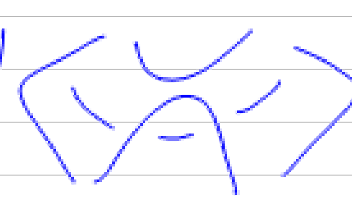
 = $+ B^2$ 

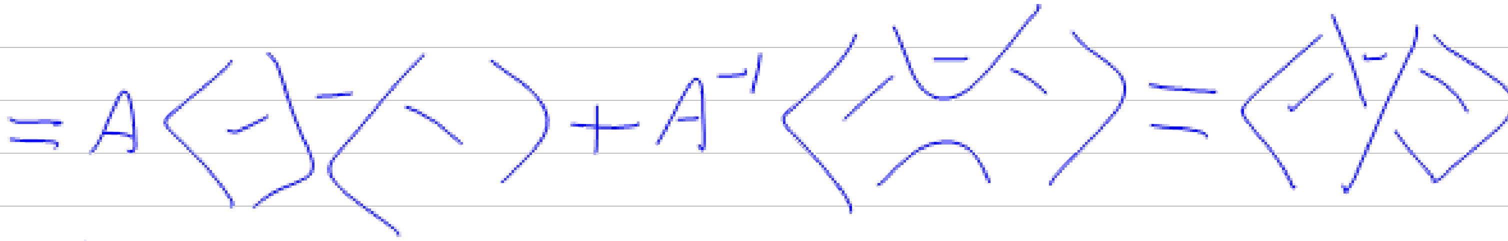
$A^2 \cdot$  $+ AB$  $+ AB$ 

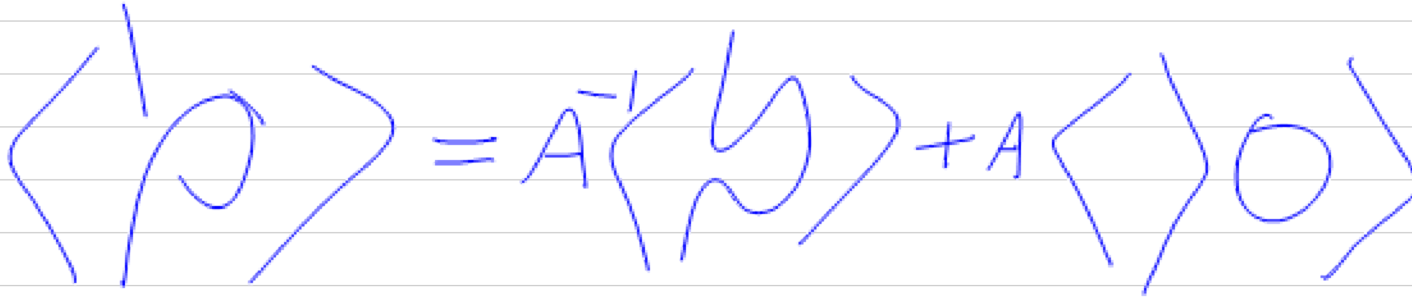
$= (A^2 + B^2 + ABd)$  $+ AB$ 

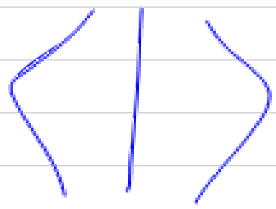
$AB = 1 \quad A^2 + B^2 + ABd = 0$

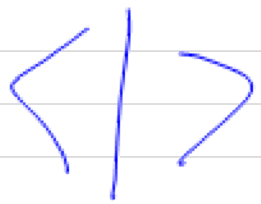
$B = A^{-1} \quad d = -A^2 - A^{-2}$ 

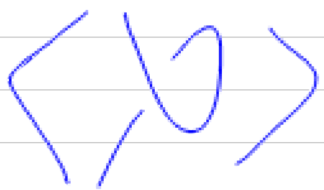
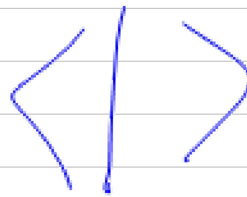
 $= A$  $+ A^{-1}$ 





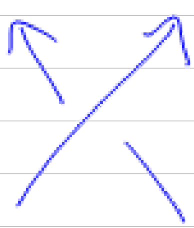
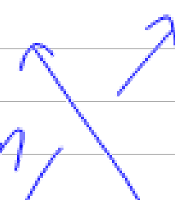
$=$  $\cdot\left(A^{-1}+A\left(-A^{2}-A^{-2}\right)\right)$

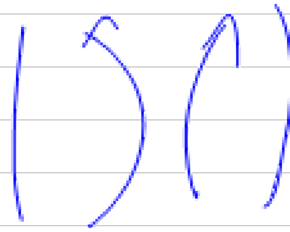
$=$  $\cdot\left(-A^{3}\right)$

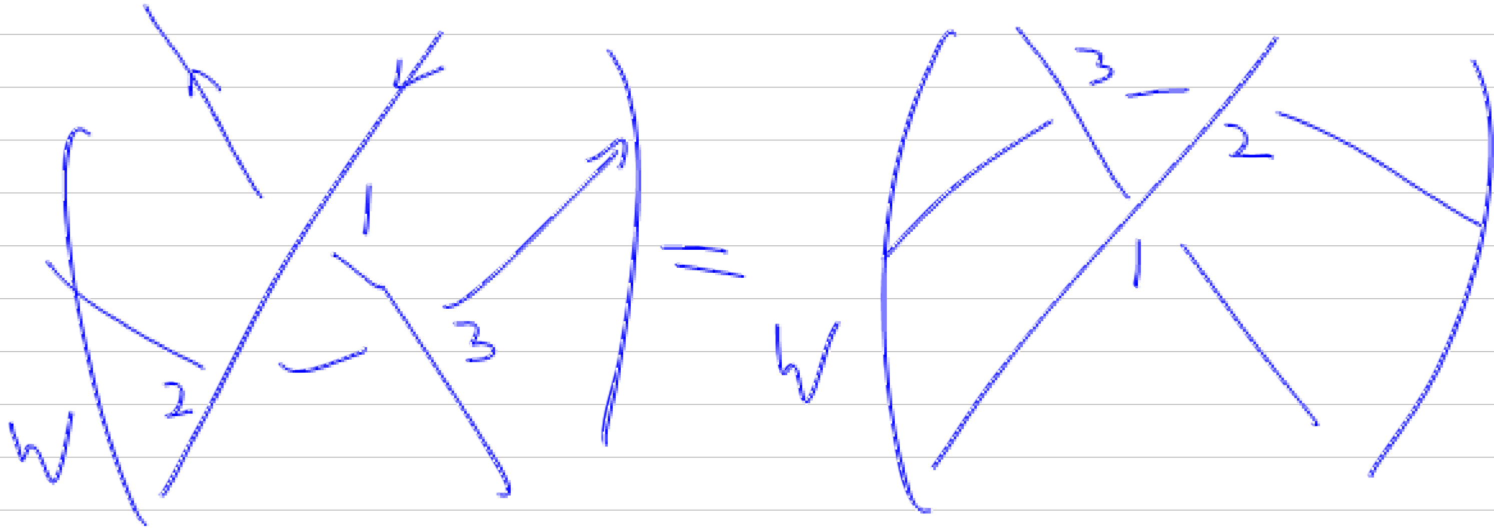
 $= \ldots =$  $\cdot\left(-A^{-3}\right)$

$$w(D) = \sum_{\substack{\text{crossings} \\ x \text{ in } D}} \operatorname{sign}(x)$$

writhe

sign  $= +1$ sign  $= -1$

w  $= w$ 



w
1
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w
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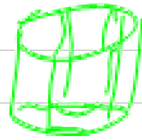
1

Hour 3, Monday Sep 14



$\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$ $\langle \bigcirc^k \rangle = (-A^2-A^{-2})^k$

invt under R2 & R3 but $\langle \rho \rangle = -A^3\langle | \rangle$ $\langle \flat \rangle = -A^{-3}\langle | \rangle$

$W(D) = \sum_{\substack{x \text{ xing} \\ \text{in } D}} \operatorname{sign}(x)$ $\operatorname{sign}(\nwarrow\!\!\!\!\nearrow) = +1$ $\operatorname{sign}(\nwarrow\!\!\!\!\nearrow) = -1$

invt under R2 & R3 but $w(\rho) = w(|)+1$ $w(\flat) = w(|)-1$.

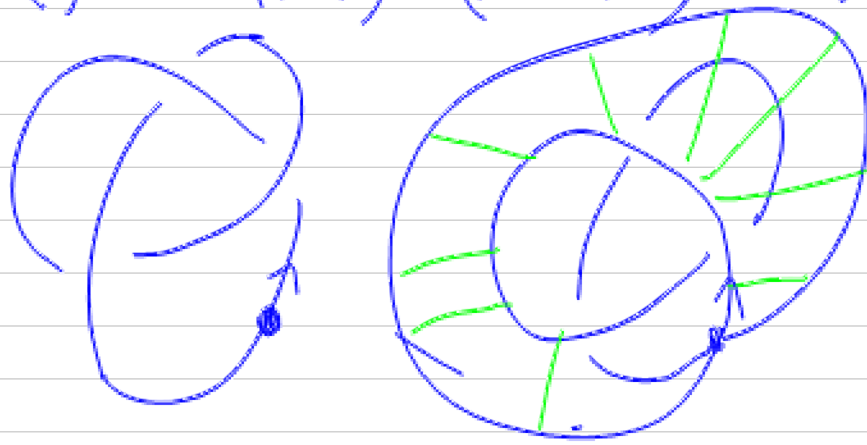
$$J(K) := (-A^3)^{-w(K)} \frac{\langle K \rangle}{d} \;/.\; A \to q^{-1/4}$$

The Jones skein relation:

$$J(\nwarrow\!\!\!\!\nearrow) = -q^{3/4}\left(q^{-1/4}J()() + q^{1/4}J(\asymp)\right)$$

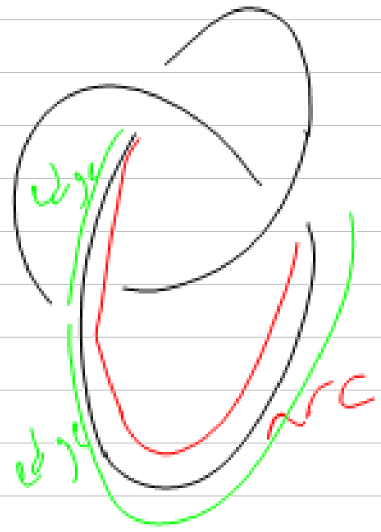
$$J(\nwarrow\!\!\!\!\nearrow) = -q^{-3/4}\left(q^{-1/4}\langle \asymp \rangle + q^{1/4}\langle)(\rangle\right)$$

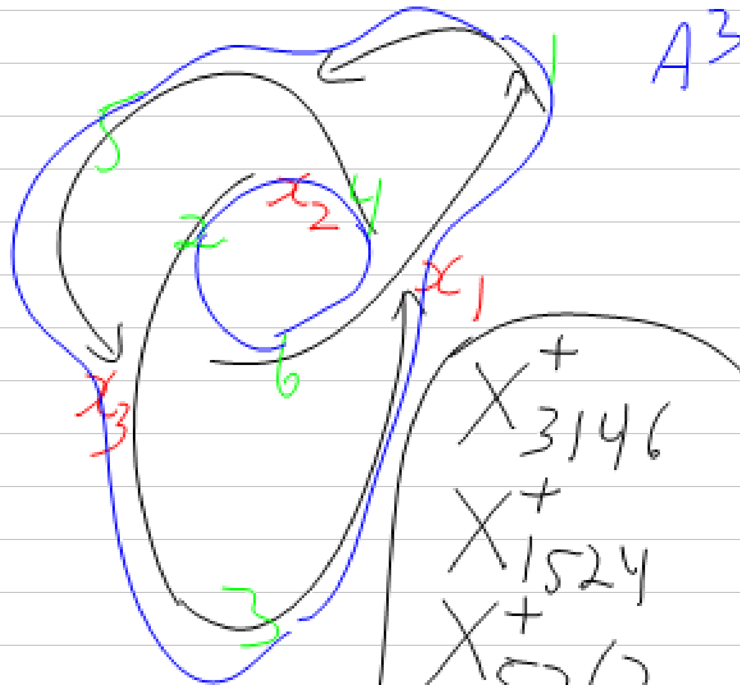
$$\Longrightarrow\; q^{-1}J(\nwarrow\!\!\!\!\nearrow) - q\,J(\nwarrow\!\!\!\!\nearrow) = (q^{1/2}-q^{-1/2})\,J(\uparrow\uparrow)$$



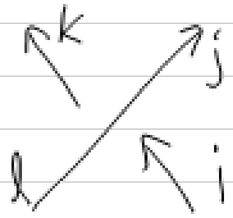
$$J(\bigcirc^k) = \left(-q^{1/2}-q^{-1/2}\right)^k$$

PD - notation



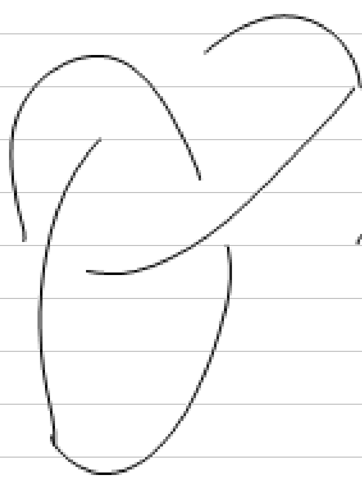


$$X^+_{3146}$$
$$X^+_{1524}$$
$$X^+_{5362}$$

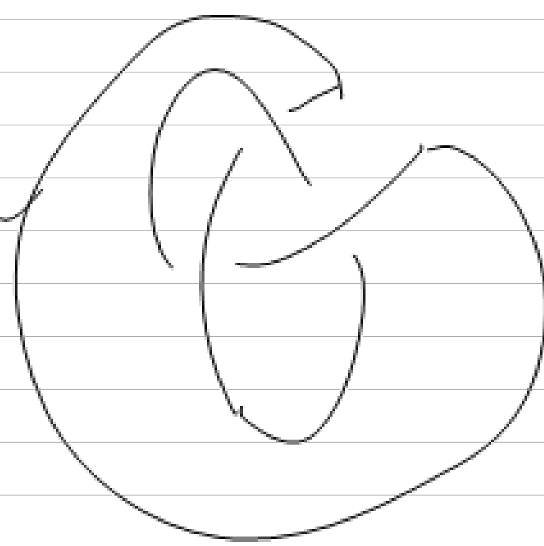


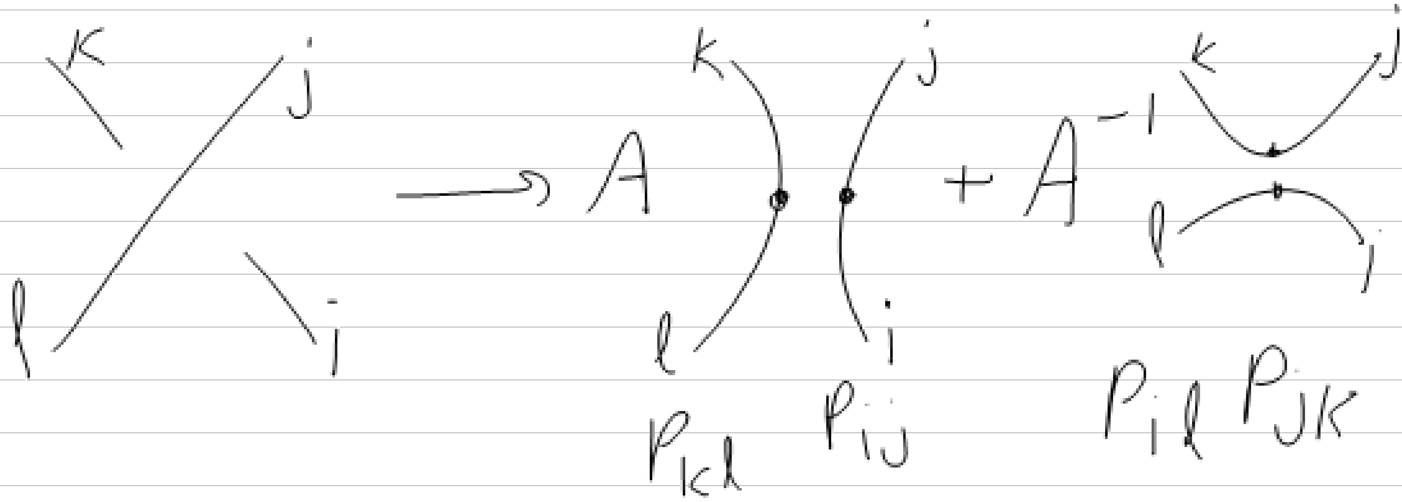
$$\longrightarrow X^s_{ijkl}$$

Exercise: This list of xing info determines D as a diagram on S^2



$\sim$

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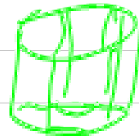
Hour 3, Monday Sep 14



$\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle \qquad \langle O^k \rangle = (-A^2-A^{-2})^k$

invt under R2 & R3 but $\langle \rho \rangle = -A^3\langle | \rangle \quad \langle \rho' \rangle = -A^{-3}\langle | \rangle$

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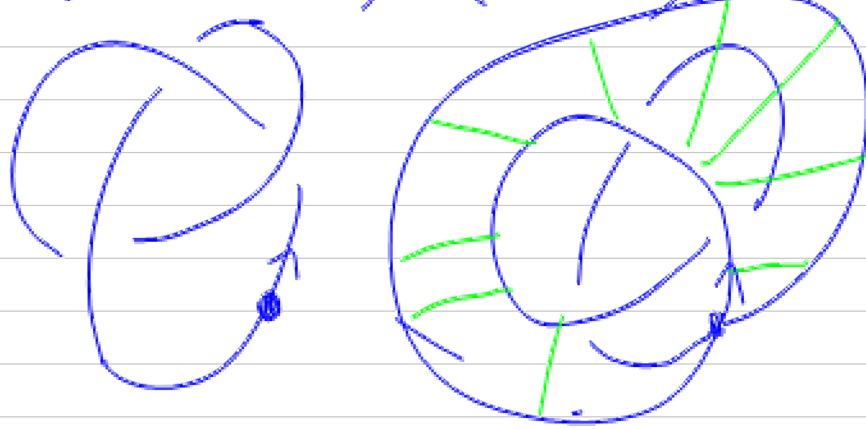
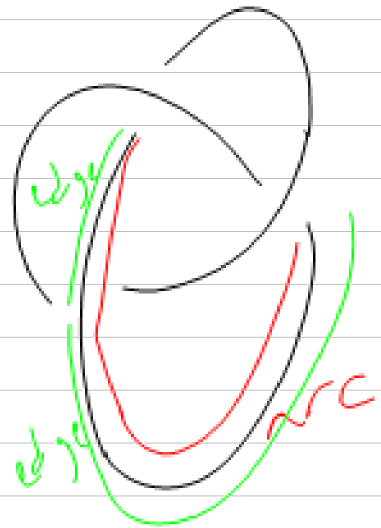
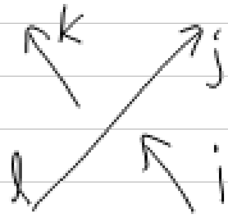


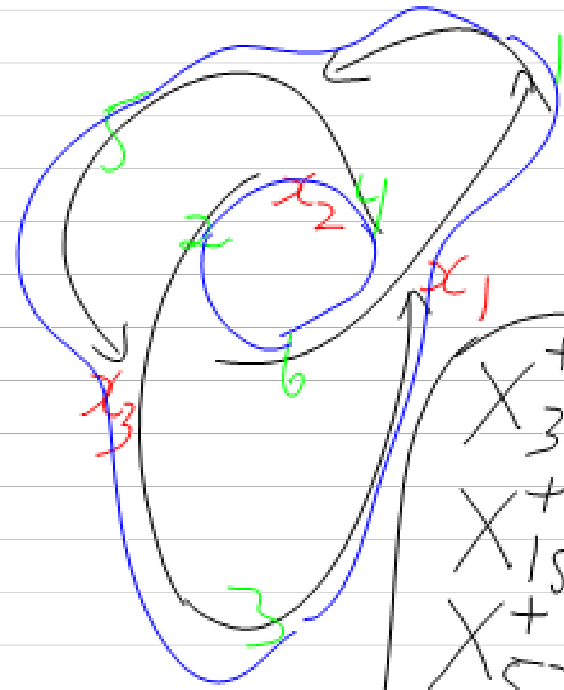
$$J(O^k) = (-q^{1/2}-q^{-1/2})^k$$

PD - notation



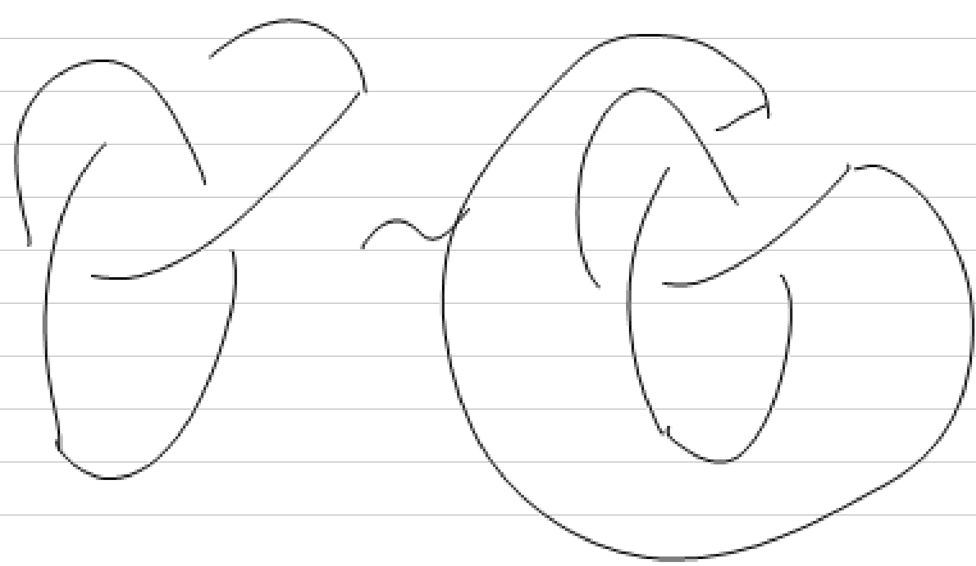


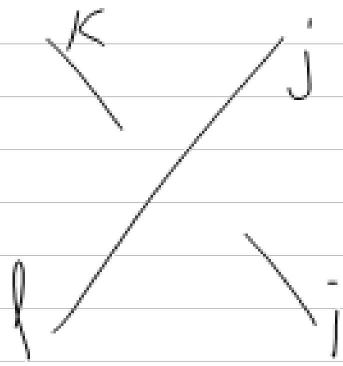
$\longrightarrow X^{s}_{ijkl}$



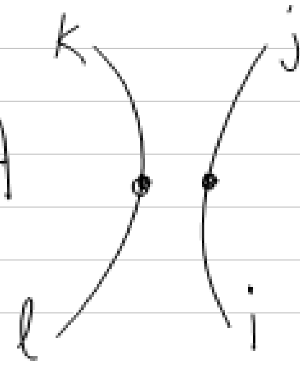
X^{+}_{3146}
X^{+}_{1524}
X^{+}_{5362}

Exercise: This list of xing info determines D as a diagram on S^2

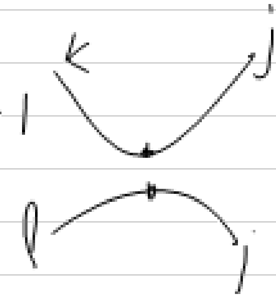




$\longrightarrow A$

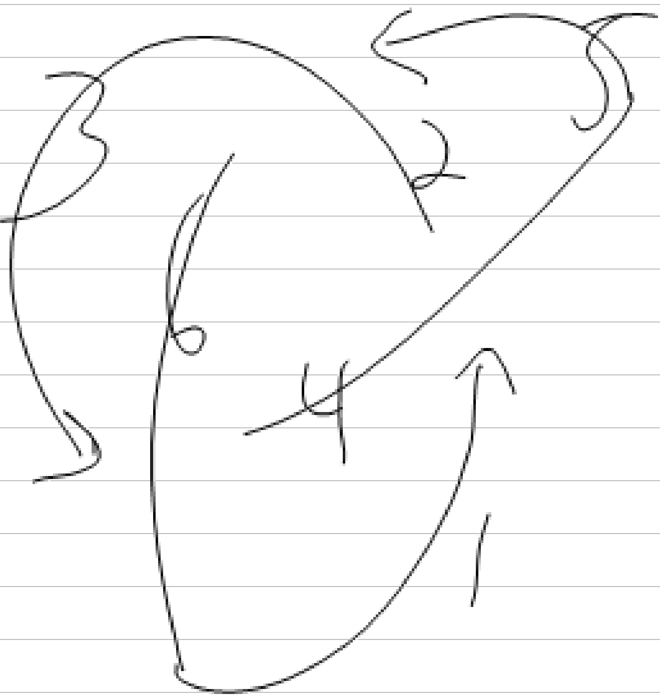


$+ A^{-1}$



$P_{kl} \quad P_{ij} \qquad P_{il} \, P_{jk}$

Hour 4, Wednesday Sep 16
HW 1 on web by midnight!



X_{1524}

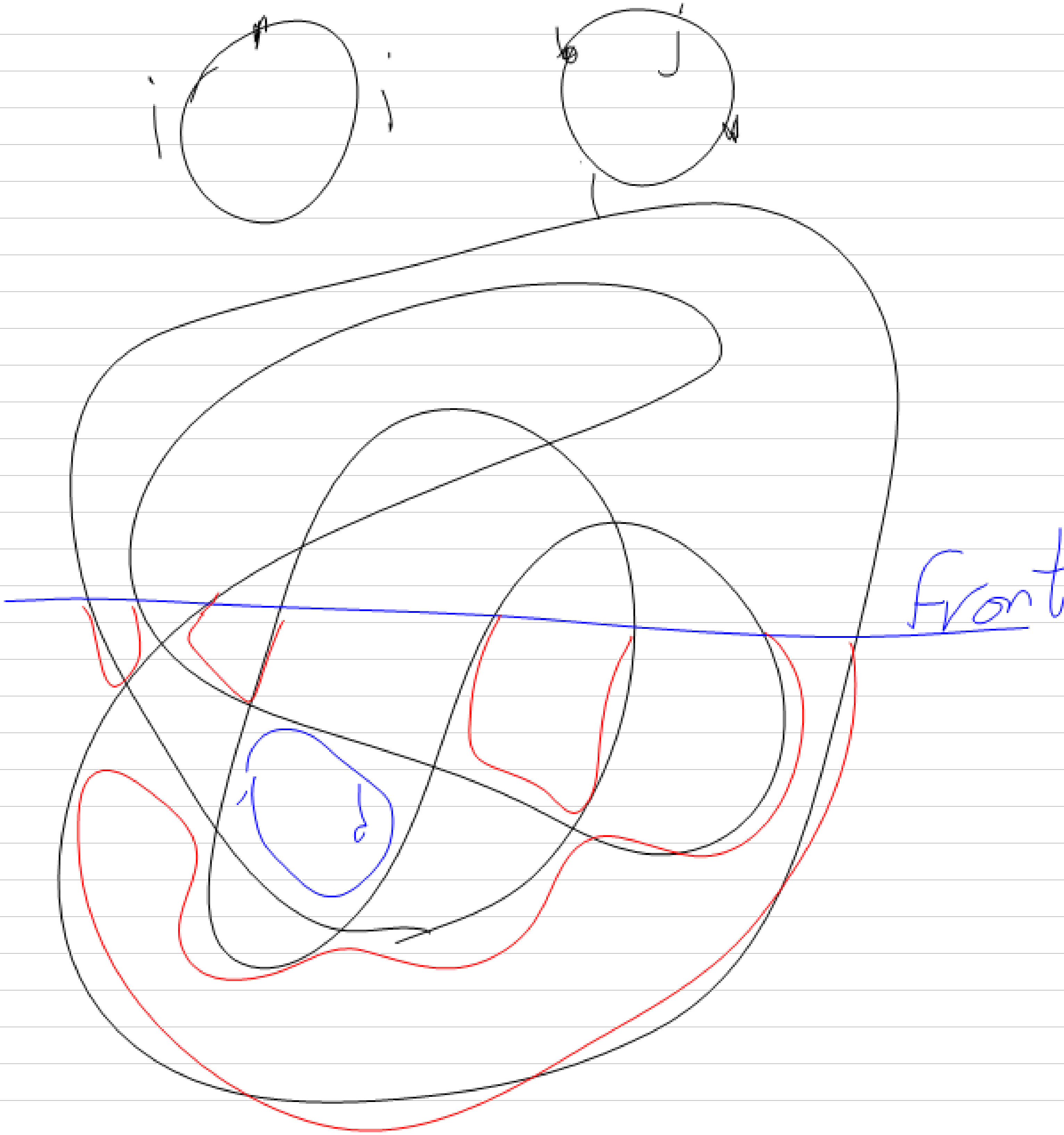
X_{5312}

n_k 3_1 4_1

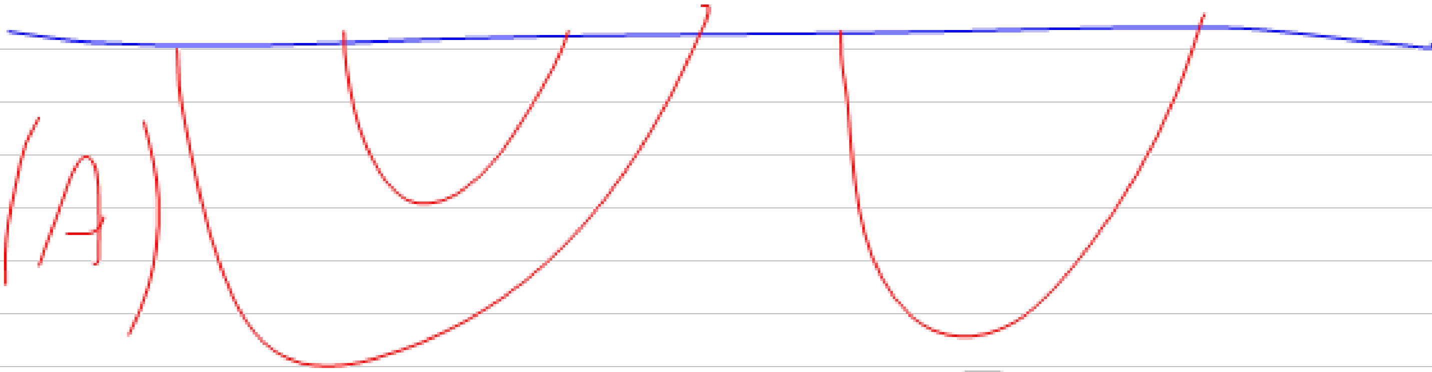
5_1 5_2

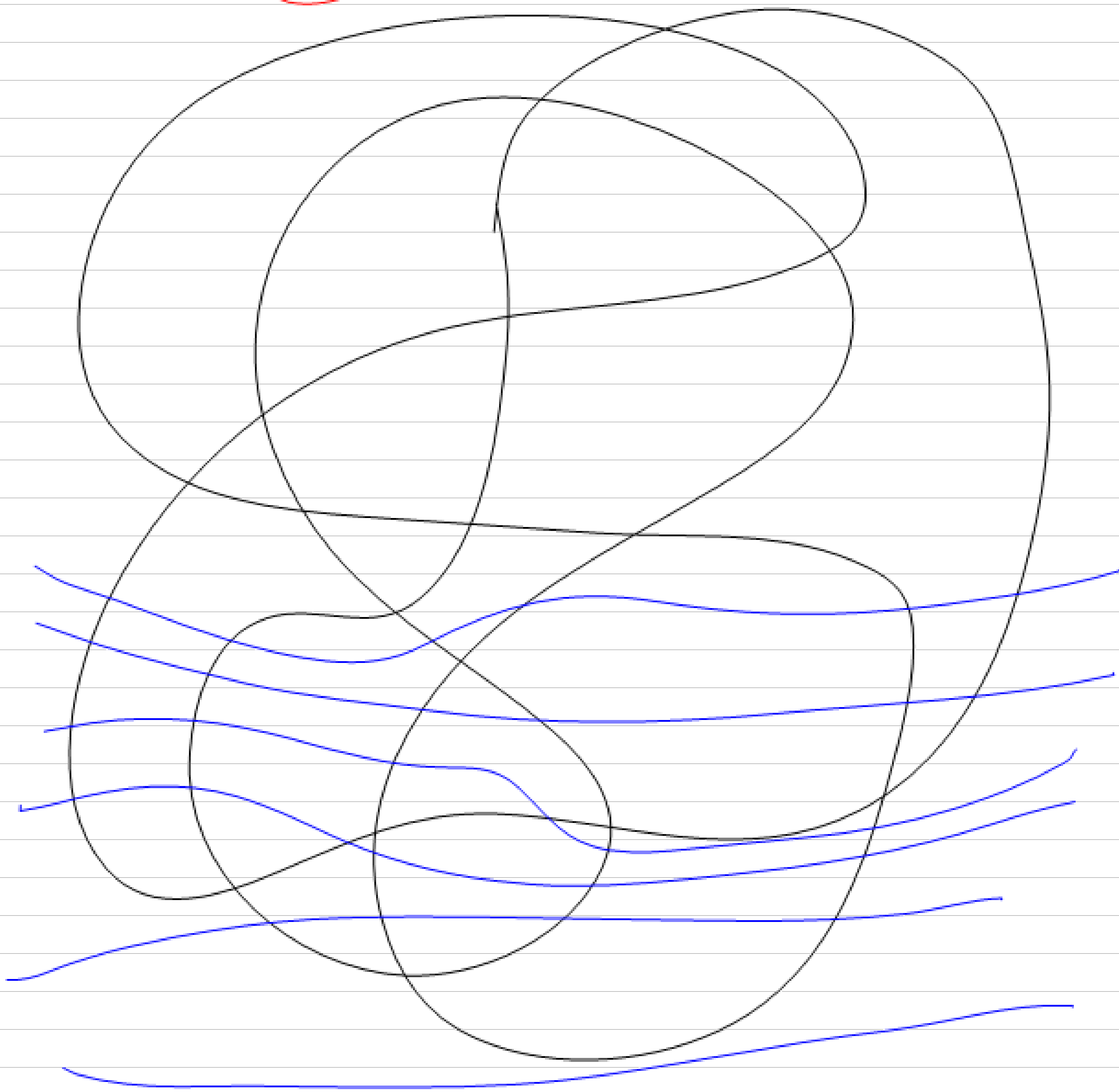
i j k

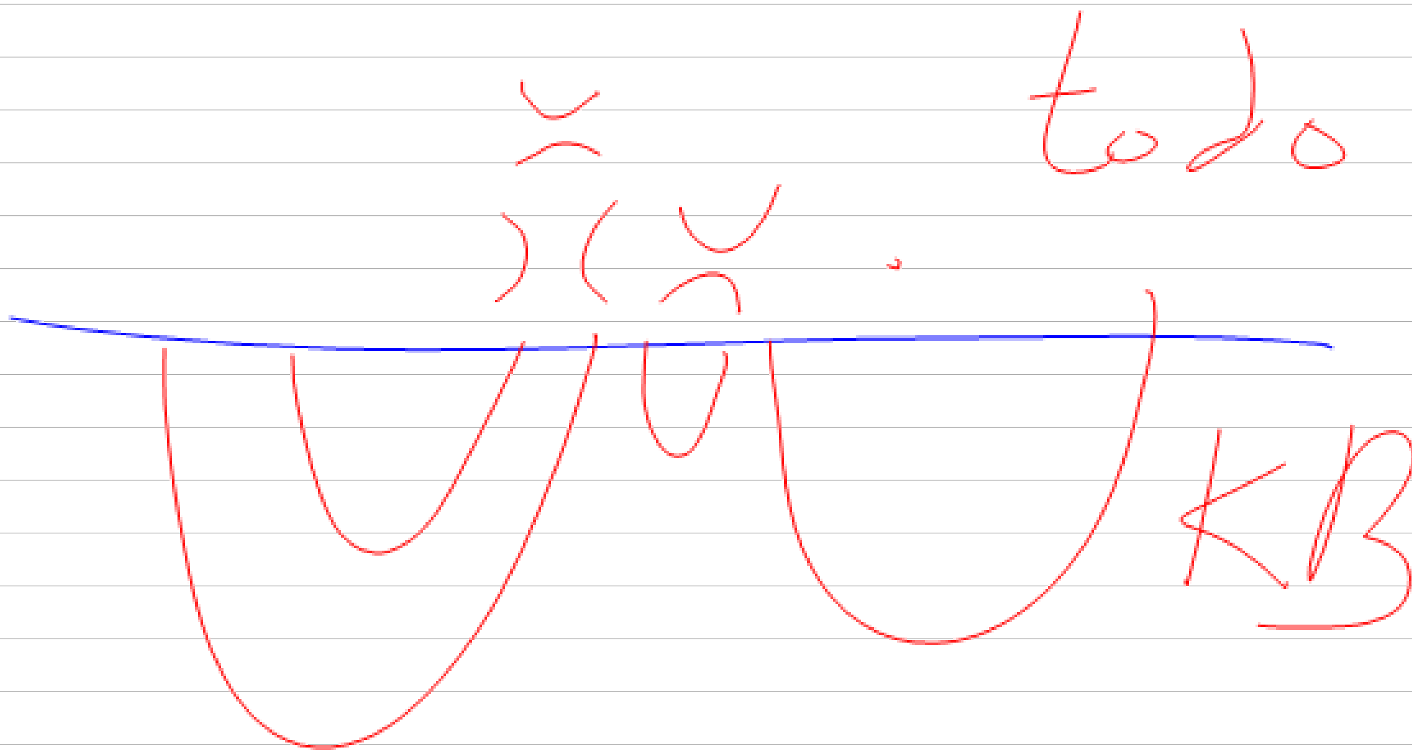
$P_{ij} P_{jk} \longrightarrow P_{ik}$

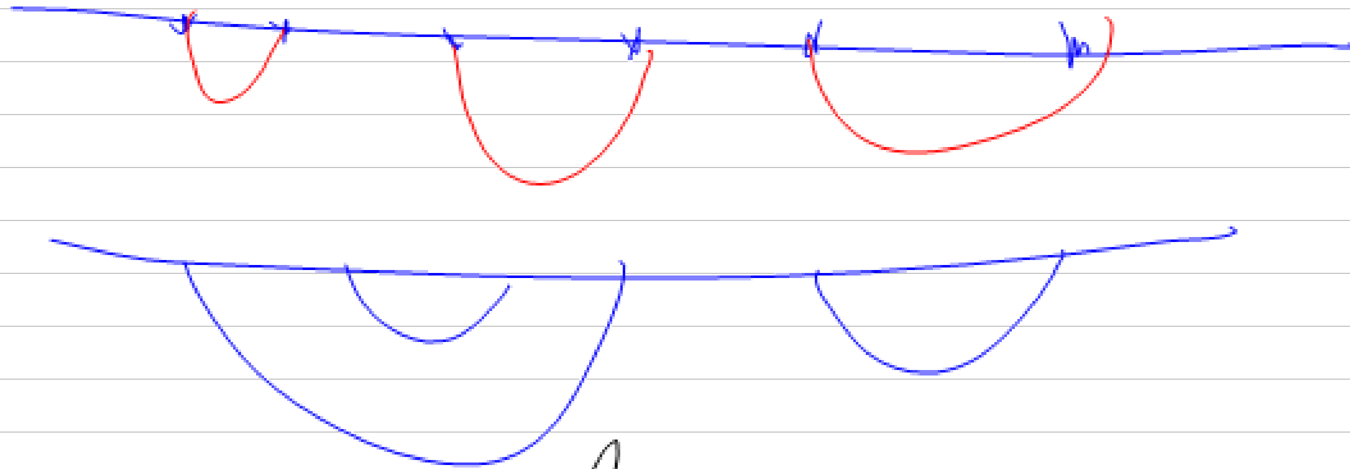

front

$P(A)$

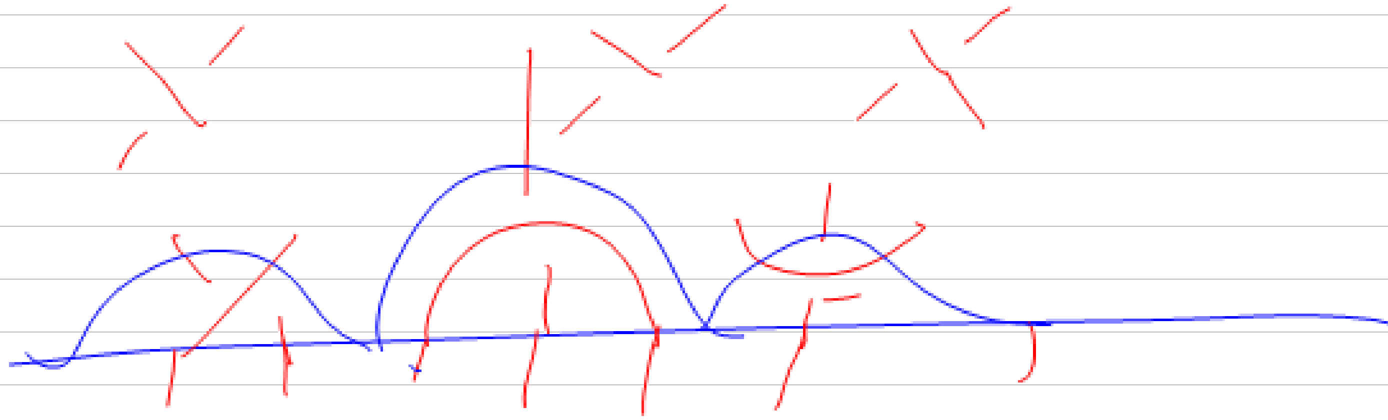


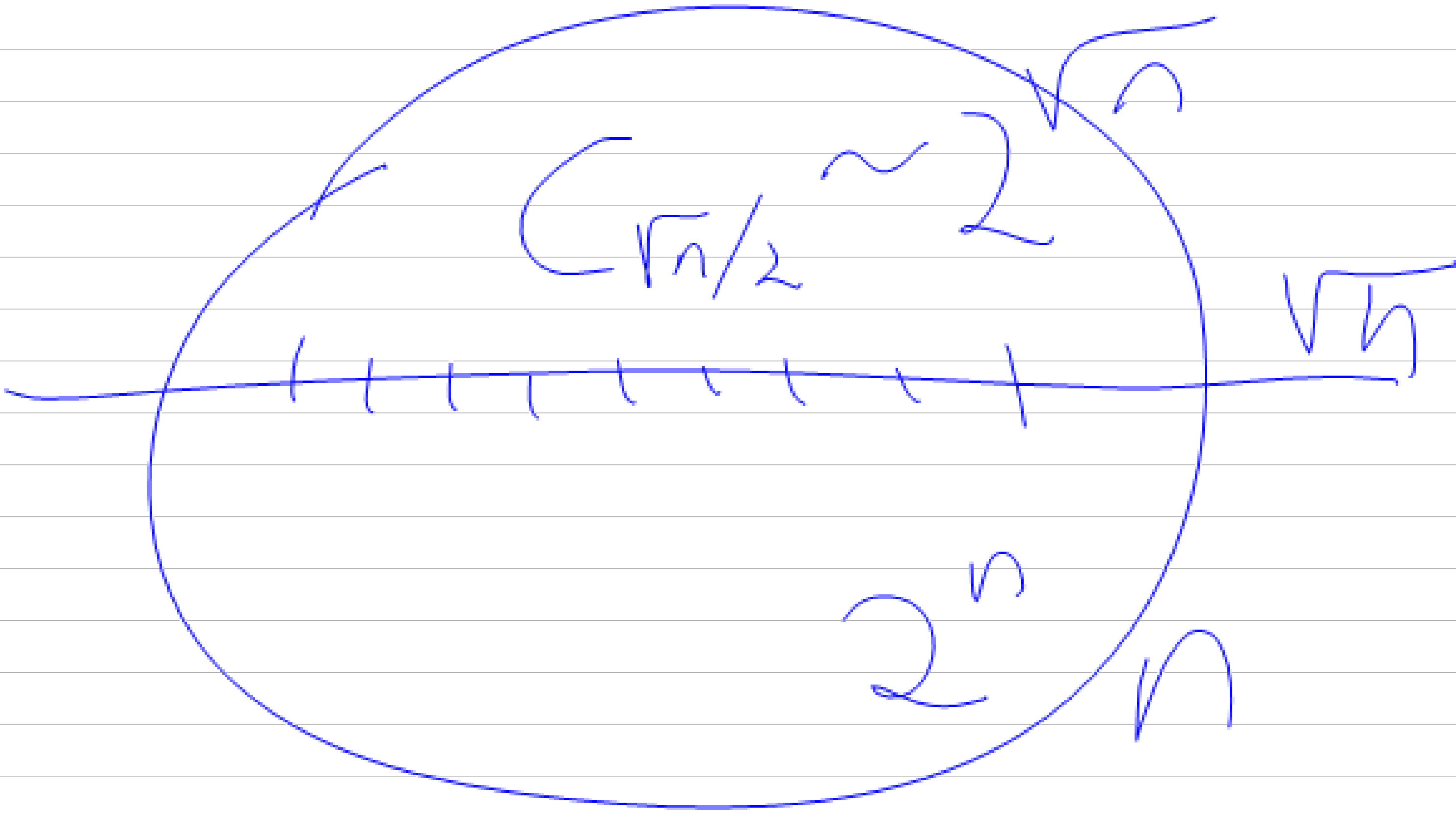






Catalan $C_{n/2}$



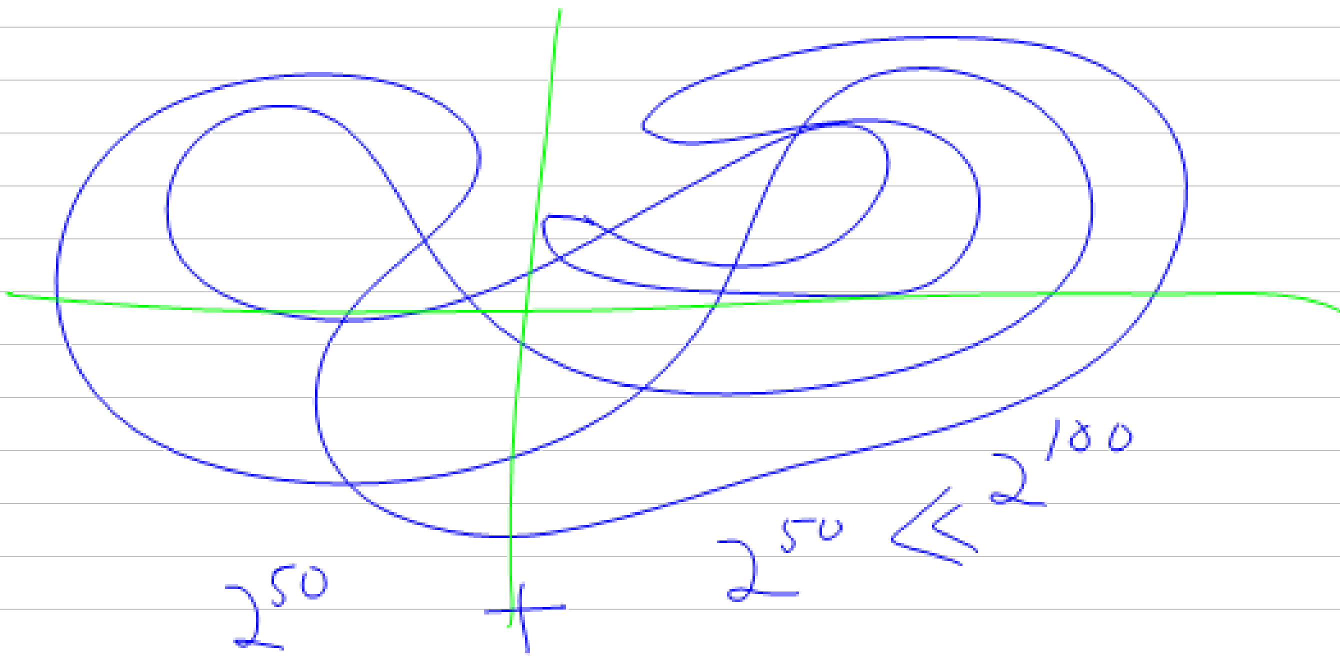

$C_{\sqrt{n}/2} \sim 2^{\sqrt{n}}$
$\sqrt{n}$
2^n
n

Today:

Clarify Wednesday's EFKB
Talk about worms in apples.
On beyond Zebra!

"The Kauffman bracket is a morphism from the planar algebra of tangles to the the Temperley-Lieb planar algebra."

"$\mathrm{Rank}_{\mathbb{Z}[A^{\pm 1}]} TL_{2n} = C_n = \frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$

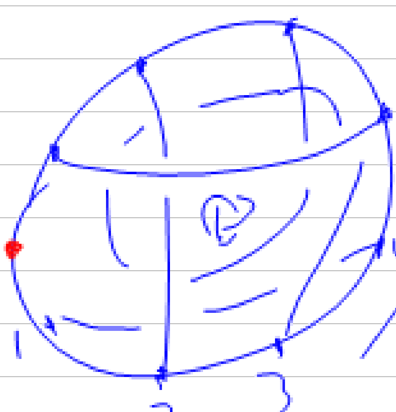


$2^{50} \ll 2^{100}$

2^{50} +

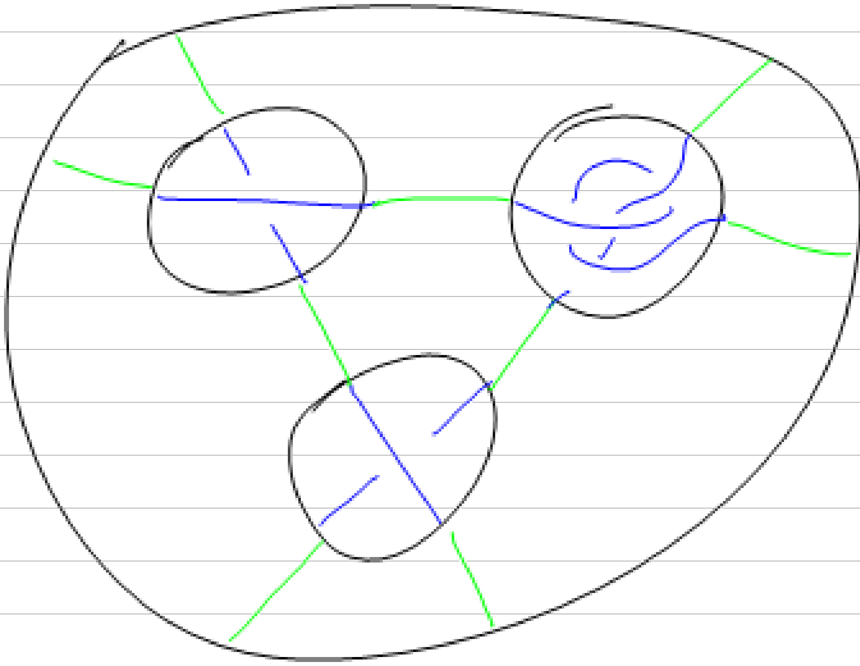
abcde...

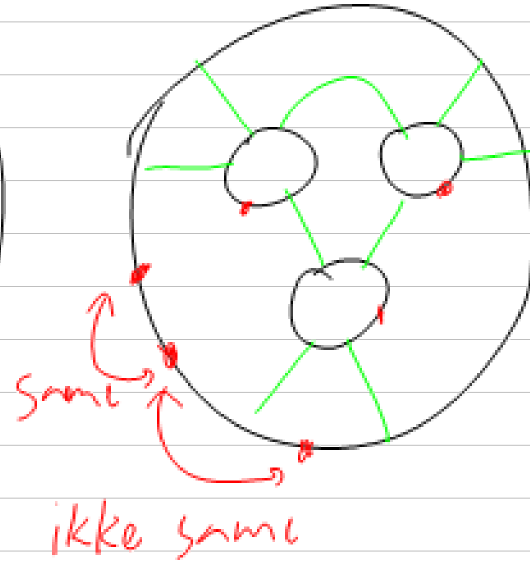
<u>Def</u> A Tangle is



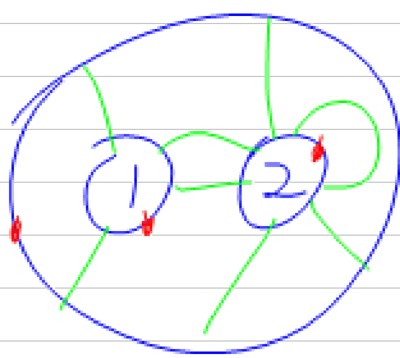
R1

R2 R3



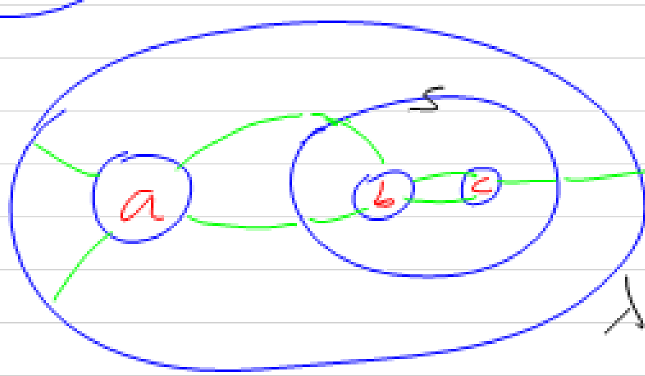


Def A planar algebra is a collection $\{P_n\}$ of sets of objects + a collection of operations labeled by "planar connection diagram"



$: P_4 \times P_7 \longrightarrow P_5$

s.t.



Remove S:

$\varphi = \overset{a}{P_4} \times \overset{b}{P_4} \times \overset{c}{P_3} \longrightarrow P_3$

Remove outside of S

$\lambda = P_4 \times P_3 \longrightarrow P_3$

remove inside of S

$\mu = P_4 \times P_3 \longrightarrow P_3$

$$\varphi(a,b,c) = \mu(a, \lambda(b,c))$$

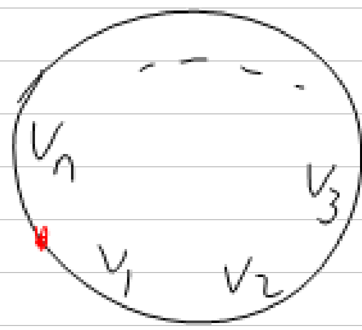
Example 1 Tangles. P_n = tangles w/ n inputs.

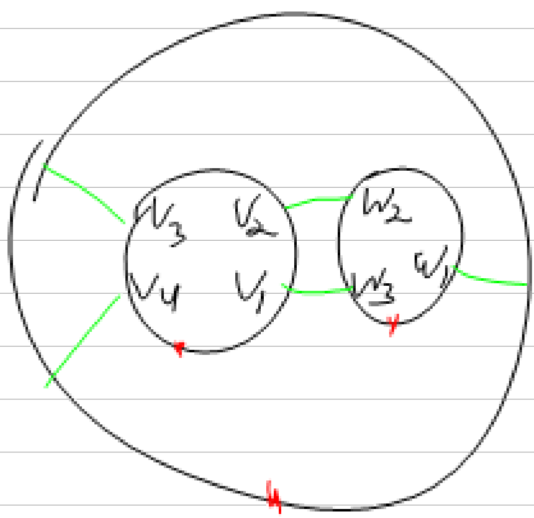
$P_{2n+1} = \emptyset$.

Ex 2 V be a f.d. V.S. w/ inner prod.

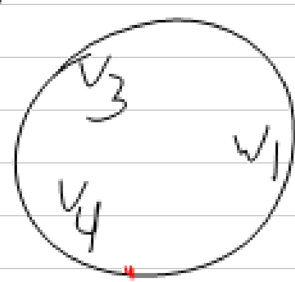
$$P_n = V^{\otimes n} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}}$$

$$= \langle v_1 \otimes v_2 \otimes v_3 \cdots \otimes v_n : v_i \in V \rangle$$

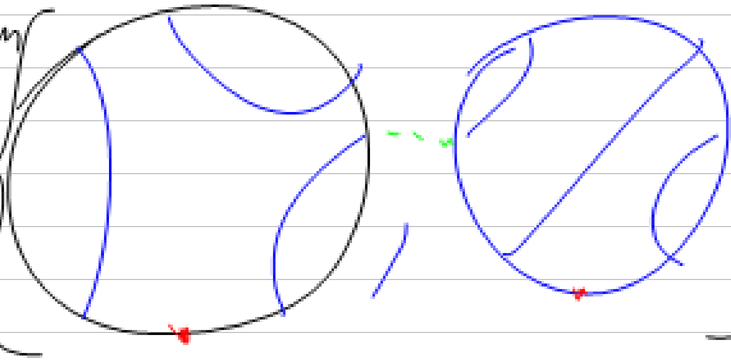


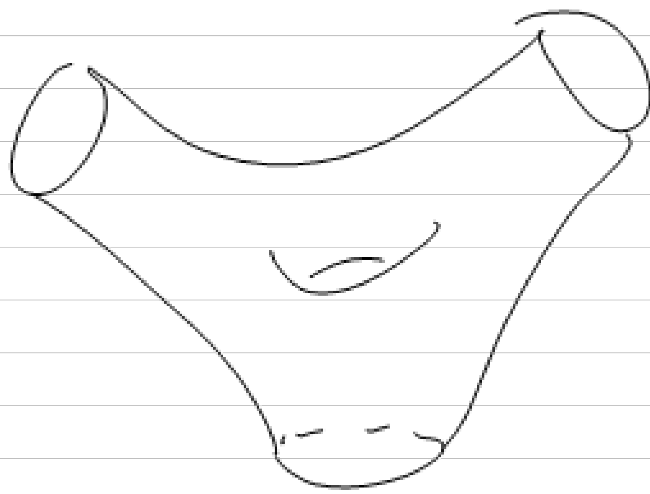


$= \langle v_2, w_2 \rangle \langle v_1, w_3 \rangle$



Example Temperley-Lieb planar alg.

$P_6 = \mathrm{Span}\Big\{$  $\Big\} \Big/ \bigcirc = d$



Last time: over $\mathbb{Z}[A^{\pm 1}]$

$$KB: \text{Planar Alg of Tangles}/R2R3 \longrightarrow TL := \{\ldots\}/\bigcirc = d = -A^2 - A^{-2}$$

$$\mathrm{Rank}_{\mathbb{Z}[A,A^{-1}]} TL_{2n} = C_n = \frac{1}{n+1}\binom{2n}{n} \sim 4^n / n^{3/2}\sqrt{\pi}$$

$$n! = \frac{n^n}{e^n}\sqrt{2\pi n}$$

C_n = How many histories lead to the score n-n in a soccer game in which team B never leads.

$$C_0 = \{(0,0)\} = 1$$

$$C_1 = \left\{\begin{matrix}00\\10\\11\end{matrix}\right\} = 1 \qquad C_2 = \left\{\begin{matrix}00\\10\\20\\21\\22\end{matrix} \quad \begin{matrix}00\\10\\11\\21\\22\end{matrix}\right\} = 2$$

$$C_3 = 5 \qquad C_4 = 14$$

$C =$

	-1	-4				0
	-1	-3	-5	-5	0	14
3	-1	-2	-2	0	5	14
2	-1	-1	0	2	5	9
1	-1	0	1	2	3	4
0		1	1	1	1	1	1
		0	1	2	3	

$C = A - B$

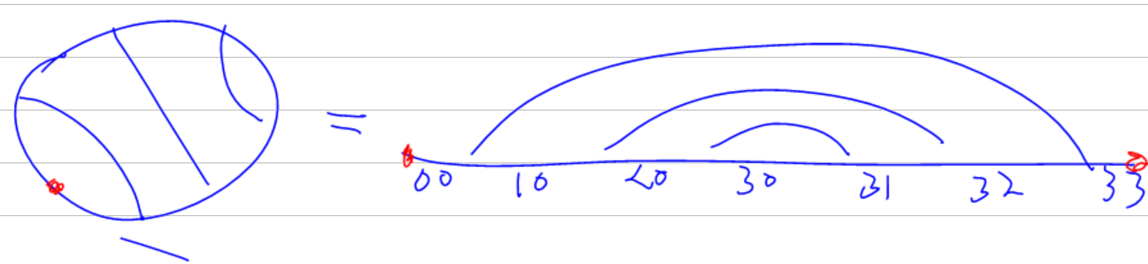
A:

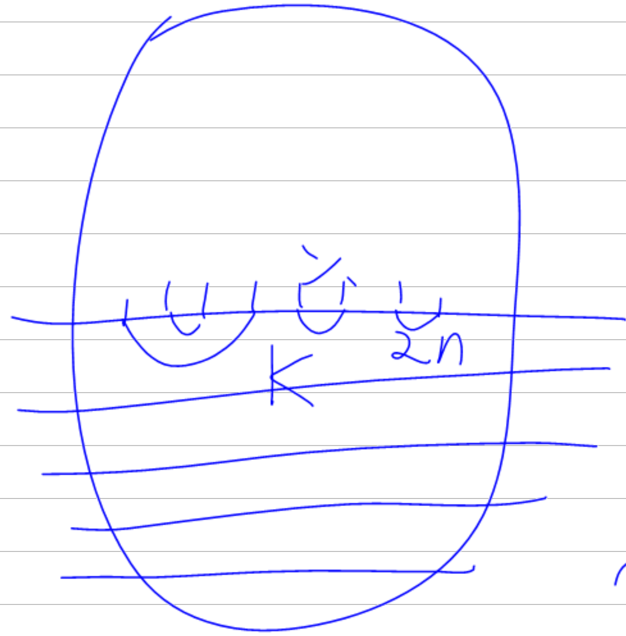
1			
1	3		
1	2	3	
1	1	1	1

B:

3			
2	3		
1	1	1	1

$$C_{n,n} = A_{n,n} - B_{n,n} = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$$

$TL_{2n} =$ 



Laurant
C_n - polys in $A^{\pm 1}$

Complexity = (# xings) C_n · (complexity of ops in $\mathbb{Z}[A^{\pm 1}]$)

$\sim 4^n \sim 4^{\sqrt{\text{xings}}}$

Khovanov homology (1999)

A graded chain complex for each knot diagrams, whose qEuler characteristic is the Jones poly, whose homology is invariant (stronger than Jones) + ... more.

finite

$$0 \to \dots \to C^{r-1} \xrightarrow{d^{r-1}} C^r \xrightarrow{d^r} C^{r+1} \to \dots \to 0$$

f.d. v.s.

$$d^r \circ d^{r-1} = 0$$

r-bndries $d^2 = d \circ d = 0$

r-cycles

$$B^r = \operatorname{im} d^{r-1} \subset \ker d^r = Z^r$$

$H^r = Z^r / B^r$ "r-th homology"

$$\chi = \sum (-1)^r \dim C^r$$

rank-nullity : $\operatorname{rank}(d^r) + \operatorname{nullity}(d^r) = \dim C^r$

$$\underset{\dim B^{r+1}}{\overset{\|}{}} + \dim Z^r = \dim C^r$$

$$\chi = \sum (-1)^r \left(\dim Z^r + \dim B^{r+1} \right)$$

$$= \sum (-1)^r \left(\dim Z^r - \dim B^r \right) = \sum (-1)^r \dim H^r$$

Hour 7, Wednesday September 23: Khovanov homology following http://drorbn.net/mo13
HW2 on web, HW1 due at 11:59PM.

Sums and products of complexes.
Graded vector spaces, q-dim, sums, prodcuts, shifts.
Graded complexes and graded Euler characteristic.
Then http://drorbn.net/mo13 ....

$A, \; B \qquad A \oplus B \qquad \dim(A\oplus B) = \dim(A) + \dim(B)$

$A, B \qquad A \otimes B$

$\{a_i\}_1^n \; \{b_j\}_1^m \qquad \langle a_i \otimes b_j \rangle_{i=1}^{n} {}_{j=1}^{m} \qquad \dim(A\otimes B) = \dim(A)\cdot\dim(B)$

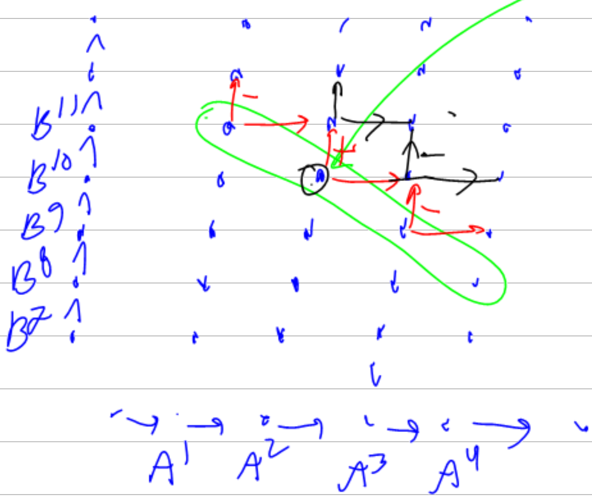
$(A^\cdot, d_A) \qquad (B^\cdot, d_B) \qquad (A^\cdot \oplus B^\cdot, (d_A, d_B))$

$(A\oplus B)^r := A^r \oplus B^r \qquad \chi(A^\cdot \oplus B^\cdot) = \chi\ldots + \chi\ldots$

$(A^\cdot \otimes B^\cdot, d_A + d_B) \qquad \chi(A^* \otimes B^*) = \chi(A)^* \times \chi(B^*)$

$$(A\otimes B)^{r} \overset{ht}{=} \bigoplus_{r_1+r_2=r} A^{r_1} \otimes B^{r_2}$$

$A^2 \otimes B^{10} \subset (A\otimes B)^{12}$



$$\underbrace{d_{A\otimes B}}_{\text{odd}}(a\otimes b) = (\overbrace{d_A}^{\text{odd}} a \otimes b) + (-1)^{ht(a)} (a \otimes \underbrace{d_B}_{\text{odd}} b)$$

$$\chi(A^* \otimes B^*) = \chi(A^\cdot)\cdot\chi(B^\cdot)$$

$$\sum (-1)^r \underbrace{\sum_{r_1+r_2=r} A^{r_1}\otimes B^{r_2}}_{\dim} = \sum_{r_1, r_2} (-1)^{r_1}(-1)^{r_2} (\dim A^{r_1}) \cdot (\dim A^{r_2})$$

Graded v.s: $V = \bigoplus_{n=n_1}^{n_2} \underbrace{V_n}_{\text{elements of deg } n}$

$$q\text{-}\dim V = \sum_n (\dim V_n)\, q^n$$

$$(V_1 \oplus V_2)_n = V_{1n} \oplus V_{2n}$$

$$(V_1 \otimes V_2)_r = \sum_{r_1 + r_2 = r} (V_1)_{r_1} \otimes (V_2)_{r_2}$$

$$q\dim(V_1 \otimes V_2) = (q\dim V_1)(q\dim V_2)$$

$$\cdots \to A^{r-1} \longrightarrow A^r \xrightarrow{d} A^{r+1} \cdots$$

$$A^r \ni a \longrightarrow da$$

$$\deg a = \deg(da) \qquad \deg d = 0$$

$$\vdots$$

$$\cdots A_7^{r-1} \longrightarrow A_7^r \longrightarrow A_7^{r+1} \cdots$$

$$\cdots A_6^{r-1} \longrightarrow A_6^r \longrightarrow A_6^{r+1} \cdots$$

$$A_5 \to A_5 \longrightarrow A_5 \cdots$$

$$\chi_q(A^*) = \sum_r (-1)^r q\dim A^r$$
$$= \sum (-1)^r q\dim H^r$$

$$\chi_q(A \otimes B) = \chi_q(A) \cdot \chi_q(B) \ldots$$

The Jones polynomial: $\bigcirc^k \mapsto (q+q^{-1})^k$

$J : \times \mapsto q)(\,-\,q^2 \asymp, \quad J : \times \mapsto -q^{-2}\asymp + q^{-1})($

$q+q^{-1} = \bigcirc = -A^2 - A^{-2}$

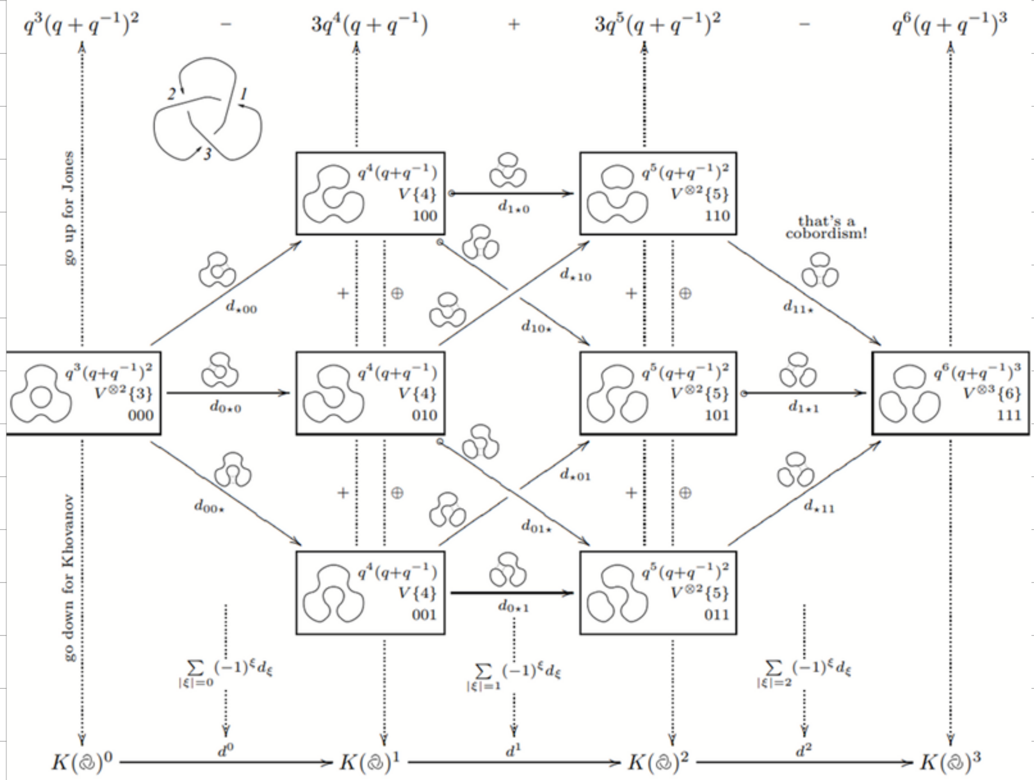
$q = -A^2 \qquad A = \sqrt{-q}$

$J(K) = \langle K \rangle (-A^{-3})^{w(K)}$

$w(\times) = +1$

$w(\times) = -1$

Example: $= q + q^3 + q^5 - q^9.$



(here $(-1)^\xi := (-1)^{\sum_{i<j}\xi_i}$ if $\xi_j = \star$) $= K$.

V is a graded v.s., $V = \bigoplus_n V_n$

$(V\{m\})_n := V_{n-m}$

$q\dim V\{m\} = q^m \cdot q\dim V$

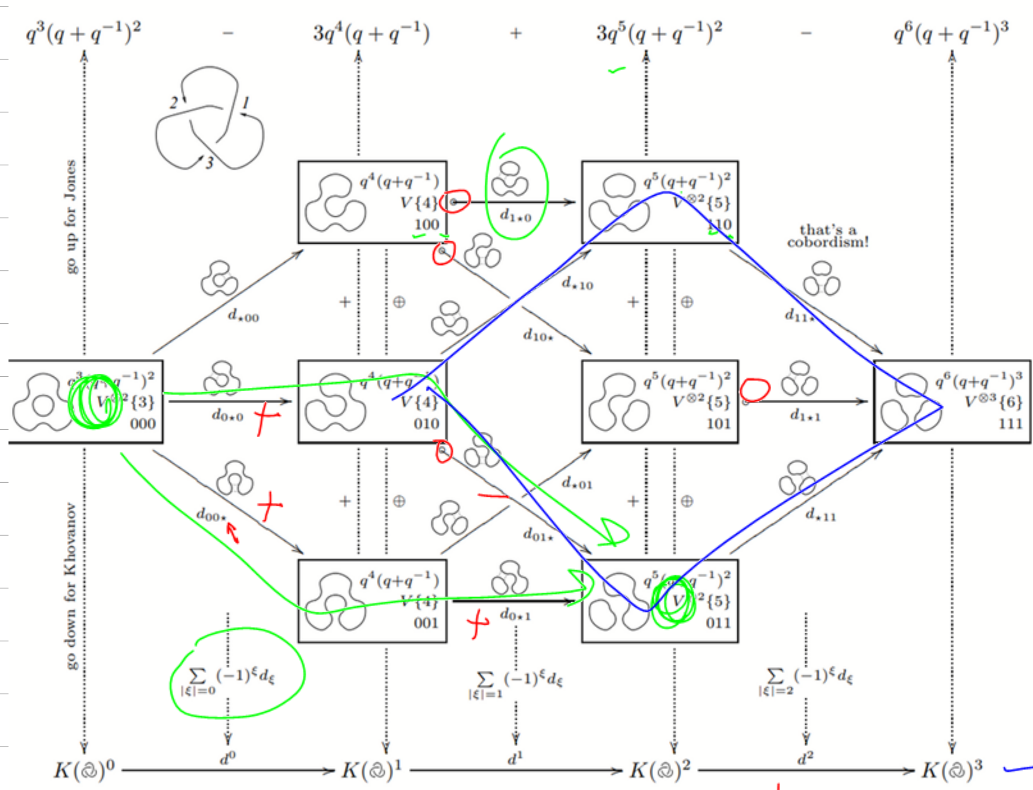
Hour 8, Friday September 25: Khovanov homology following http://drorbn.net/mo13

Riddle: $\exists$? Cont. $f:\mathbb{R}\to\mathbb{R}$ s.t. $f\circ f=\cos$?

The Jones polynomial: $\bigcirc^k \mapsto (q+q^{-1})^k$

$J: \times \mapsto q)(-q^2 \asymp$, $\quad J: \times \mapsto -q^{-2}\asymp + q^{-1})($,

Example: $\qquad = q+q^3+q^5-q^9.$



(here $(-1)^\xi := (-1)^{\sum_{i<j}\xi_i}$ if $\xi_j=\star$) $\qquad = K(\text{trefoil})$.

$V \quad q\dim = q+q^{-1}$

$V=\langle v_+, v_-\rangle$

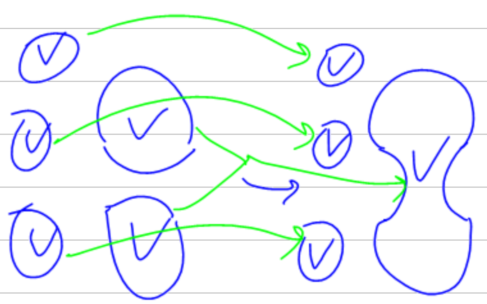
$\deg(v_\pm)=\pm 1$

$V^{\otimes 3}\{6\}$

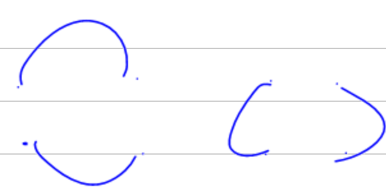
$d^2=0$

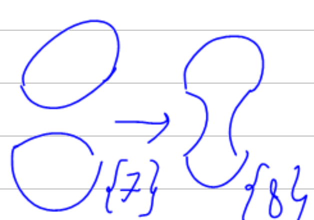
$\xrightarrow{\chi_q} J(\text{trefoil})$

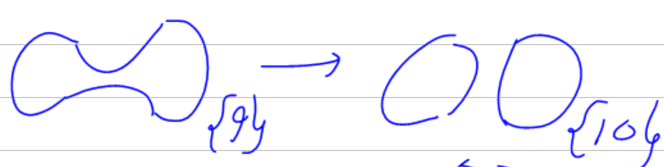
JOU: why signs work and where they came from.



$V^{\otimes 5} \to V^{\otimes 4}$

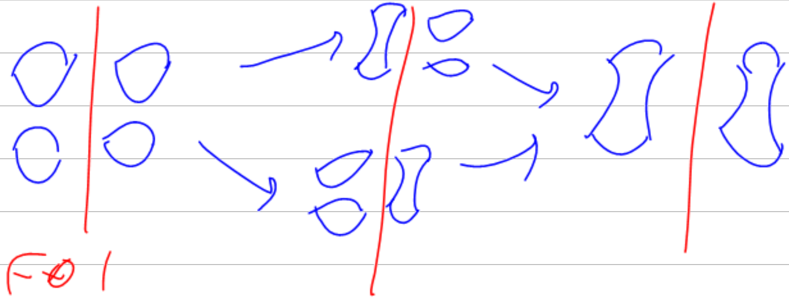






$m: V^{\otimes 2}\to V \qquad \Delta: V \to V^{\otimes 2}$

Cond #1: $\deg(m)=\deg(\Delta)=-1$



F01

109

Hour 9, Monday September 28: Khovanov homology following http://drorbn.net/mo13 (Power outage at 2PM!)

The Jones polynomial: $\bigcirc^k \mapsto (q+q^{-1})^k$

$J : \overcrossing \mapsto q)(- q^2 \asymp, \quad J : \undercrossing \mapsto -q^{-2}\asymp + q^{-1})($

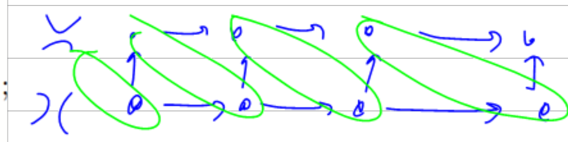
$\dashrightarrow \; 0 \longrightarrow \underset{\text{ht } 0}{V^{\otimes k}} \longrightarrow 0 \ldots$

Khovanov: $K(L)$ is a chain complex of graded $\mathbb{Z}$-modules;

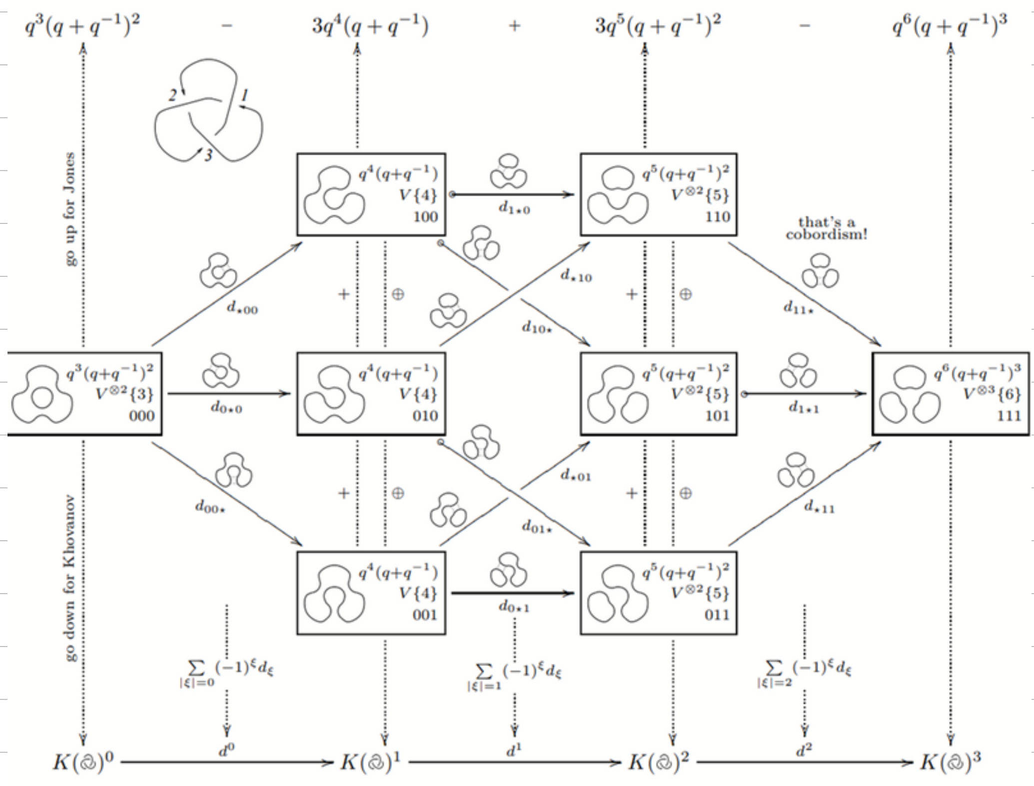
$V = \operatorname{span}\langle v_+, v_-\rangle$; $\deg v_\pm = \pm 1$; $q\dim V = q + q^{-1}$;

$K(\bigcirc^k) = V^{\otimes k}$; $K(\overcrossing) = \text{Flatten}\left(0 \to \underset{\text{height } 0}{K()(\{1\}}} \to \underset{\text{height } 1}{K(\asymp)\{2\}} \to 0\right)$;

$K(\undercrossing) = \text{Flatten}\left(0 \to \underset{\text{height } -1}{K(\asymp)\{-2\}} \to \underset{\text{height } 0}{K()(\{-1\}}} \to 0\right)$;



Example: $= q + q^3 + q^5 - q^9$.



(here $(-1)^\xi := (-1)^{\sum_{i<j}\xi_i}$ if $\xi_j = \star$) $= K(\text{trefoil})$.

Need:

$(\bigcirc\bigcirc \longrightarrow \text{merged}) \longrightarrow (V \otimes V \xrightarrow{m} V)$

$(\text{merged} \longrightarrow \bigcirc\bigcirc) \longrightarrow (V \xrightarrow{\Delta} V \otimes V)$

Such that:

1. $\deg m = \deg \Delta = -1$.
2.

Aside: Hopf Algebras: $(a+b)\cdot c = ac + bc$

Algebra: A a v.s. w/ $m : V \otimes V \longrightarrow V$

$$\begin{tikzcd} a\otimes b\otimes c & (ab)\otimes c \\ V\otimes V\otimes V \arrow[r, "m\otimes 1"] \arrow[d, "1\otimes m"'] \arrow[dr, phantom, "\circlearrowright"] & V\otimes V \arrow[d, "m"] \\ V\otimes V \arrow[r, "m"] & V \\ a\otimes (bc) \arrow[r] & a(bc) = (ab)\cdot c \end{tikzcd}$$

"Associativity"

$$\begin{tikzcd} a\otimes b \arrow[rr] & & b\otimes a \\ V\otimes V \arrow[rr, "\sigma"] \arrow[dr, "m"'] & \circlearrowright & V\otimes V \arrow[dl, "m"] \\ & V & \end{tikzcd}$$

"Commutative law"

Co-algebra C w/ $\Delta : C \to C\otimes C$

s.t.

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowright"] & C\otimes C \arrow[d, "\Delta\otimes 1"] \\ C\otimes C \arrow[r, "1\otimes\Delta"] & C\otimes C\otimes C \end{tikzcd}$$

"Co-associativity"

Co-commutative: if

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C\otimes C \arrow[rr, "\sigma"] & & C\otimes C \end{tikzcd}$$

commutes.

Claim If A is a F.d. algebra, then $C = A^*$ is a co-algebra. $m : A\otimes A \to A$

Indeed $\Delta = m^* : A^* \to A^*\otimes A^*$

$C \to C\otimes C$

Hopf Algebra A v.s. which is both an algebra & a co-alg at the same time (has m, Δ; satisfying both assoc. & co-assoc)

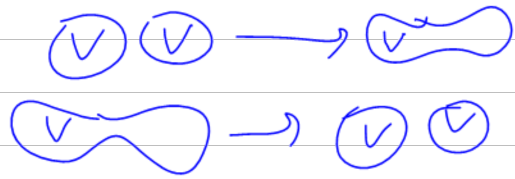
[Forgetting axioms related to units & "inverses"]

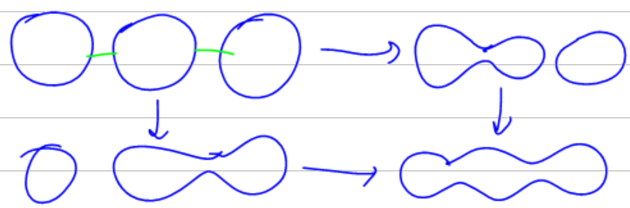
and s.t. $(a\otimes b)(c\otimes d) = (ac\otimes bd)$

$\Delta : A \to A\otimes A$ is a morphism of algebras.

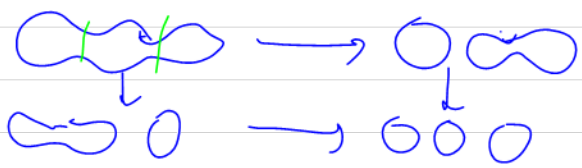
$m: V\otimes V \to V$ s.t.
$\Delta: V \to V \otimes V$

1. $\deg m = \deg \Delta = -1$
2. m is commutative.
3. Δ is co-commutative
4. m is associative
5. Δ co-associative

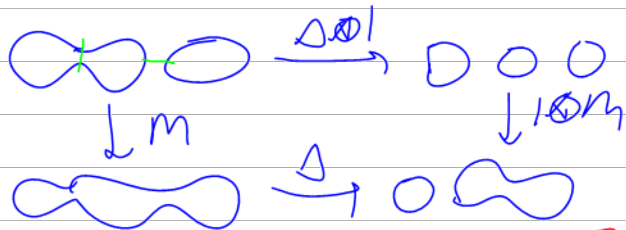




$$\begin{array}{ccc} V\otimes V\otimes V & \xrightarrow{m\otimes 1} & V\otimes V \\ \downarrow{\scriptstyle 1\otimes m} & \circlearrowright & \downarrow{\scriptstyle m} \\ V\otimes V & \xrightarrow[m]{} & V \end{array}$$



$$\begin{array}{ccc} V & \to & V\otimes V \\ \downarrow & & \downarrow \\ V\otimes V & \to & V\otimes V\otimes V \end{array}$$



6. $\Delta\otimes 1 /\!/ 1\otimes m = m /\!/ \Delta$

2 → 2+7   1

$m:$
$V_+\otimes V_+ \longrightarrow V_+$
$V_+\otimes V_- \longrightarrow V_-$
$V_-\otimes V_- \longrightarrow 0$

$\Delta: V_+ \longrightarrow (V_+\otimes V_- + V_-\otimes V_+)$
$V_- \longrightarrow (V_-\otimes V_-)$

W1: $n>0$, $a_i>0$, $\sum a_i = b$ known to W1 & W2
$\prod a_i = a$ age of W1,
W2: If I knew n & a I'd know a_i
W1: No you couldn't
W2: I know a; what's b?

Hour 10, Wednesday September 30: Khovanov homology following http://drorbn.net/mo13

HW3 on web
HW2 due tonight

The Jones polynomial: $\bigcirc^k \mapsto (q+q^{-1})^k$

$J : \overcrossing \mapsto q)(- q^2 \asymp, \quad J : \undercrossing \mapsto -q^{-2} \asymp + q^{-1})($

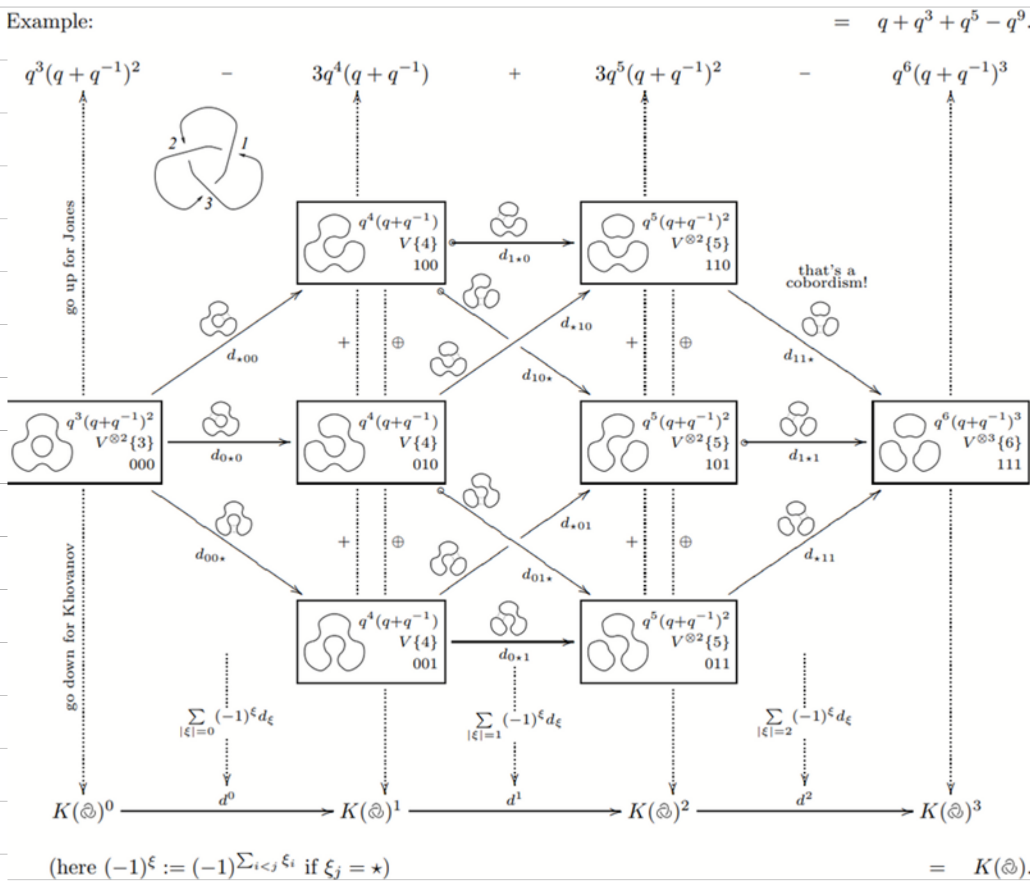
Today: The most disturbing open problem about Khovanov homology.

Khovanov: $K(L)$ is a chain complex of graded $\mathbb{Z}$-modules;
$V = \operatorname{span}\langle v_+, v_-\rangle$; $\deg v_\pm = \pm 1$; $q\dim V = q+q^{-1}$;

$$K(\bigcirc^k) = V^{\otimes k};\quad K(\overcrossing) = \text{Flatten}\left(0 \to \underset{\text{height } 0}{K()(\{1\}} \to \underset{\text{height } 1}{K(\asymp)\{2\}} \to 0\right);$$

$$K(\undercrossing) = \text{Flatten}\left(0 \to \underset{\text{height } -1}{K(\asymp)\{-2\}} \to \underset{\text{height } 0}{K()(\{-1\}} \to 0\right);$$

Example:



Need:

$(\bigcirc\bigcirc \longrightarrow \bigcirc\!\!\bigcirc) \longrightarrow (V\otimes V \xrightarrow{m} V)$

$(\bigcirc\!\!\bigcirc \longrightarrow \bigcirc\bigcirc) \longrightarrow (V \xrightarrow{\Delta} V\otimes V)$

$$m : \begin{cases} v_+\otimes v_- \mapsto v_- & v_+\otimes v_+ \mapsto v_+ \\ v_-\otimes v_+ \mapsto v_- & v_-\otimes v_- \mapsto 0 \end{cases}$$

$$\Delta : \begin{cases} v_+ \mapsto v_+\otimes v_- + v_-\otimes v_+ \\ v_- \mapsto v_-\otimes v_- \end{cases}$$

Such that:
1. $\deg m = \deg \Delta = -1$.
2. m is commutative and associative
3. Δ is co-commutative and co-associative.
4. A funny "Frobenius compatibility" of m & Δ holds.

1. $v_+(v_- v_+) \overset{?}{=} (v_+ v_-) v_+$ ✓

2. $v_+ \xrightarrow{\Delta} v_+v_- + v_-v_+ \xrightarrow{1\otimes\Delta} v_+v_-v_- + v_-(v_+v_- + v_-v_+)$ ✓
$\xrightarrow{\Delta\otimes 1}$

Claim $(V, m) \cong \mathbb{Z}[x]/x^2=0$ $\quad 1 \to v_+$, $x \to v_-$

Thm Suppose $\mathcal{C}' \subset \mathcal{C}$ is an inclusion $)\sim)($ of complexes

$$\begin{array}{lccccccc}
\mathcal{C}' & & C'^{r-1} & \to & C'^{r} & \to & C'^{r+1} & \\
& & \downarrow & & \downarrow & & \downarrow & \\
\mathcal{C} & \cdots \to & C^{r-1} & \to & C^{r} & \to & C^{r+1} & \to \cdots \\
\mathcal{C}/\mathcal{C}' & & & \to & C^r/C'^r & \to & &
\end{array}$$

A. If $H(\mathcal{C}')=0$ then $H(\mathcal{C}/\mathcal{C}')=H(\mathcal{C})$
"$\mathcal{C}'$ is acyclic"

B If $H(\mathcal{C}/\mathcal{C}')=0$ then $H(\mathcal{C})=H(\mathcal{C}')$

Pf Whenever you have a short exact seq. of cmplxs:

$$0 \to \mathcal{C}' \to \mathcal{C} \to \mathcal{C}/\mathcal{C}' \to 0$$

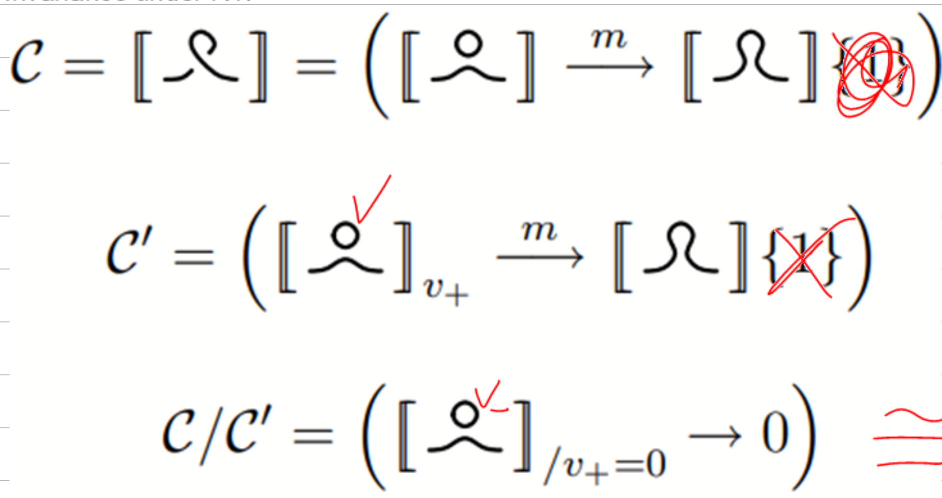
there is a corresponding long exact seq of H's:

$$\begin{array}{l}
\to \cancel{H^{r-1}(\mathcal{C}/\mathcal{C}')} \to \\
\to \cancel{H^{r}(\mathcal{C}')} \to H^r(\mathcal{C}) \to \cancel{H^r(\mathcal{C}/\mathcal{C}')} \to \\
\to \cancel{H^{r+1}(\mathcal{C}')} \to
\end{array}$$

A

B

Invariance under R1:



v_+ v_- v $v \otimes v \otimes v$

Invariance under R2:

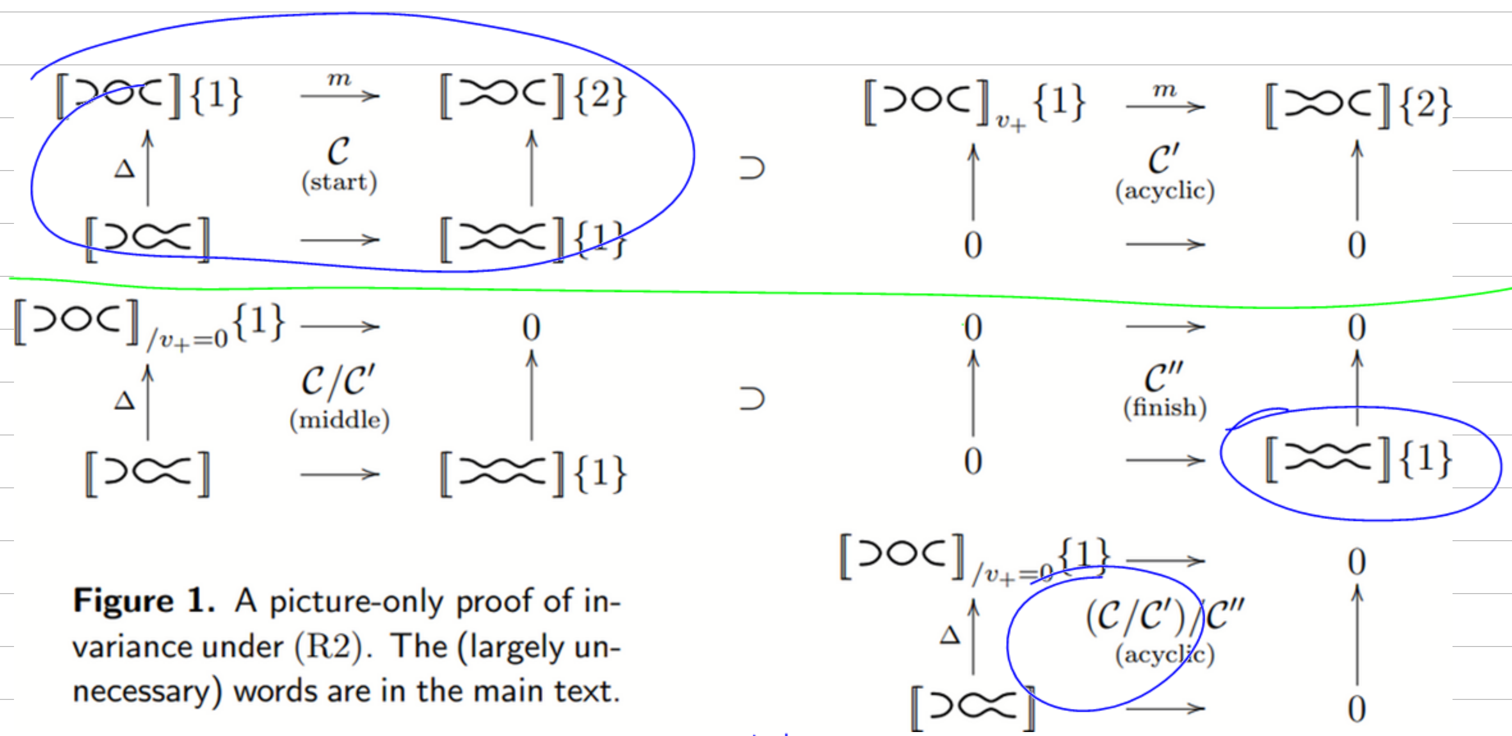


Figure 1. A picture-only proof of invariance under (R2). The (largely unnecessary) words are in the main text.

$\Delta: v_- \to v_- \otimes v_-$ $\xrightarrow{\text{mod by } v_+=0 \text{ on left}}$ $\to v_- \otimes v_-$

$v_+ \to v_- v_+ + v_+ v_-$ $\to v_- \otimes v_+$

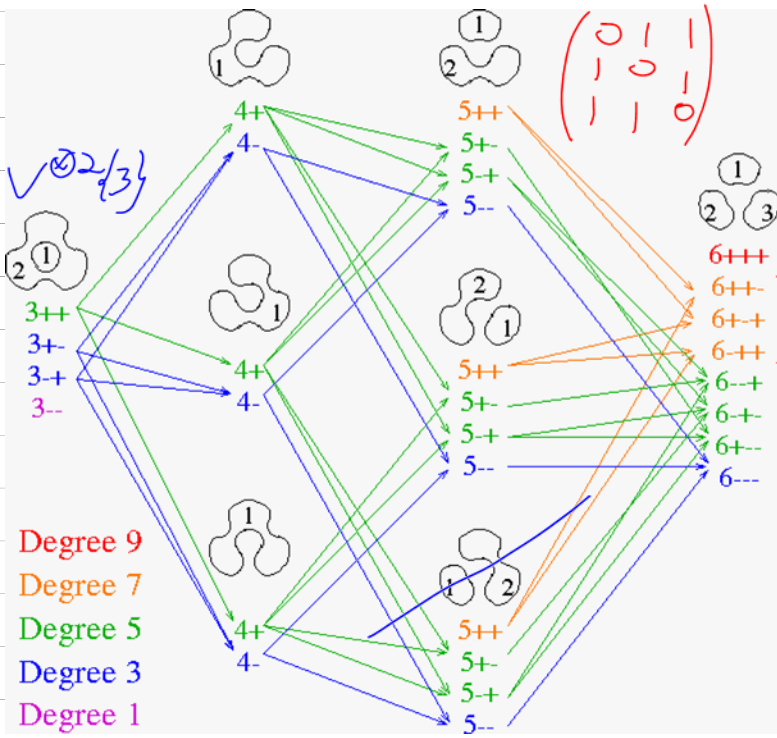
Invariance under R3?
http://www.math.toronto.edu/~drorbn/papers/Categorification/Categorification.pdf or wait a bit.

Thm $Kh(K) = V \Rightarrow K = \bigcirc$

Pf ~~wrongest~~ ever (morally)
Way too complicated.

Hour 11, Friday October 2: Last class on "plain" Khovanov homology.

Open problems: 1. 3D interpretation?
2. Why $Kh(K)=Kh(O) \Rightarrow K=O$?



$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

```
In[1]:= << KnotTheory`
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.

In[2]:= Kh[PD@Mirror[Knot[3, 1]]][q, t]
••• KnotTheory: Loading precomputed data in PD4Knots`.
••• KnotTheory: The Khovanov homology program JavaKh-v2 is an update of
    Jeremy Green's program JavaKh-v1, written by Scott Morrison in
    2008 at Microsoft Station Q.

Out[2]= q + q^3 + q^5 t^2 + q^9 t^3
```

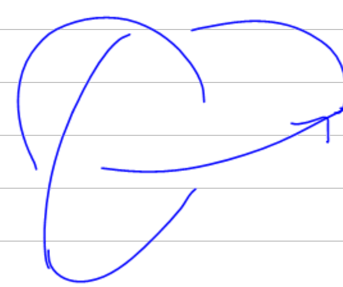
$Kh = \sum t^r q \dim H^r$

So coeff of $t^r q^d$ is $\dim (H^r)_d$

$Kh/_{t \to -1} = J$

Parts of homological alg make sense even if ker & im don't 0 | Polynomials on {knots} or sl_2 V_2

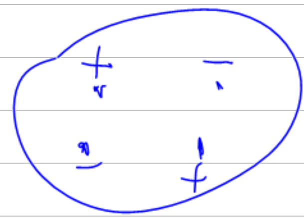
Let V (for Vassiliev) be an inv. of oriented knots in oriented $\mathbb{R}^3/S^3$. V can be extended to 1-singular knots via:



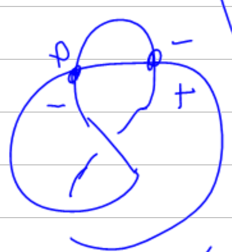
$V^{(1)}$ is "first derivative" of V

$$V^{(m)}(\underbrace{\times\times\cdots\times}_{m}) = V^{(m-1)}(\nearrow\times\cdots\times) - V^{(m-1)}(\nwarrow\times\cdots\times)$$

"m-singular knots"

 + analysis

$$V(\times\times) = V(\overset{+}{\nearrow}\overset{+}{\nearrow}) - V(\overset{+}{\nearrow}\overset{-}{\nwarrow}) - V(\overset{-}{\nwarrow}\overset{+}{\nearrow}) + V(\overset{-}{\nwarrow}\overset{-}{\nwarrow})$$



$$V^{(m)}(\overbrace{\underbrace{\times\cdots\times}_{m}}^{K}) = \sum_{s\in\{-1,+1\}^m} \left(\prod s_i\right) V(K_s)$$

<u>Def</u> V is poly of deg m,

if $V^{(m+1)} \equiv 0$

$$V : \left\{ \begin{array}{l} \text{oriented knots} \\ \text{in oriented } \mathbb{R}^3 \end{array} \right\} \longrightarrow A \quad \left(\begin{array}{c} \text{an Abelian} \\ \text{group} \end{array} \right)$$

$$V^{(m)}(\underbrace{\times \cdots \times}_{m}) := V^{(m-1)}(\nearrow\!\!\!\!\nwarrow \times \cdots \times) - V^{(m-1)}(\nwarrow\!\!\!\!\nearrow \times \cdots \times)$$

"V of type m" means $V^{(m+1)} = 0 = V(\underbrace{\times \cdots \times}_{>m})$

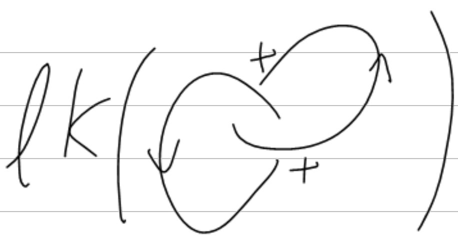
<u>Example 0</u> $V \overset{(0)}{\equiv} C$ of type 0:

$$V^{(1)} = V^{(0)}(\nearrow\!\!\!\!\nwarrow) - V^{(0)}(\nwarrow\!\!\!\!\nearrow) = C - C = 0$$

<u>Example 1</u> If L is a 2-component link,

$$lk(L) = \frac{1}{2} \sum_{\substack{x : \text{ xings} \\ \text{between} \\ \text{the two} \\ \text{components}}} (-1)^x$$

linking number of K

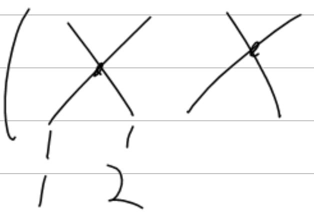


$$= \frac{1}{2}(1+1) = 1$$

invt under $\widetilde{R1}$ $\overbrace{R2 \quad R3}^{\text{already checked}}$

$R1$ doesn't count

$$lk(\times) = 0 \qquad lk(\times) = 1$$

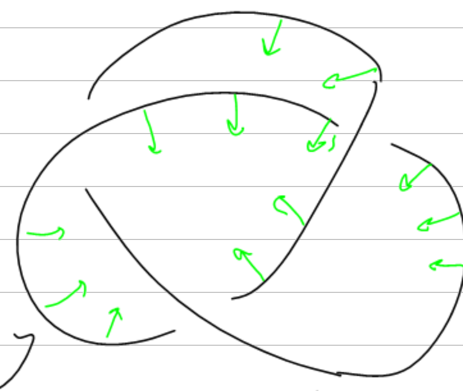
$lk\Big($  $\Big) = \begin{matrix} 0 - 0 \\ 1 - 1 \end{matrix} = 0$

$\Rightarrow lk$ is of type 1.

If L is a link w/ n components, labeled $1, \dots, n$, let l_{ij} be the linking number of comp $\#i$ w/ comp $\#j$:

$$l_{ij} = \frac{1}{2} \sum_{\substack{x:\ \text{xing} \\ \text{between } i \\ \& \ j}} (-1)^{x}$$

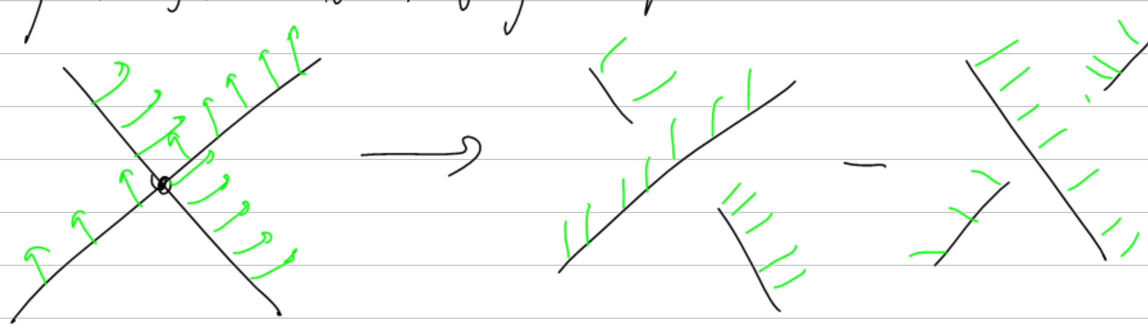
l_{ij} is of type 1.



<u>Example 3</u> Framed Knot:

A knot w/ an up-to-homotopy choice of a $\neq 0$ normal at every pt.

A framed knot differs from an unframed one by just one integer parameter.



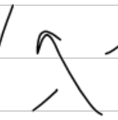
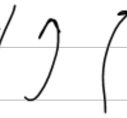
<u>Def</u> If K is a framed knot,

$$\underset{\text{self linking}}{sl(K)} = lk(K, K^+)$$

K^+ ← the push of K in the dir of the framing.

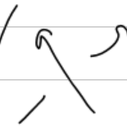
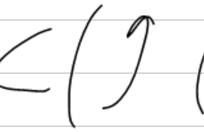
<u>Exercise</u> sl is of type 1.

<u>Exercise</u> What's the relationship between $sl(K)$ & $w(K)$

$q^{-1} J($  $) - q J($  $) = (q^{1/2} - q^{-1/2}) J($  $)$

The Conway polynomial:

$$C : \{\text{knots \& links}\} \longrightarrow \mathbb{Z}[z]$$

$C($  $) - C($  $) = z \cdot C($  $)$

$C($  $)$ $=$

$$C(O^k) = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

<u>Thm</u> This gives a well-defined invts.

<u>Thm</u> Write $C(L)(z) = \sum_{m=0}^{\infty} V_m(L) z^m$

$V_m(L)$: invts in $\mathbb{Z}$.

Then V_m is of type m.

$$C(\underbrace{\times \cdots \times}_{m+1}) = z\,C(\uparrow\uparrow \times \cdots \times)$$

$$\vdots$$

$$= z^{m+1} C(\uparrow\uparrow \cdots \uparrow\uparrow)$$

So $V_m(\underbrace{\times \cdots \times}_{m+1})$ = coeff of z^m in $C(\underbrace{\times \cdots \times}_{m+1}) = 0$

<u>Example 5</u>

$$q^{-1}J(\times_+) - qJ(\times_-) = (q^{1/2}-q^{-1/2})J(\uparrow\uparrow)$$

$$J(L)(q=e^x) = \sum_{m=0}^{\infty} V_m(L)\,x^m$$

$V_m(L)$ is of type m.

$$(1-x+\cdots)J(\times_+) - (1+x+\cdots)J(\times_-) = (x+\cdots)J(\uparrow\uparrow)$$

$$J(\times) = J(\times_+) - J(\times_-) = \begin{matrix}(x+\cdots)J(\times_+)\\ (x+\cdots)J(\times_-)\\ (x+\cdots)J(\uparrow\uparrow)\end{matrix} = x\,(\text{Junk})$$

$$J(\underbrace{\times \cdots \times}_{m+1}) = x^{m+1}(\text{Junk})^{m+1} \qquad \square$$

$q = (1 + x + \cdots\cdots$

$q^{-1} = (1 - x + \cdots\cdots \quad)$

Hour 13, Wednesday October 7: The basics of finite type invariants.

HW 2 will not be marked.
HW3 due at midnight (extensions will be considered)
HW 4 on web soon.

$$V \in \mathcal{V}_m \iff V^{(m+1)} = 0 \iff V(\underbrace{\times \cdots \times}_{>m}) \equiv 0$$

$\mathcal{V}_m$: v.s. of type m invts

A: v.s. or $\mathbb{Z}$-module $\supset \mathcal{V}_{m-1}$

$$V^{(m+1)} \equiv 0 \implies V^{(m)} \text{ is "constant"}$$

"independent of the embedding into the ambient space $\mathbb{R}^3$"

$V \in \mathcal{V}_m \implies$

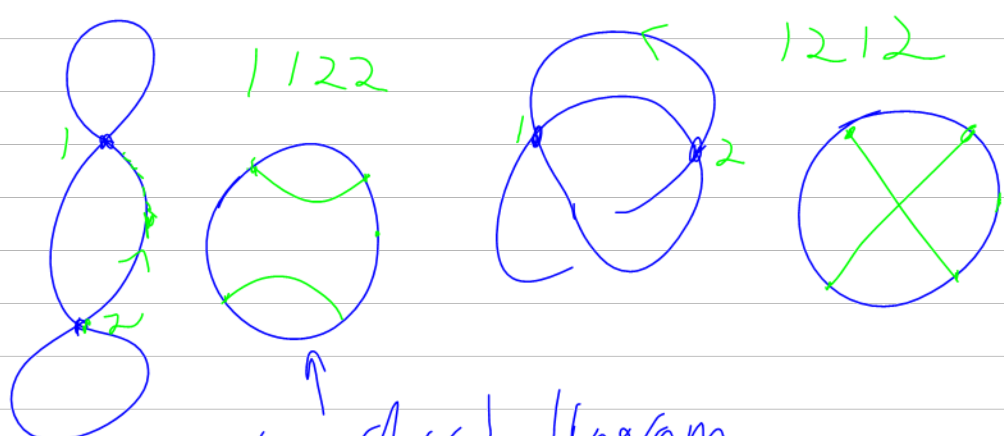
$$V(\underbrace{\times \cdots \times}_{m} \diagup\!\!\!\diagdown) - V(\times \cdots \times \diagdown\!\!\!\diagup) = V(\underbrace{\times \cdots \times}_{m} \times) = 0$$

So

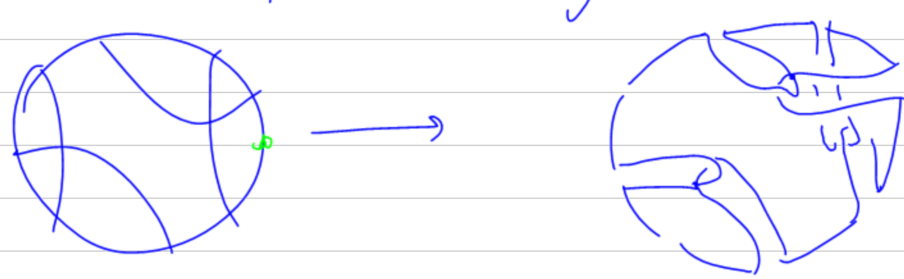
$$V^{(m)} : \{m\text{-singular knots}\} \big/ \diagup\!\!\!\diagdown = \diagdown\!\!\!\diagup \longrightarrow A$$

$\xrightarrow[\text{ex.}]{1\text{-}1?}$ {m chord diagrams}

$m=2$



m-chord diagram



$W_V = V^{(m)} : \{m\text{-chord diagrams}\} \longrightarrow A$

"weight system of V"

$W_V : \mathcal{D}_m \longrightarrow A$ $\qquad$ $\mathcal{D}_m = A\langle m\text{-chord diagrams}\rangle$

Cor V_m is an A-module of finite rank.

Pf $\qquad V_{m-1} \longrightarrow V_m \xrightarrow{v \mapsto W_v} \mathcal{D}_m^*$

V_{m-1}: finite rank by induction $\qquad$ $\mathcal{D}_m^*$: finite rank $\qquad$ onto?

$(V_0 = \{\text{const}\})$

$v_1 \quad v_2$

$W_{v_1} = W_{v_2}$

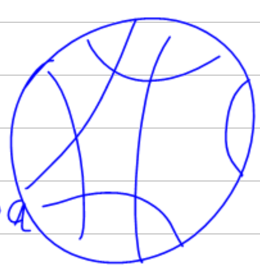
$v_1^{(m)} = v_2^{(m)}$

$(v_1 - v_2)^{(m)} = 0$

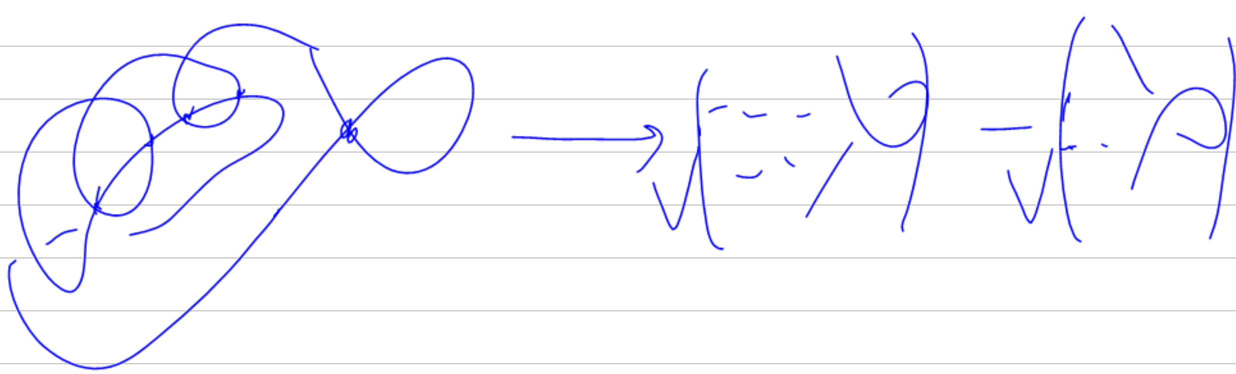
$v_1 - v_2 \in V_{m-1}$

W_V satisfies two relation:

FI: Framing Independence

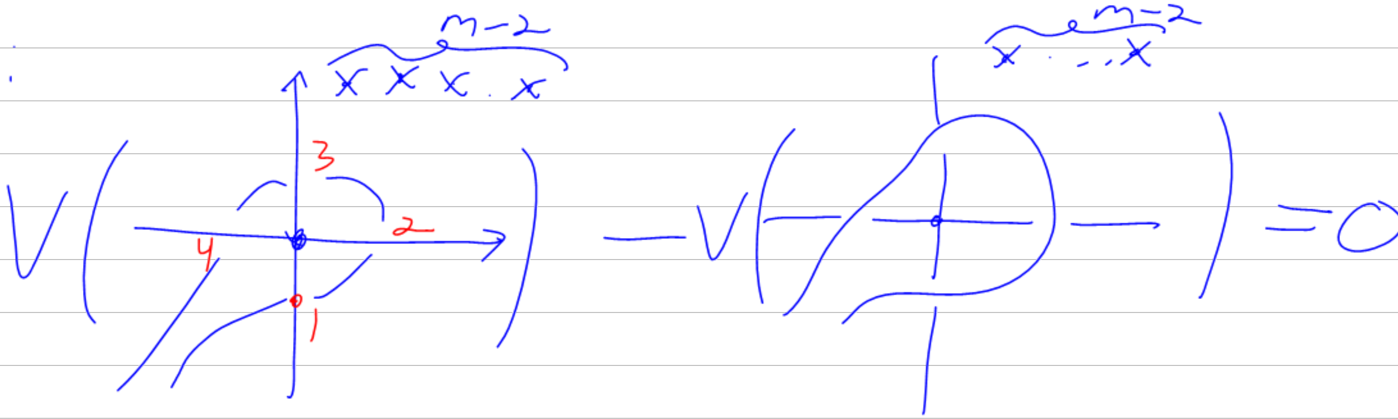


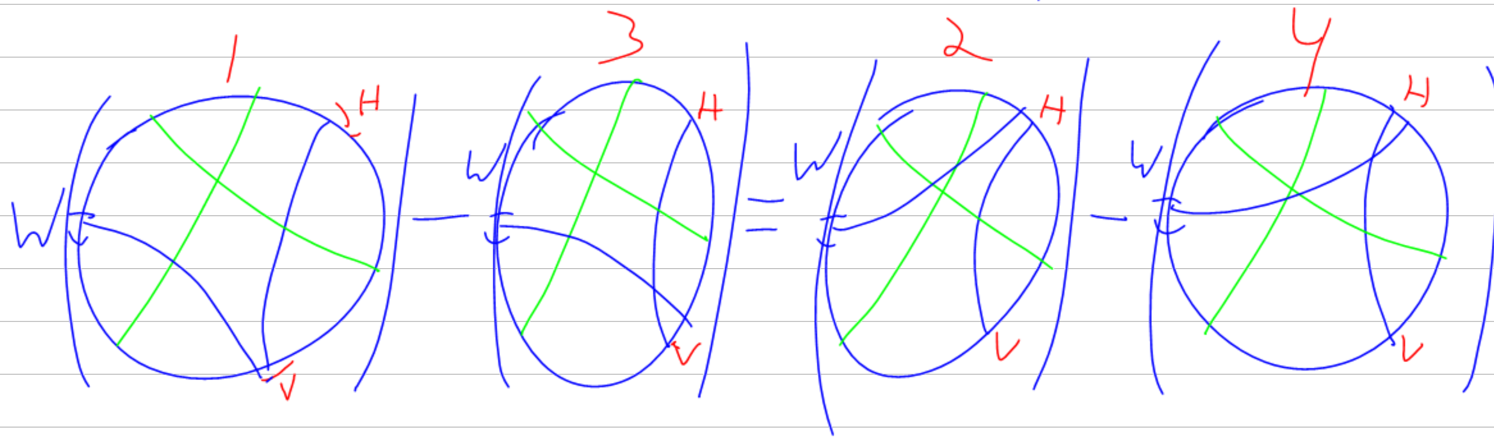
$\xrightarrow{W_V} 0$



$= 0$

4T:





Thm "The Fundamental Thm of F.T. Invariants": Over $\mathbb{Q}$, this is all.

If $W \in \mathcal{D}_m^*$ satisfies FT & 4T, then $\exists V \in \mathcal{V}_m$ s.t. $W_V = W$.

over $\mathbb{Q}$: $$\mathcal{V}_m / \mathcal{V}_{m-1} \cong \left(\mathcal{D}_m \Big/ \substack{FI \\ 4T}\right)^*$$

No class on Monday!

The Fundamental Thm of FTI (Kontsevich+....): $\forall W \in \mathcal{W}_m := (\underset{\mathcal{A}^r_m}{\underbrace{\mathcal{D}_m/FI, 4T}})^*$

$\exists V \in \mathcal{V}_m$ s.t. $W = W_V \ (\sim V^{(m)})$.

m	0	1	2	3	4	5	6	7	8	9	10	11	12
dim $\mathcal{A}^r_m$	1	0	1	1	3	4	9	14	27	44	80	132	232
dim $\mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	184	316	548

$\mathcal{A}_m := \mathcal{D}_m/4T$

Def $\mathcal{A}^{(r)} = \bigoplus_{m=0}^{\infty} \mathcal{A}^{(r)}_m$ $\quad \boxed{\hat{\mathcal{A}}^{(r)} := \prod_{m=0}^{\infty} \mathcal{A}^{(r)}_m}$

$(\mathcal{A}_m)^* \sim$ invts of type m

$(\mathcal{A})^* \sim$ invts of all type invts $\sim (\mathcal{K})^*$ → Knots

$\mathcal{K}$ has a commutative product.

$\mathcal{K}(\bigcirc) \cong \mathcal{K}(\uparrow)$:

long knots

clearly has a product

$-\boxed{K_1}- \# -\boxed{K_2}- = -\boxed{K_1}-\boxed{K_2}\rightarrow$

$\quad\quad = -\boxed{K_2}-\boxed{K_1}\rightarrow$?

Commutative!

$\mathcal{K}^*$ also a commutative algebra

$\mathcal{A}_0 = \{\bigcirc\}$

dim $\mathcal{A}_0 = 1$ $\mathcal{A} = \bigoplus_{m=0}^{\infty} \mathcal{A}_m$ $m \geq 0$

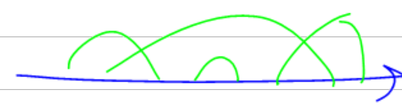
Thm $\mathcal{A}$ is a connected graded commutative & co-commutative bi-algebra:

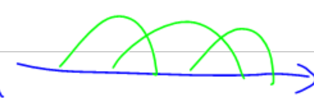
$m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

$$\begin{tikzcd} \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[r, "m\otimes 1"] \arrow[d, "1\otimes m"'] & \mathcal{A}\otimes\mathcal{A} \arrow[d, "m"] \\ \mathcal{A}\otimes\mathcal{A} \arrow[r, "m"] & \mathcal{A} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathcal{A}\otimes\mathcal{A} \arrow[d, "\Delta\otimes 1"] \\ \mathcal{A}\otimes\mathcal{A} \arrow[r, "1\otimes\Delta"] & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \end{tikzcd}$$

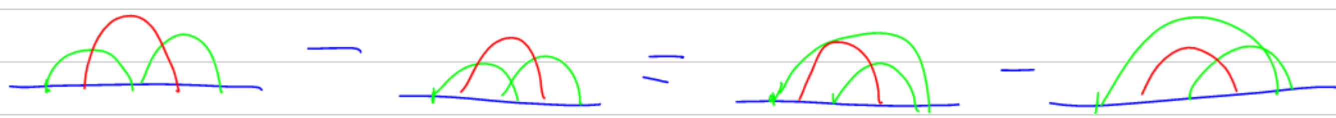
$$\begin{tikzcd} \mathcal{A}\otimes\mathcal{A} \arrow[rr, "\sigma"] \arrow[dr, "m"'] & & \mathcal{A}\otimes\mathcal{A} \arrow[dl, "m"] \\ & \mathcal{A} & \end{tikzcd}$$

Compatibility

$K(\bigcirc) \cong K(\uparrow)$ FI:  $= 0$

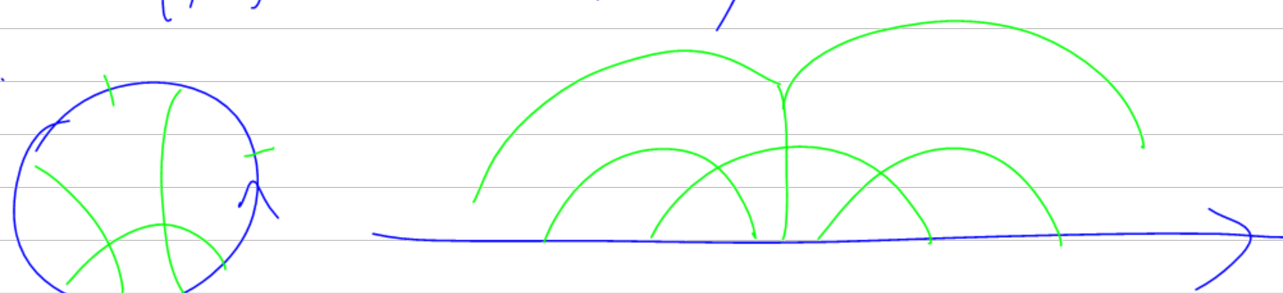
Thm $\mathcal{A}(\bigcirc) \cong \mathcal{A}(\uparrow) = \langle$  $\rangle / 4T$

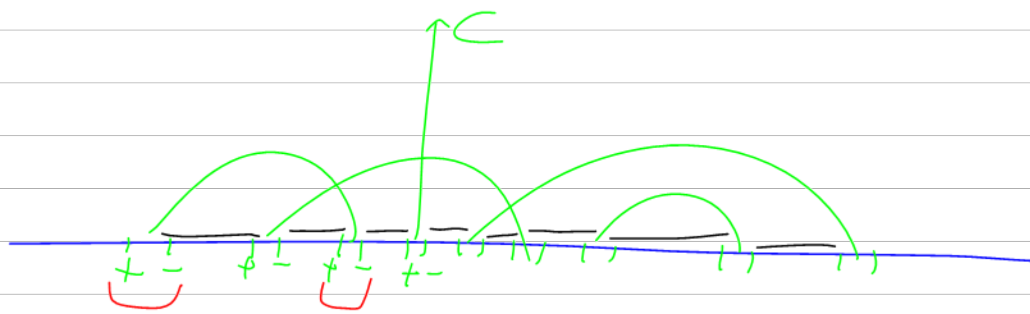
4T



Pf $\mathcal{A}(\uparrow) \longrightarrow \mathcal{A}(\bigcirc)$ obvious.

$\Leftarrow$:





S = sum of all ways of connecting C to the hooks, with signs $\in \mathcal{A}$

$=$ sum over chords (sum of ~~4 hooks~~ next to its ends) + 2 ~~further connections~~ $= 0$

$=$ sum over edges (~~sum over $+$ & $-$ hooks on on inc. edge~~) + cont from hooks at ends

clearly associative.

<u>Prop 1</u> If $V_1 \in \mathcal{V}_{m_1}$ & $V_2 \in \mathcal{V}_{m_2}$ then

$$V_1 \cdot V_2 \in \mathcal{V}_{m_1+m_2}$$

$\xrightarrow{f}\ \xrightarrow{g}$

$g \circ f$

$f /\!/ g := g \circ f$

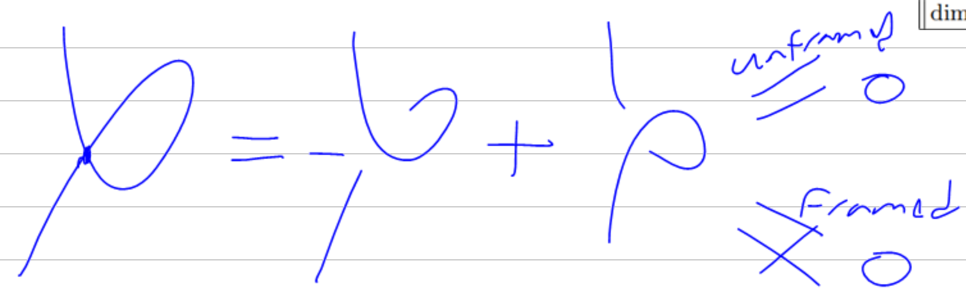
<u>Prop 2</u> $\exists!\ \square : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$

$$W_{V_1 \cdot V_2} = \square /\!/ W_{V_1} \cdot W_{V_2}$$

$$\mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \xrightarrow[W_{V_2}]{W_{V_1} \otimes} R \otimes R \longrightarrow K$$

Hour 15, Wednesday October 14: more on \calA
HW4 due by midnight. By then I hope HW5 will be on web and HW3 will be marked.

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}^r_m$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	184	316	548
$\dim \mathcal{P}_m$	0	1	1	1	2	3	5	8	12	18	27	39	55



$$C : \{\text{links}\} \longrightarrow \mathbb{Z}[z]$$

$$C_0\begin{pmatrix}k \text{ comp}\\ \text{link}\end{pmatrix} = \delta_{k1}$$

$$C(\underbrace{O \ldots O}_{k}) = \delta_{k1}$$

$$C(\times) = C(\times) - C(\times) = z\,C()()$$

$$C(K) = \sum_{m=0}^{\infty} c_m(K)\cdot z^m$$

is of type m.

$$W_{c_m} : \mathcal{A}_m \longrightarrow \mathbb{Z}$$

m-singular knot whose underlying c.d. is D.

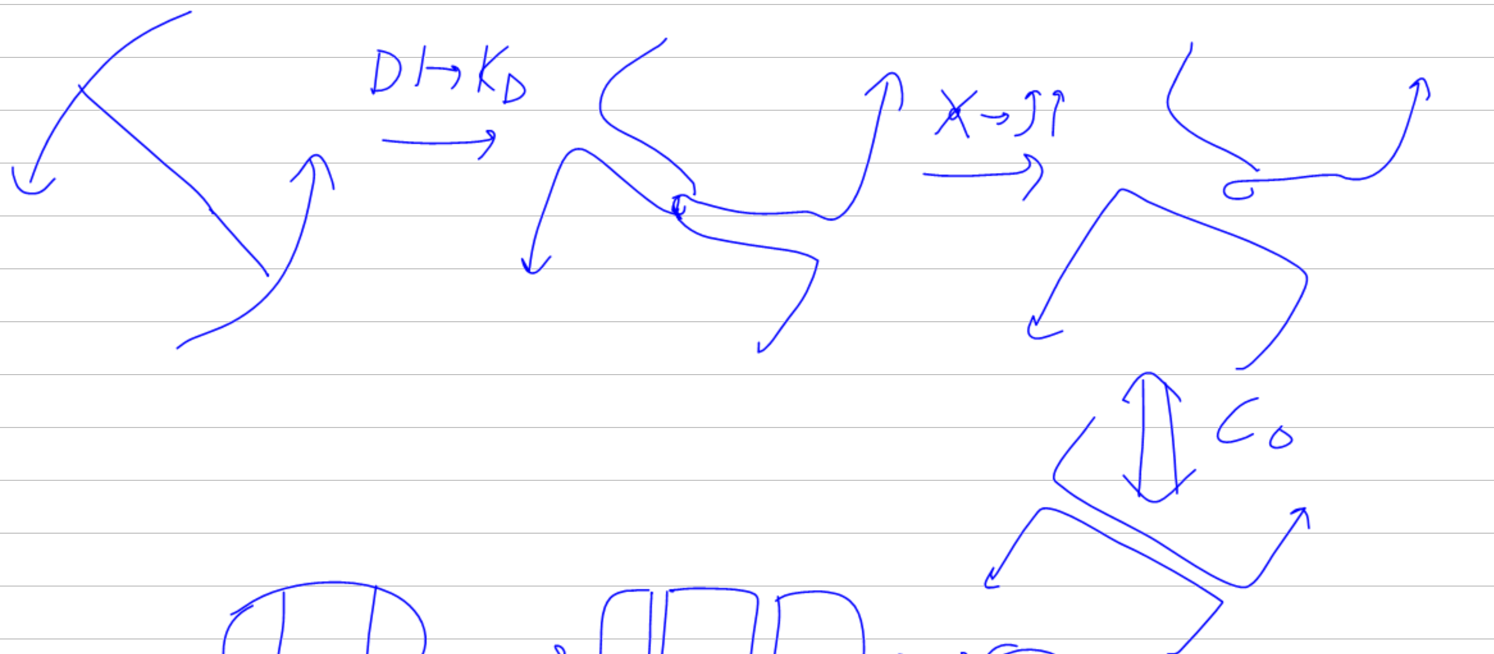
$$W_{c_m}(\underbrace{\bigoplus}_{m \text{ chords}} = D) = c_m(K_D)$$

$$= \mathrm{Coeff}_{z^m}\left(C\left(K_D / \asymp \to)(\right)\right)$$

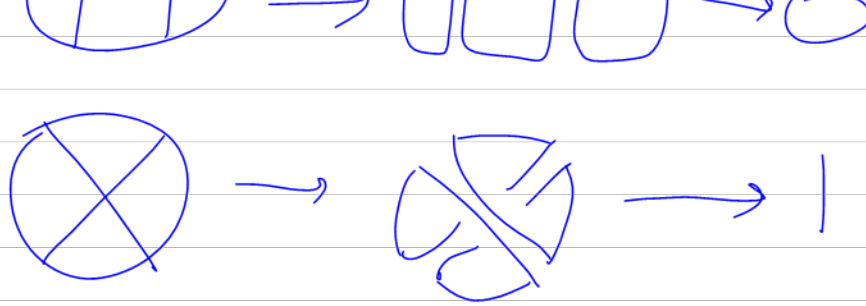
$$= \mathrm{Coeff}_{z^0}\left(C\left(K_D / \asymp \to)(\right)\right)$$

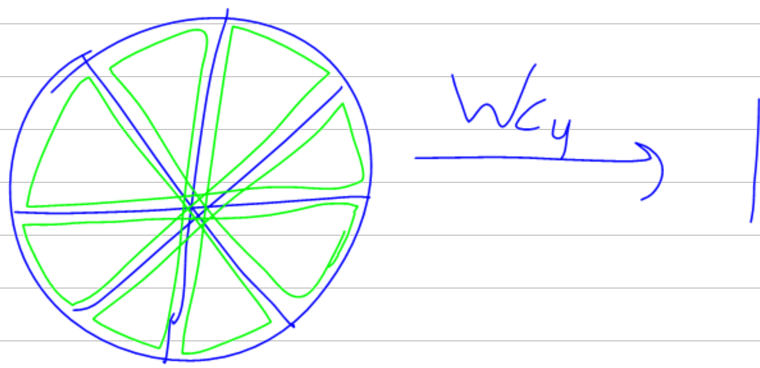
$$= c_0\left(K_D / \asymp \to)(\right)$$

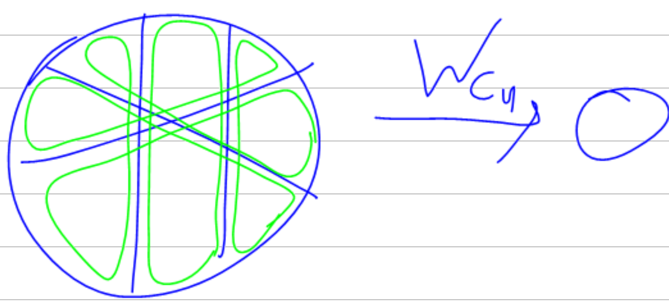
$$W_{C_n}(D) = \begin{cases} 1 & \text{if } K_D / X \to)(\text{ is connected.} \\ 0 & \text{otherwise} \end{cases}$$



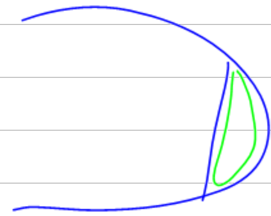
W_{C_2}:





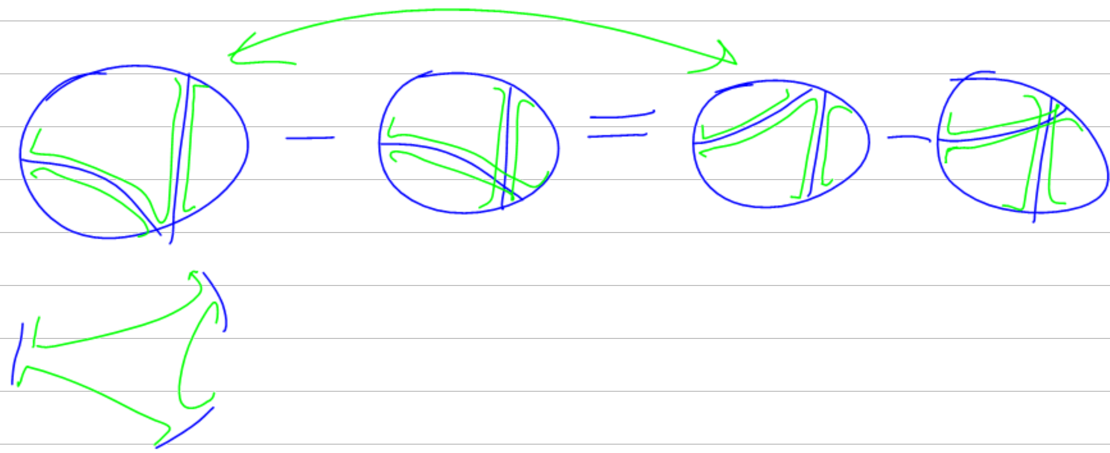


FI



$\to 0$

4T



Prop If $V_1 \in \mathcal{V}_{m_1}$ & $V_2 \in \mathcal{V}_{m_2} \Rightarrow V_1 \cdot V_2 \in \mathcal{V}_{m_1+m_2}$.

Prop $\exists! \; \square : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

$$W_{V_1 \cdot V_2} = \square // W_{V_1} \cdot W_{V_2}$$

$$\partial\partial\partial\partial\partial\partial\partial\partial(x^3 \cdot x^4) \qquad (fg)' = f'g + fg'$$

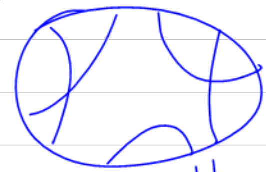
$\deg f = 2$ $\quad$ $\deg g = 3$

$$\partial\partial\partial\partial\partial(f \cdot g)$$

= sum of all ways of splitting 5 ∂ into 2 to left & 3 to rgt.

V_1 of type 2

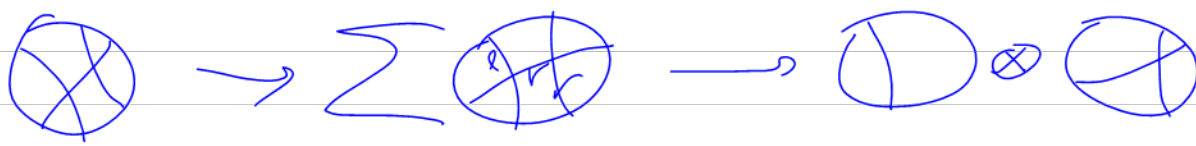
V_2 of type 3



$$W_{\underset{2+3}{V_1 \cdot V_2}}(D) = \sum_{\substack{\text{splitting of } D \\ \text{into "left" } D_l \\ \text{\& "right" } D_r}} W_{V_1}(D_L) \cdot W_{V_2}(D_R)$$

$\| \, D$

$$\square : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$



$$W_{V_1 \cdot V_2}(D) = \square \,/\!/\, W_{V_1} \cdot W_{V_2}$$

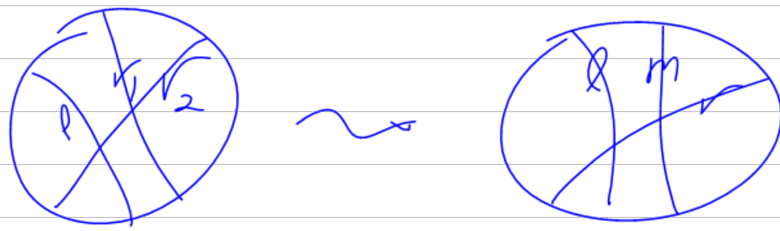
<u>Claims</u> 1. This is right.

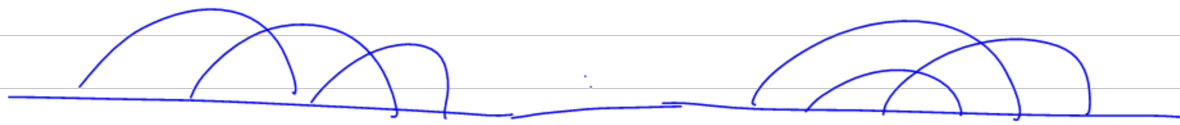
2. $\square : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ descends mod 4T: $\square : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$.

<u>Thm</u> $(\mathcal{A}, m, \square, \bar{\epsilon}, \bar{\eta})$ is
co-unit unit

a connected graded commutative co-commutative bi-algebra.

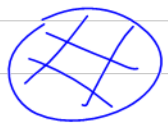
$$\square \mathbin{/\!/} 1 \otimes \square = \square \mathbin{/\!/} \square \otimes 1$$





m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}^r_m$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	184	316	548
$\dim \mathcal{P}_m$	0	1	1	1	2	3	5	8	12	18	27	39	55

Thm $(\mathcal{A}, m, \Box, \eta, \epsilon)$ is a connected graded commutative co-commutative bialgebra; $\dim \mathcal{A}_m < \infty$.

$\mathcal{A} = \{$  $\}/4T \xleftarrow{Z} \sim K$

1·1·1·1 3·1
1·1·2
2·2

Unit: $1 \in A$: $1a = a \cdot 1 = a$

alt: $\eta : \mathbb{Q} \longrightarrow A$ "$\alpha \longmapsto \alpha \cdot 1$"

$$\begin{tikzcd}
\alpha \otimes a & \mathbb{Q} \otimes A \arrow[r, "\eta \otimes I"] \arrow[dr] & A \otimes A \arrow[d, "m"] & \alpha 1 \otimes a \\
& \alpha a & A & \alpha 1 a
\end{tikzcd}$$

Co-unit: $\epsilon : A \longrightarrow \mathbb{Q}$ (in a co-algebra)

$$\begin{tikzcd}
& A \arrow[dl, "\Delta"'] \arrow[dr, "\sim"] & \\
A \otimes A \arrow[rr, "\epsilon \otimes I"] & & \mathbb{Q} \otimes A
\end{tikzcd}$$

circle, not zero

In $\mathcal{A}$, $\eta : \mathbb{Q} \longrightarrow \mathcal{A}$ is $\alpha \longmapsto \alpha \cdot \bigcirc$

ϵ:

$$\epsilon(D) = \begin{cases} 1 & \text{if } D = \bigcirc \\ 0 & \text{if } \deg D \geq 1 \end{cases}$$

Milnor-Moore Thm (60s) (co-commutative version)

If A is ......, then A is a graded polynomial algebra ....

$\mathbb{Q}[P_1\ P_2\ P_3 \ldots\ldots]$ $\deg P_i \geq 1$

$\mathbb{Q}[x,y]$ $\deg x = 1$, $\deg y = 2$ $\deg xy^2 = 5$

$\Box P_i = 1 \otimes P_i + P_i \otimes 1$

m, ϵ

$\Box(xy^2) = \Box(x) \cdot (\Box(y))^2$

$= (x\otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y)^2 = \ldots$

If P is a graded vector space,

$S(P) = \langle p_1 \otimes p_2 \otimes \ldots \otimes p_n \rangle_{p_i \in P} \Big/ \forall \sigma \in S_n \ \ p_1 \otimes \ldots \otimes p_n = p_{\sigma 1} \otimes \ldots \otimes p_{\sigma n}$

If $p \in P$ decree that $\Box(p) = p \otimes 1 + 1 \otimes p$.

$S(\text{v.s.}\langle p_1 \ldots p_n \ldots \rangle) = \mathbb{Q}[p_1 \ldots p_n]$.

MM: $A \underset{\substack{\uparrow \\ \text{as a bialgebra}}}{\cong} S(P(A))$

$P(A) : \{ p \in A : \Box p = p \otimes 1 + 1 \otimes p \}$

$\mathbb{Q}[x, y]$ $\qquad \deg x = 1$, $\deg y = 2$

$y + 0.01 \cdot x^2$

$\mathcal{A} = \langle \ominus, \otimes - ⦶, \ldots \rangle$

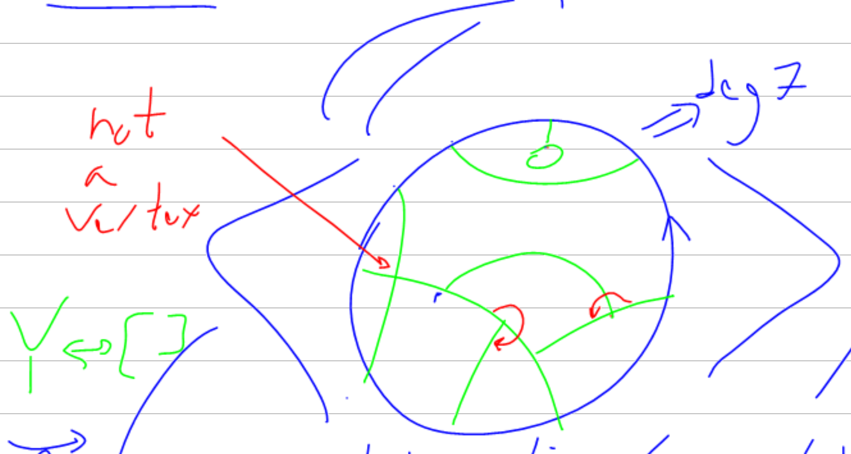
$\mathcal{A}^r = \mathcal{A} / \langle \ominus \rangle$ $\qquad) = 0$

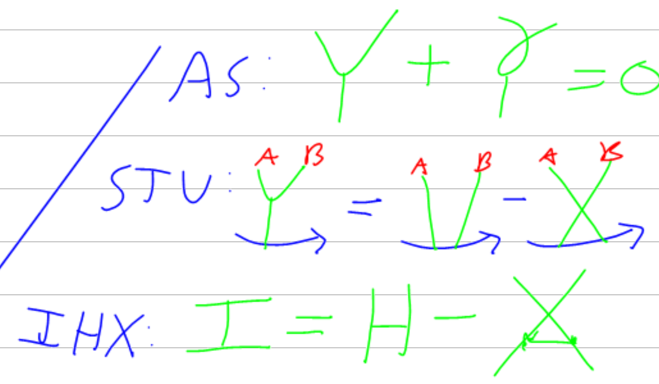
$$\mathcal{A} \xrightarrow{proj} \mathcal{A}^r \xrightarrow{HWI} \mathcal{A}$$

$\mathcal{A}^r = \mathbb{Q}\langle \otimes - ⦶, \ldots \rangle$

$\mathcal{A} \otimes \begin{pmatrix} \text{Lie Algebras} \\ \text{\& reps} \end{pmatrix} \longrightarrow \mathbb{Q}$

<u>Thm</u> $\mathcal{A} \cong \mathcal{A}^t$ $\longrightarrow$ trivalent temporary





oriented vertices / connected / degree $= \frac{1}{2}$(total # of verts.)

mult.

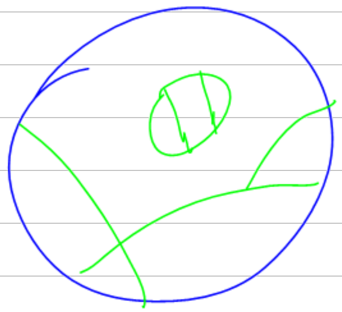
$$L \quad [\,,\,] : L \otimes L \to L \qquad [A,B] = AB - BA$$

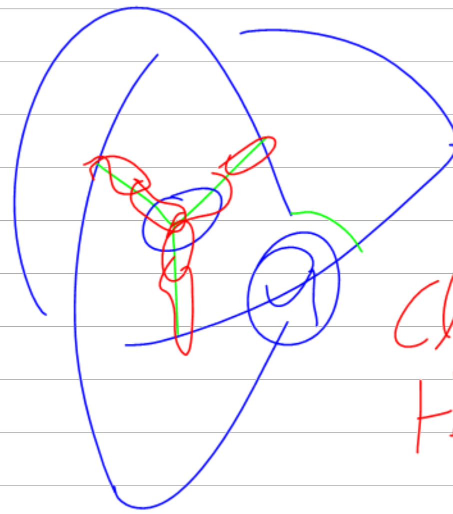
$$[a,b] = -[b,a]$$

$$0 = [a,[b,c]] + [b,[c,a]] + [c,[a,b]]$$

"Jacobi-identity"

Not connected:

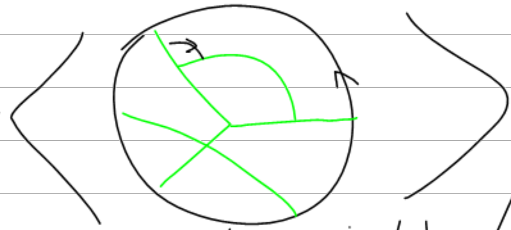




Claspers
Habiro.

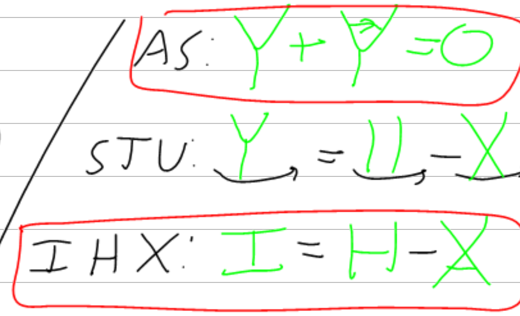
Hour 17, Monday October 19: \calA^t and Lie Algebras

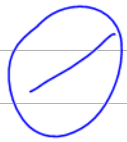
Thm $\mathcal{A} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathcal{A}^t := $



$\mathcal{A} = \mathcal{D}/4T$

* connected, oriented vertices, $\deg = \frac{1}{2}(\text{verts})$



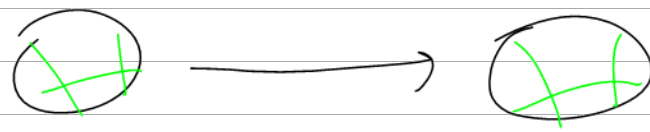
$\theta =$ 

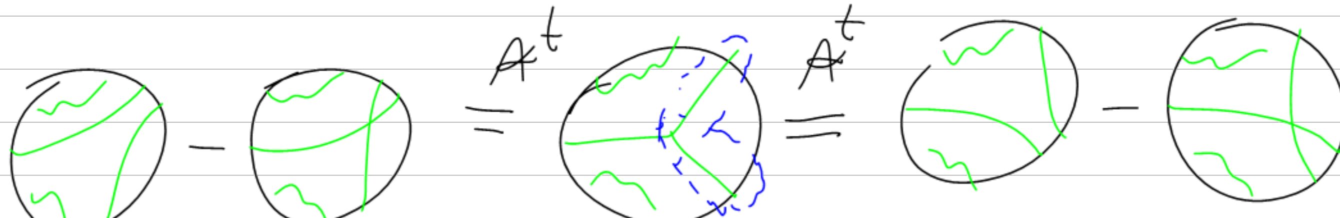
$\dim \mathcal{A}_m = \dim \mathcal{A}^r_m + \dim \mathcal{A}_{m-1}$

Polys in $\theta, x, y \ldots$ — Polys w/ no θ — Poly of deg 1 less in $\theta, x, y, \ldots$

$\mathcal{A} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathcal{A}^t$

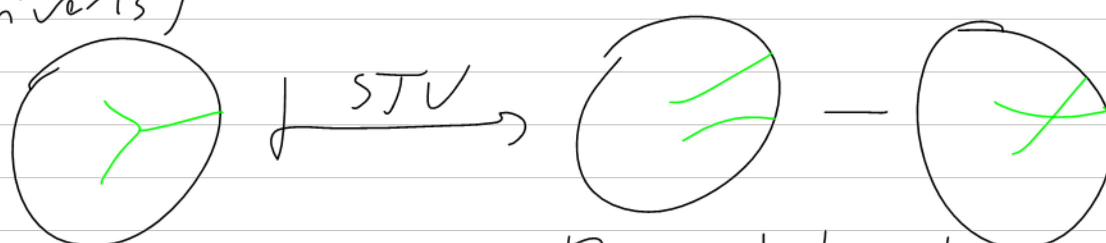
ϕ:



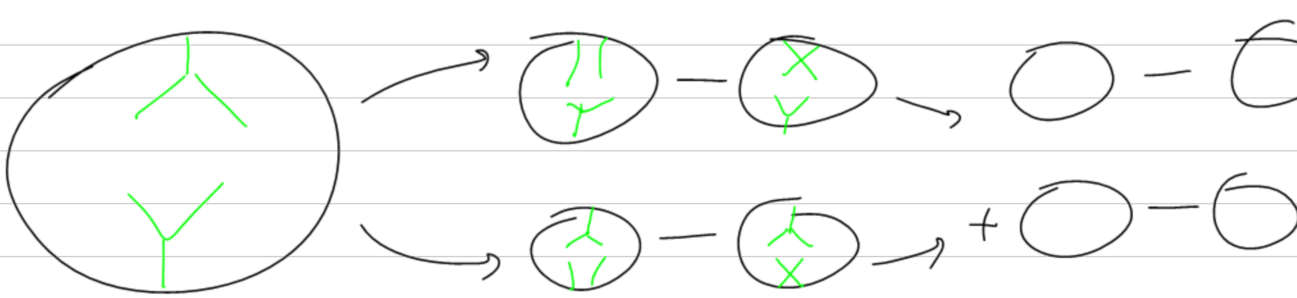


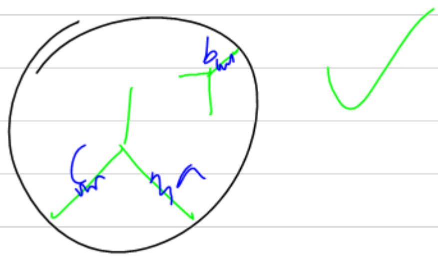
ψ(digs w/ no green verts) = itself. So ϕ is well def.

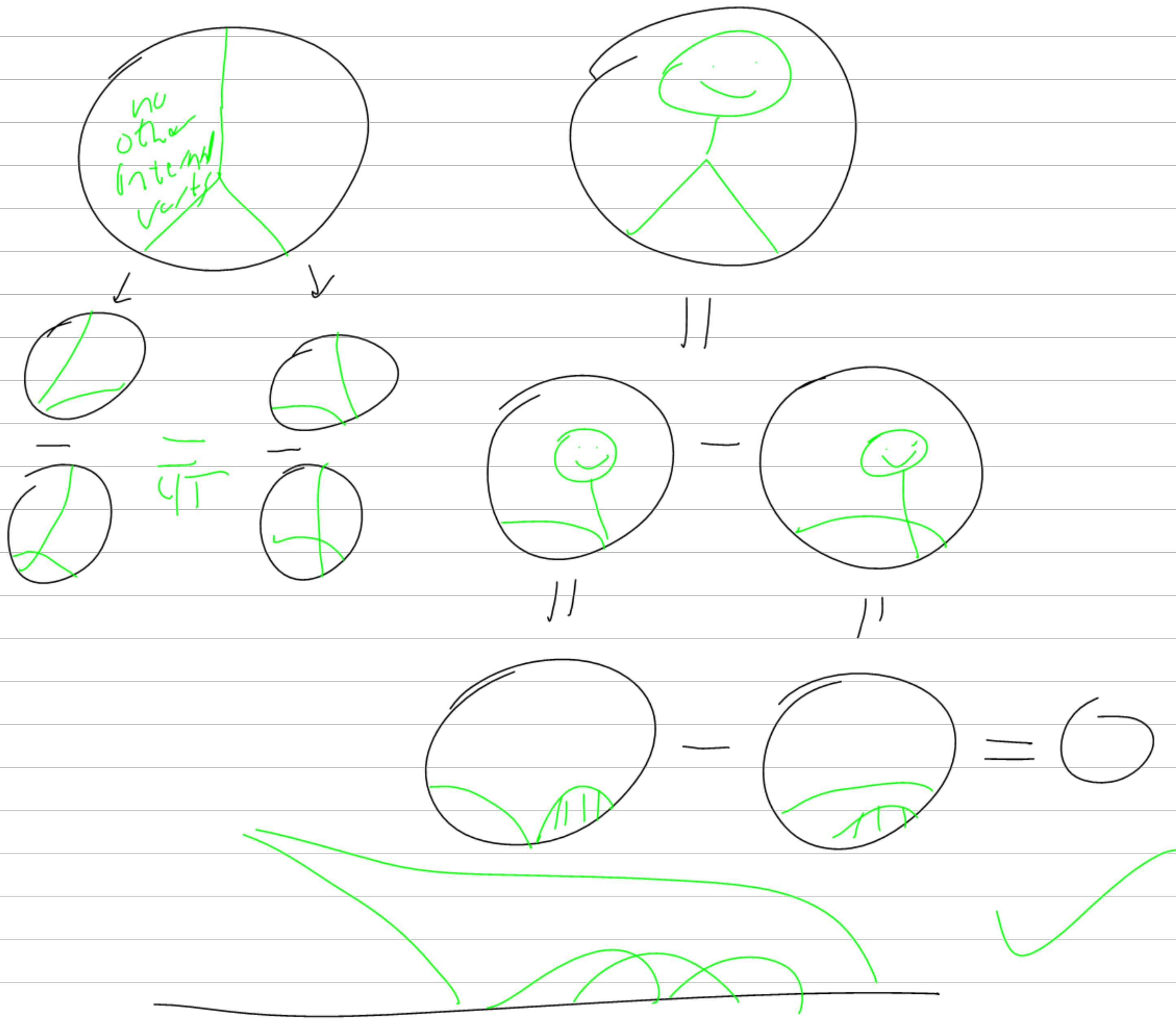
otherwise ψ:



Then iterate...



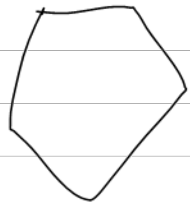




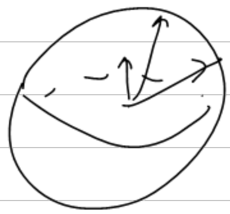
<u>Def</u> A Lie Alg is a V.S. L along with a bilinear map $[\,,\,]: L^{\otimes 2} \to L$ s.t.

1. $[x,y]+[y,x]=0$
2. $0=[x,[y,z]]+[y,[z,x]]+[z,[x,y]]$

↑ Jacobi identity



$\longrightarrow D_{10}$



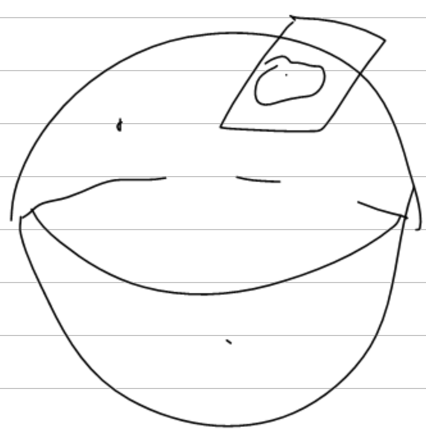
$S^2 \longrightarrow O(3) = \begin{pmatrix} | & | & | \end{pmatrix} = A$

$\{A : A^T A = I\}$

$SO(n) = \left\{ A \in M_{n\times n}(\mathbb{R}) : \begin{array}{l} A^T A = I \\ \det A = 1 \end{array} \right\}$

$GL(n) = \{$all invertible $n \times n$ matrices$\}$

$SL(n) = \{A \in GL(n) : \det(A) = 1\}$



Enough to understand

$T_1 G = L$

↑ Lie algebra of the group.

A metric on L is a non-degenerate symmetric, invariant, bilinear form on L:

$$\langle\, , \rangle : L \otimes L \longrightarrow \mathbb{Q}$$

s.t.

1. $\langle x, y\rangle = \langle y, x\rangle$
2. $\langle [z, x], y\rangle + \langle x, [z, y]\rangle = 0$
3. non-deg: if $x \neq 0$ then $\exists\, y$ s.t. $\langle x, y\rangle \neq 0$.

Examples

$$gl(n) = \{A : A \in M_{n\times n}\}$$

$$[A, B] = AB - BA.$$

$$[A, [B, C]] = [A, BC - CB] =$$

$$= \cancel{ABC} - \cancel{ACB} - \cancel{BCA} + \cancel{CBA}$$

$$\dots = \cancel{BCA} - BAC - CAB + \cancel{ACB}$$

$$\dots = CAB - \cancel{CBA} - \cancel{ABC} + BAC$$

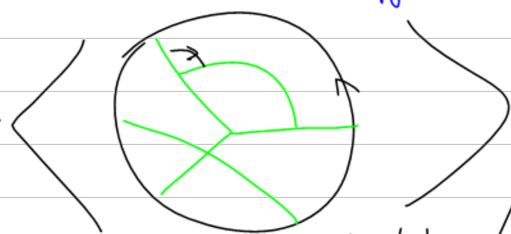
$$\langle A, B\rangle := tr\, A\cdot B = tr\, B\cdot A = \langle B, A\rangle$$

$$\langle [C, A], B\rangle + \langle A, [C, B]\rangle \overset{?}{=} 0$$

$$\mathrm{tr}\left(\cancel{CAB} - \cancel{ACB}\right) + \mathrm{tr}\left(\cancel{ACB} - \cancel{ABC}\right)$$

$$\forall A \neq 0 \ \exists B \text{ s.t. } \langle A, B \rangle \neq 0$$

Hour 18, Wednesday October 21: Lie Algebraic Weight Systems, gl(N).
By midnight: HW5 due, HW6 assigned.

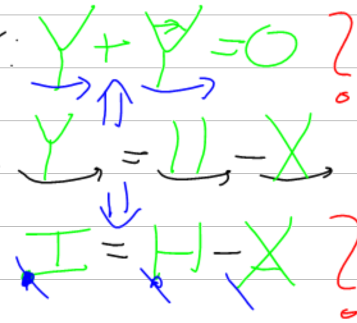
Thm $\mathcal{A} \cong \mathcal{A}^t := \Big\langle$  $\Big\rangle \Big/$ AS, STU, IHX

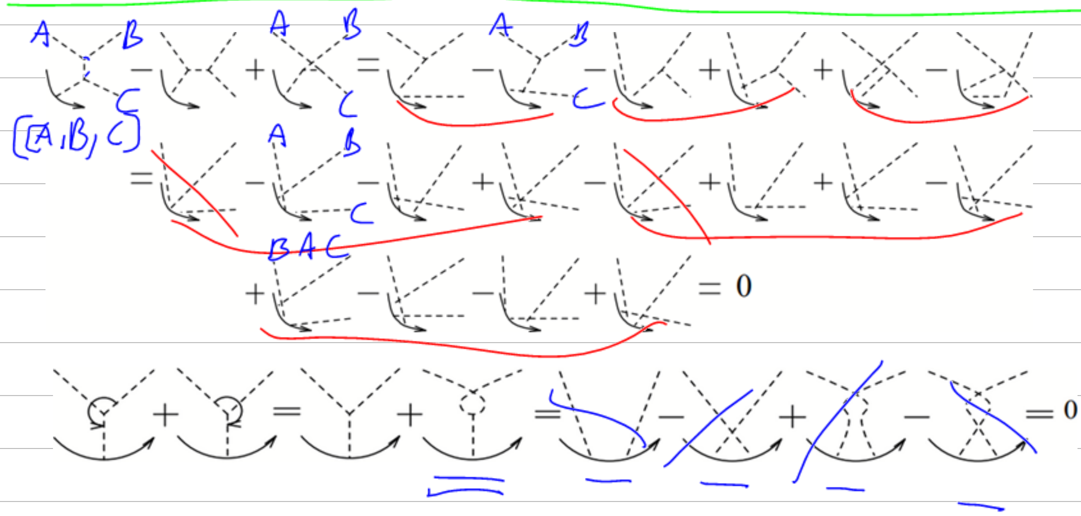
$\mathcal{A} = \mathcal{D}/4T$

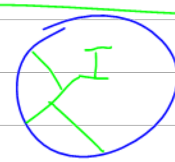
δ_{ij}

* connected, oriented vertices, $\deg = \frac{1}{2}(\text{vertices})$

AS: STU: IHX:







Reminders

$\mathfrak{g}$: a metrized Lie algebra

$\langle\, , \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{Q}$ sym, non-deg, inv.

$$\langle [Z,X], Y\rangle + \langle X, [Z,Y]\rangle = 0$$

R: representation of $\mathfrak{g}$

$\rho : \mathfrak{g} \xrightarrow[\text{of Lie alg.}]{\text{morphism}} \mathrm{End}(R) \sim \mathfrak{gl}_{\dim(R)}$

v.s.

$$\rho : \mathfrak{g} \to M_{n\times n}(\mathbb{Q})$$

$$\rho([X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$$

Construction/
Thm Given a f.d. Lie alg $\mathfrak{g}$,
and a f.d. representation thereof R
$\exists\ W_{\mathfrak{g},R} : \mathcal{A} \longrightarrow \mathbb{Q}$

[A F.T. invariant of knots for each m.]

Construction:

$$\mathfrak{g} = \langle X_a \rangle_{a=1}^{\dim \mathfrak{g}} \qquad R = \langle e_\alpha \rangle_{\alpha=1}^{\dim R}$$

$$[X_a, X_b] = \sum \underbrace{f_{ab}^{c}}_{AS} X_c = f_{ab}^{c} X_c$$

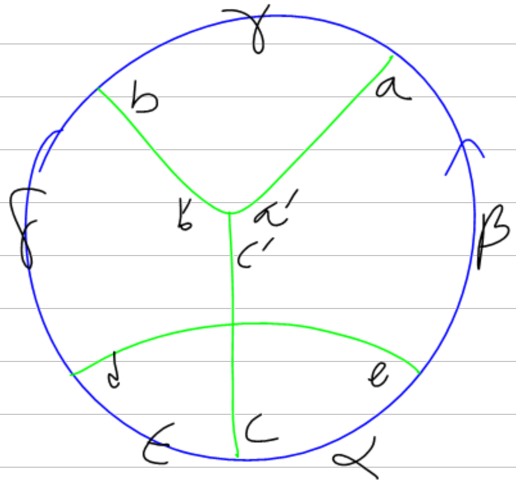
"the structure constants"

$$\langle X_a, X_b \rangle = t_{ab} \ \text{sym} \qquad t_{ab} t^{bc} = \delta_a^c \ \text{sym}$$

$$f_{abc} = \langle [X_a, X_b], X_c \rangle = f_{ab}^{d} t_{dc}$$

totally AS

$$X_a e_\alpha = \rho(X_a) e_\alpha = r_{a\alpha}^{\beta} e_\beta$$



$$\longrightarrow \sum_{\substack{a'b'c' \\ abcde \\ \alpha\beta\gamma\delta\epsilon}} F_{a'b'c'}\, t^{aa'}\, t^{bb'}\, t^{cc'}\, t^{ed}\; r^{\gamma}_{a\beta}\, r^{\delta}_{b\gamma}\, r^{\epsilon}_{d\delta}\, r^{\alpha}_{c\epsilon}\, r^{\beta}_{e\alpha}$$

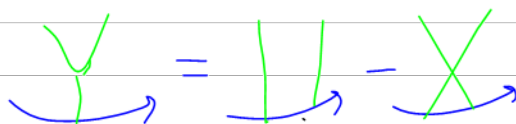
MNN $\quad \bar{M}\bar{N}$ $\quad \in \mathbb{Q}$

$F_{\dots} \in g^* \otimes g^* \otimes g^* \qquad t^{\cdot\cdot} \in g \otimes g \qquad r \in g^* \otimes R^* \otimes R$

AS STU IHX

$\widetilde{AS}$

AS.



Jacobi Id.

$\rho([X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$

"representations represent" $\square$

F_{abc}

t^{ab}

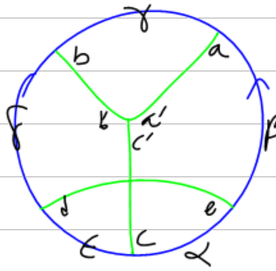
$r^{\beta}_{a\alpha}$

$\longrightarrow \int\int\int\int$

Hour 19, Friday October 23: gl(N) and maybe also sl(2).

$\mathfrak{g} = \langle X_a \rangle_{a=1}^{\dim \mathfrak{g}}$ $\quad R = \langle e_\alpha \rangle_{\alpha=1}^{\dim R}$

$F_{abc} = \langle [X_a, X_b], X_c \rangle$ $\quad \langle X_a, X_b \rangle = t_{ab}$ $\quad t_{ab} t^{bc} = \delta_a^c$ $\quad X_a e_\alpha = \sum r_{a\alpha}^{\beta} e_\beta$



$$W_{\mathfrak{g},R}: \longrightarrow \sum_{\substack{a'b'c' \\ abcde \\ \alpha\beta\gamma\delta\epsilon}} f_{a'b'c'} t^{aa'} t^{bb'} t^{cc'} t^{ed} \cdot r_{a\beta}^{\gamma} r_{b\gamma}^{\delta} r_{d\delta}^{\epsilon} r_{c\epsilon}^{\alpha} r_{e\alpha}^{\beta} \in \mathbb{Q}$$

AS IHX
STU

$\mathfrak{gl}_N$ w/ defining rep.

$$X_{ij} = \begin{array}{c} \\ i \end{array}\begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix}_{j} \qquad a \longleftrightarrow (ij)$$

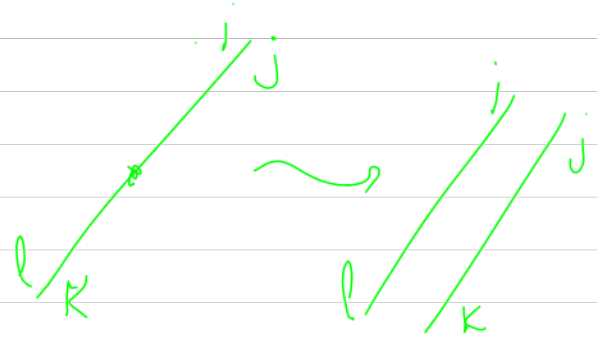
$$X_{ij} X_{kl} = \delta_{jk} X_{il}$$

$$[X_{ij}, X_{kl}] = \delta_{jk} X_{il} - \delta_{il} X_{kj} \qquad (AB)_{ik} = \sum A_{ij} B_{jk}$$

$$t_{(ij)(kl)} = tr(\underbrace{X_{ij} X_{kl}}_{\delta_{jk} X_{il}}) = \delta_{jk} \delta_{il}$$

$$t^{(ij)(kl)} = \delta_{jk} \delta_{il} \qquad t^{(ij)(kl)} t_{(kl)(mn)} = \sum_{k,l} \delta_{jk} \delta_{li} \delta_{lm} \delta_{kn} = \delta_{jn} \delta_{im}$$

$\updownarrow$



$$f_{abc} = f_{(ij)(kl)(mn)} = \langle [X_{ij}, X_{kl}], X_{mn} \rangle$$

$$= \langle \delta_{jk} X_{il} - \delta_{il} X_{kj}, X_{mn} \rangle$$

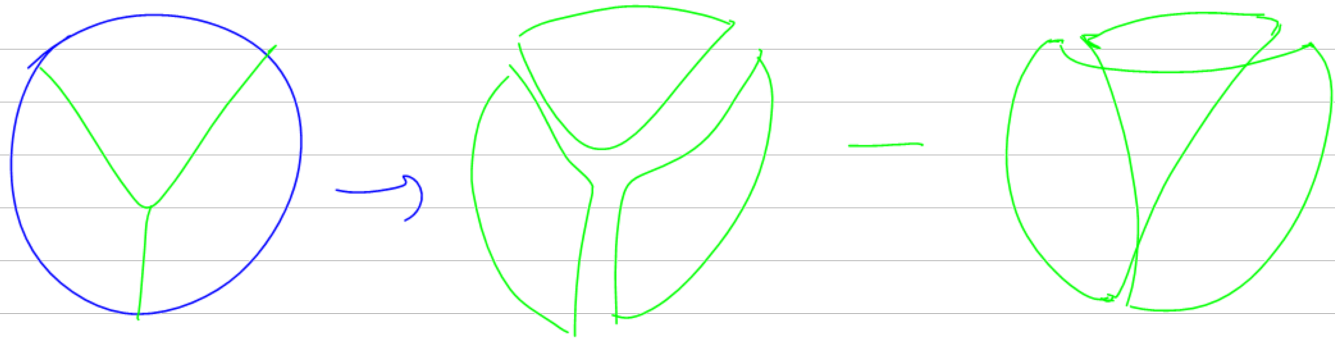
$$= \delta_{jk}\delta_{in}\delta_{lm} - \delta_{il}\delta_{kn}\delta_{jm}$$

$$\{e_\alpha\} \xrightarrow{R} = \langle e_i \rangle \qquad e_i = i\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

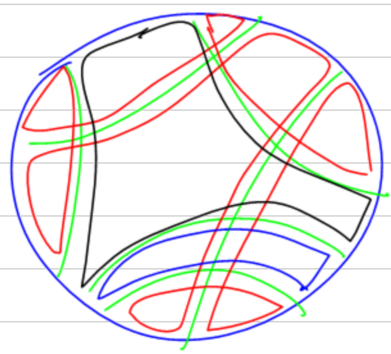
$$X_a e_k = X_{ij} e_k = \delta_{jk} e_i = \sum_l \delta_{kl}\delta_{il} e_l$$

$$V_{(ij)k}^{\ l} = \delta_{jk}\delta_{il}$$

$$N^2 - N^2 = 0$$



$$= N^3 - N$$



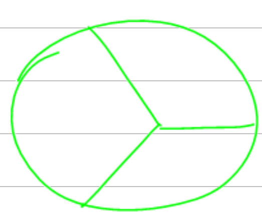
$$\longrightarrow N^3$$

$$sl_2 \cong so_3 = \langle X_1, X_2, X_3 \rangle$$

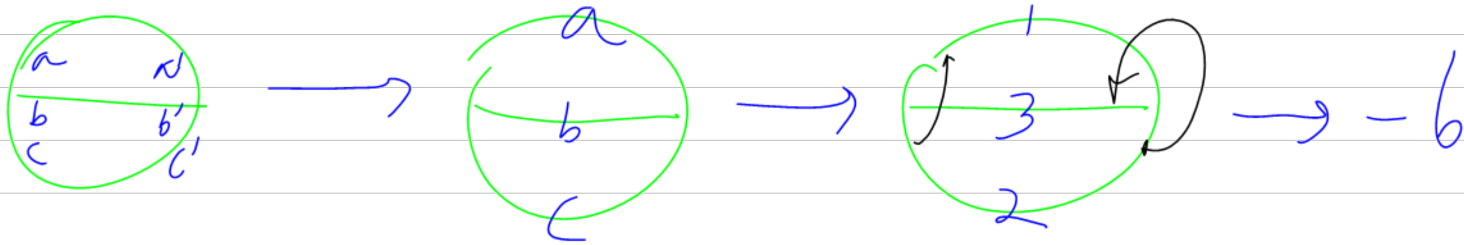
$$X_1 \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad X_2 \to \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad X_3 \to \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$f_{abc} = \epsilon_{abc} = \begin{cases} (-1)^{abc} & \text{if } [abc] \text{ is a permutation} \\ 0 & \text{otherwise} \end{cases}$$

$$t^{ab} \sim \delta_{ab}$$



$$\begin{array}{ccc} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ {[a} & b & c] \end{array}$$



<u>Thm</u> If D is green-only, and planar, all contributions come w/ same sign.

$W_{sl(2)} = W_{so(3)}(D) = \pm \#$ edge 3-colorings of D

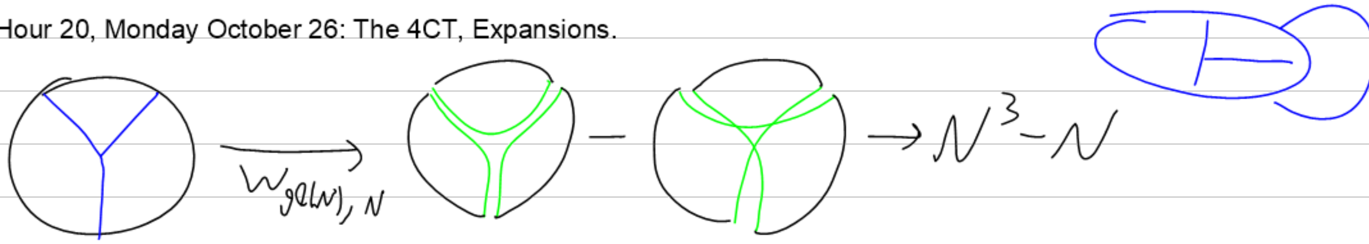
$= \pm \#$ face 4-colorings of D $= \pm \#$ 4 colorings of the map complementary to D

statement about W equiv to ~~th~~ the 4CT.

$$W_\emptyset : \mathcal{A} \longrightarrow \mathbb{Q} \qquad W_\emptyset(D) = \begin{cases} 1 & \text{if } D = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$W_1, W_2 \in \mathcal{A}^* \quad (W_1 \cdot W_2)(a) = (W_1 \otimes W_2)(\square a)$$

$$W(\partial_\emptyset a) = (W \cdot W_\emptyset)(a) = (W \otimes W_\emptyset)(\square a)$$



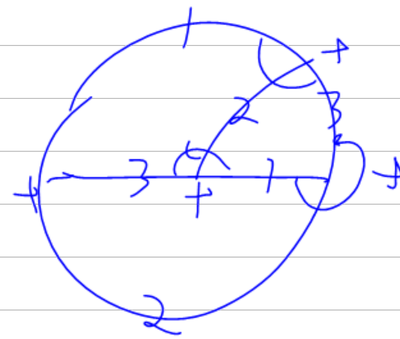
HW6 Q2 If D is blue, $|W^{top}_{gl(N)}(D)| = $ # planar embeddings of D

Last bit of H19: If D is blue & planar, $|W_{sl(2)}(D)| \propto$ # edge 3-colorings of D.

$sl_2 \sim so(3)$ $\quad X_1, X_2, X_3$

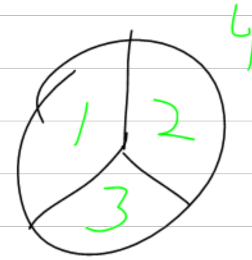
$$[X_i, X_j] = \sum_k \epsilon_{ijk} X_k$$

$$t^{ab} \propto f^{ab}$$



If D is planar

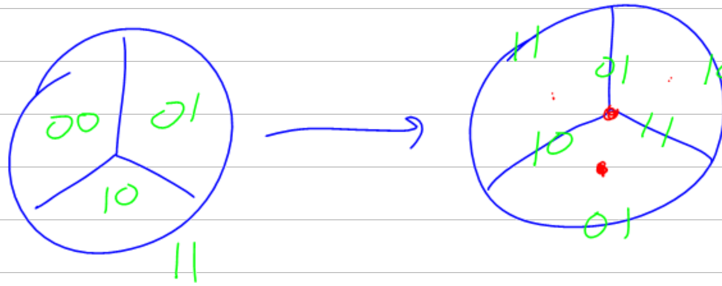
$$4(\#\text{ edge 3-colorings of } D) = \#\text{ face 4-colorings of } D$$

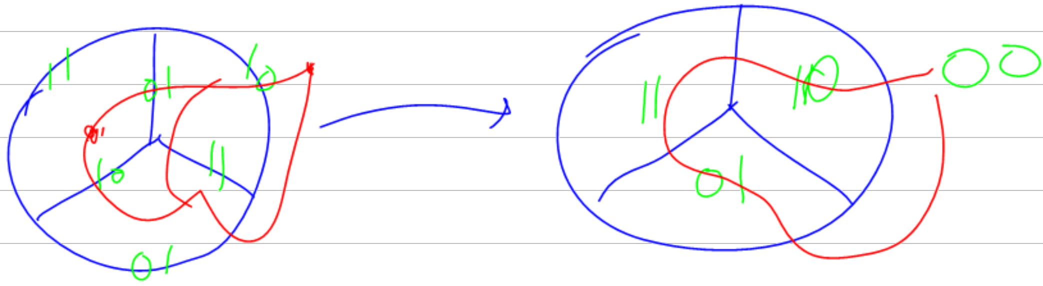


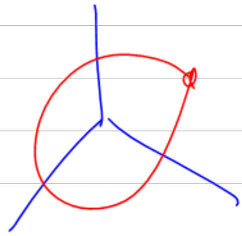
use face colours $\mathbb{Z}/2 \times \mathbb{Z}/2$: 00, 01, 10, 11

use edge colours $\mathbb{Z}/2 \times \mathbb{Z}/2 \setminus \{00\}$: 01, 10, 11

face 4-colorings $\longrightarrow$ edge 3-colorings





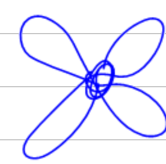


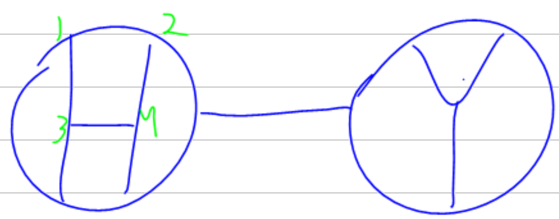
$$\begin{array}{c} 01 \\ 10 \\ 11 \\ \hline 00 \end{array}$$

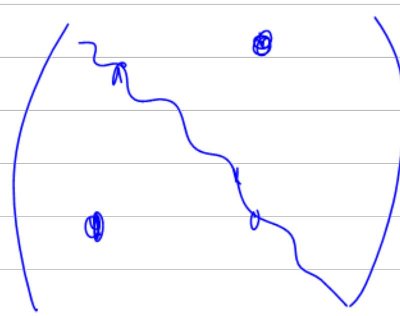
So 4CT $\iff$ $\left(\begin{array}{c}\#\text{ planar} \\ \text{embeddings} \\ \text{of } D\end{array}\right) \neq 0 \implies \left(\begin{array}{c}\#\text{ face} \\ \text{4-colorings} \\ \text{of } D\end{array}\right) \neq 0$

$$\iff \left[W_{sl(2)}(D) = 0 \implies W^{\text{tor}}_{gl(N)}(D) = 0 \right]$$

Sounds reasonable!







$$W_{g,R} : \mathcal{A} \longrightarrow \mathbb{Q}$$

HW6Q1

$$\begin{tikzcd} \mathcal{A}^r \arrow[r, "P"] \arrow[rr, bend right, "\overline{W}_{g,R}"'] & \mathcal{A} \arrow[r, "W_{g,R}"] & \mathbb{Q} \end{tikzcd}$$

Prop $\begin{pmatrix}\text{Fund. thm} \\ \text{For framed} \\ \text{knots}\end{pmatrix} \iff \exists\, Z : \underset{\substack{\| \\ \mathcal{K}}}{\{\text{knots}\}} \to \hat{\mathcal{A}}$

s.t.

$$\underset{\substack{\uparrow \\ m\text{-singular}}}{Z(K)} = \underset{\substack{\uparrow \\ \text{chord diag} \\ \text{of } K}}{D_K} + \text{higher degrees}$$

Pf $\Leftarrow$: Suppose such Z is given, suppose $W \in \mathcal{A}_m^*$

$$\begin{tikzcd} \mathcal{K} \arrow[r, "Z"] \arrow[rr, bend right, "V"'] & \mathcal{A} \arrow[r, "W", "\text{supported on deg } m"'] & \mathbb{Q} \end{tikzcd}$$

$$W_V(D) \xlongequal[\text{s.t. } D_K = D]{\text{Fix } K} V(K) = W(Z(K))$$

$$= W(D_K + \text{h.o.}) = W(D) + 0$$

$\Rightarrow$: Let $(a_{m,i})_{i=1}^{d_m}$ be a basis of $\mathcal{A}_m$

Let $W_{m,i}$ be the dual basis of $\mathcal{A}_m^*$

$$W_{m,i}(a_{m,j}) = \delta_{ij}$$

By Fund thm, Find $V_{m,i}$ s.t. $W_{V_{m,i}} = W_{m,i}$

Define $Z : \mathcal{K} \longrightarrow \mathcal{A}$ as follows:

$$Z(K) = \sum_{m \geq 0} \sum_{i=1}^{d_i} V_{m,i}(K)\, a_{m,i} \in \mathcal{A}$$

Suppose that K is m-singular.

$$Z(K) = \sum_{m'=0}^{\infty} \sum_{i=1}^{d_i} V_{m',i}(K)\, a_{m',i}$$

$$= (h.o.) + \sum_{i=1}^{d_i} V_{m,i}(K)\, a_{m,i}$$

$$= (h.o.) + \sum_{i} \cancel{W_{m,i}}\, (D_K)\, a_{m,i}$$

$$W_{m,i}$$

$$= (h.o.) + D_K \quad \square$$

Homework Assignment 6

$$\frac{\mathbb{Q}[x,y]}{\langle x \rangle} \xrightarrow{ev_{x=0}} \mathbb{Q}[x,y]$$

Question 1. Let $\Theta : \mathcal{A} \to \mathcal{A}$ be the multiplication operator by the 1-chord diagram θ, and let $\partial_\theta = \frac{d}{d\theta}$ be the adjoint of multiplication by W_θ on $\mathcal{A}^\star$, where W_θ is the obvious dual of θ in $\mathcal{A}^\star$. Let $P : \mathcal{A} \to \mathcal{A}$ be defined by

$$P = \sum_{n=0}^{\infty} \frac{(-\Theta)^n}{n!} \partial_\theta^n.$$

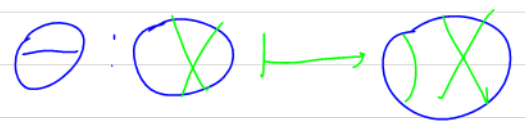
$$\frac{\mathcal{A}}{\langle \theta \rangle} \longrightarrow \mathcal{A}$$

Verify the following assertions, but submit only your work on assertions 4,5,7,11:

1. $[\partial_\theta, \Theta] = 1$, where $1 : \mathcal{A} \to \mathcal{A}$ is the identity map and where $[A, B] := AB - BA$ for any two operators.
2. P is a degree 0 operator; that is, $\deg Pa = \deg a$ for all $a \in \mathcal{A}$.
3. ∂_θ satisfies Leibnitz' law: $\partial_\theta(ab) = (\partial_\theta a)b + a(\partial_\theta b)$ for any $a, b \in \mathcal{A}$.
4. P is an algebra morphism: $P1 = 1$ and $P(ab) = (Pa)(Pb)$.
5. Θ satisfies the co-Leibnitz law: $\Box \circ \Theta = (\Theta \otimes 1 + 1 \otimes \Theta) \circ \Box$ (why does this deserve the name "the co-Leibnitz law"?).
6. P is a co-algebra morphism: $\eta \circ P = \eta$ (where η is the co-unit of $\mathcal{A}$) and $\Box \circ P = (P \otimes P) \circ \Box$.
7. $P\theta = 0$ and hence $P\langle\theta\rangle = 0$, where $\langle\theta\rangle$ is the ideal generated by θ in the algebra $\mathcal{A}$.
8. If $Q : \mathcal{A} \to \mathcal{A}$ is defined by
$$Q = \sum_{n=0}^{\infty} \frac{(-\Theta)^n}{(n+1)!} \partial_\theta^{(n+1)}$$
then $a = \theta Q a + P a$ for all $a \in \mathcal{A}$.
9. $\ker P = \langle\theta\rangle$.
10. P descends to a Hopf algebra morphism $\mathcal{A}^r \to \mathcal{A}$, and if $\pi : \mathcal{A} \to \mathcal{A}^r$ is the obvious projection, then $\pi \circ P$ is the identity of $\mathcal{A}^r$. (Recall that $\mathcal{A}^r = \mathcal{A}/\langle\theta\rangle$). $\to$ Q
11. $P^2 = P$.

$$f \mapsto f|_{\hat{x} - \hat{x}\,\partial^n}$$

$$= \sum_{n\geq 0} \frac{(-x)^n}{n!} (\partial_x)^n f$$



$$\partial_\theta = W_\theta^*$$

$W_\theta(\theta) = 1 \qquad W_\theta\left(\text{anything not of deg 1}\right) = 0$

$T : V \to W$, u

$T^* : W^* \to V^*$, w

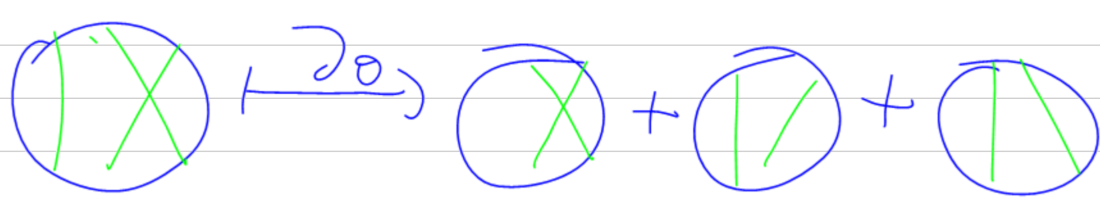
$w(Tu) = (T^* w)(u)$

$\varphi(\partial_\theta D) = (W_\theta \cdot \varphi)(D)$

$\varphi \in \mathcal{A}^*$ $\quad = (W_\theta \otimes \varphi)(\Box D)$ $\quad \mathcal{A}\otimes\mathcal{A}$

$= \varphi$(all diag obtained from D by dropping one chord)

$\Longrightarrow \partial_\theta D =$ (Sum of all ways of dropping one chord from D)



$$\partial_x x^5 = \sum x \cancel{x} x x x = 5x^4$$

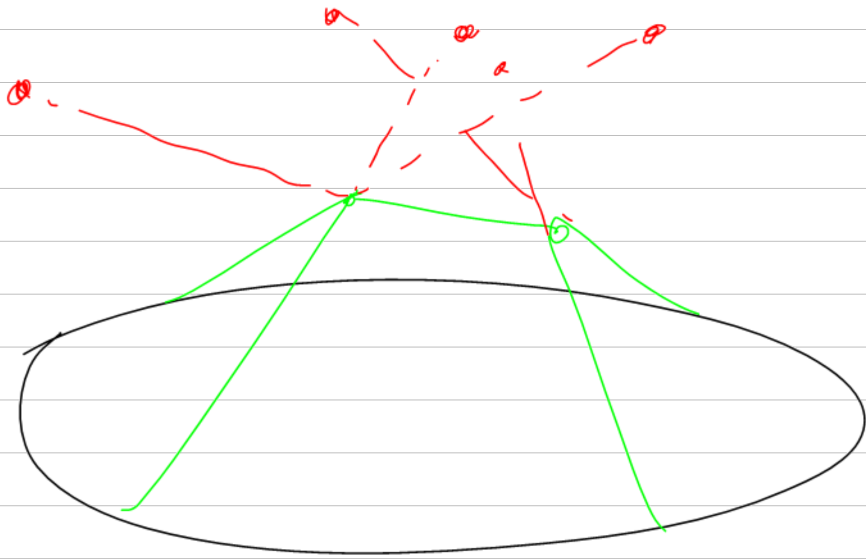
$$[\partial_x, \hat{x}] = I$$

heisenberg Commutation relation

$$[\partial_\theta, \hat{\theta}] = I$$

$$N^3$$

$$W_{gl(N)}(\quad)$$

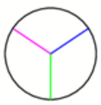
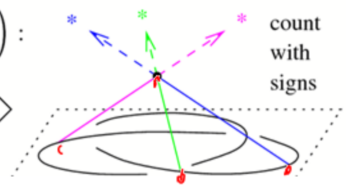


Hour 21, Wednesday October 28: Expansions and Cosmic Coincidences.
No HW assigned this week! But HW6 is due. Should we discuss it?

Recall: We are seeking an "expansion/UFTI";

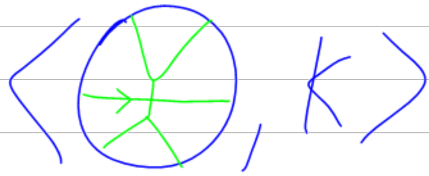
$Z: \mathcal{K} \to \mathcal{A}$ s.t. $Z(K) = D_K + \text{h.o.}$

$\langle D, K\rangle_{\Pi} := \begin{pmatrix}\text{The signed Stonehenge}\\ \text{pairing of } D \text{ and } K\end{pmatrix}$: count with signs

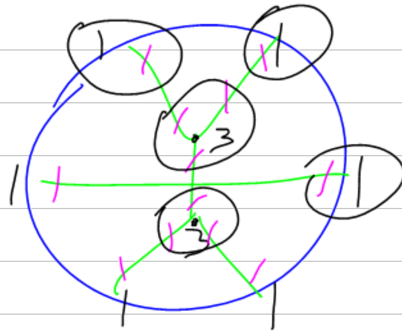
$D=$  $K=$  $\Rightarrow$ 

unknowns: $S^1 \times S^1 \times S^1 \times \mathbb{R}^3 \sim 6$

eqns: $S^2 \times S^2 \times S^2 \sim 6$



$1+1=2$

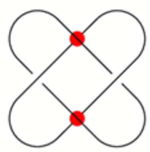


$F: M^k \to (N^k, n)$

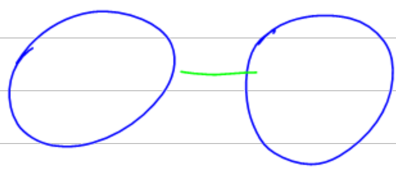
$x \mapsto n$

The Gaussian linking number

$lk(\bigcirc\!\!\bigcirc) = \sum_{\substack{\text{vertical}\\ \text{chopsticks}}} (\text{signs}) = \langle \bigcirc\!-\!\bigcirc, \bigcirc\!\!\bigcirc \rangle_{\Pi}$



C.F. Gauss



The generating function of all cosmic coincidences:

$$Z(K) := \lim_{N\to\infty} \sum_{\text{3-valent } D} \frac{\langle D, K\rangle_{\Pi} D}{2^c c! \binom{N}{e}} \cdot \begin{pmatrix} \cancel{\text{framing-}} \\ \cancel{\text{dependent}} \\ \cancel{\text{counter-term}} \end{pmatrix} \in \mathcal{A}(\circlearrowleft)$$

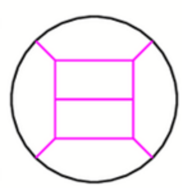
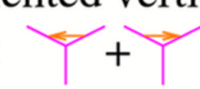
D. Thurston

IHX, STU, AS

N := # of stars

c := # of chopsticks

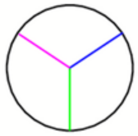
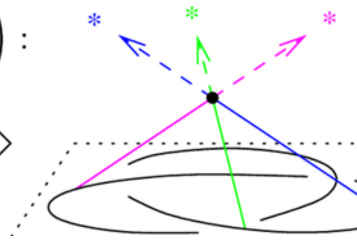
e := # of edges of D

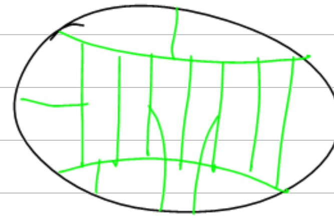
$\mathcal{A}(\circlearrowleft)$:= Span $\left\langle\right.$  $\left.\right\rangle$ / oriented vertices AS:  $=0$ & more relations

dbnsymb

$Z: \mathcal{K} \to \mathcal{A} = \mathcal{D}/\text{AS, IHX, STU}$

$\langle D, K\rangle_{\Pi} := \begin{pmatrix}\text{The signed Stonehenge}\\ \text{pairing of } D \text{ and } K\end{pmatrix}$: count with signs

$D=$  $K=$  $\Rightarrow$ 



The generating function of all cosmic coincidences:

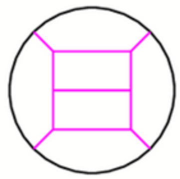
$$Z(K) := \cancel{\lim_{N\to\infty}} \sum_{\text{3-valent } D} \frac{\langle D, K\rangle_{\Pi} D}{2^{e}\cancel{c!\binom{N}{e}}} \cdot \cancel{\begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix}} \in \mathcal{A}(\circlearrowleft)$$

D. Thurston

N := # of stars

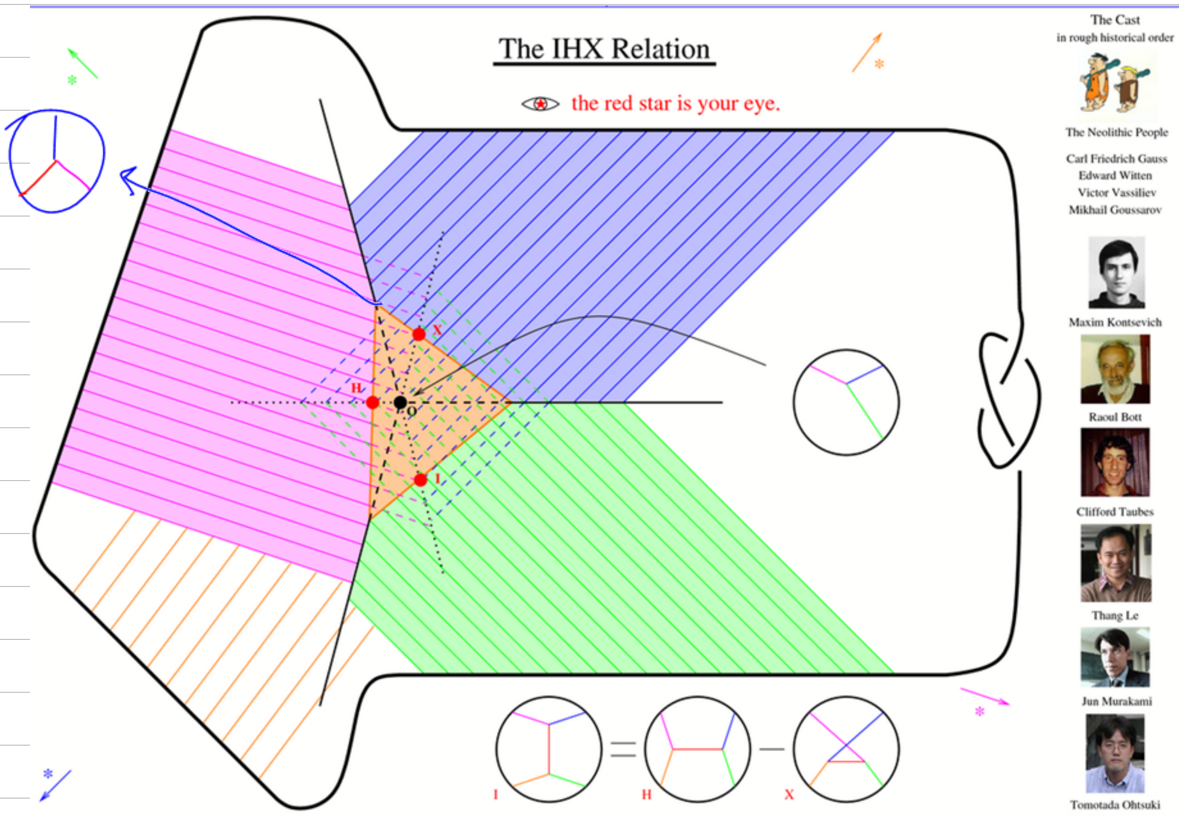
c := # of chopsticks

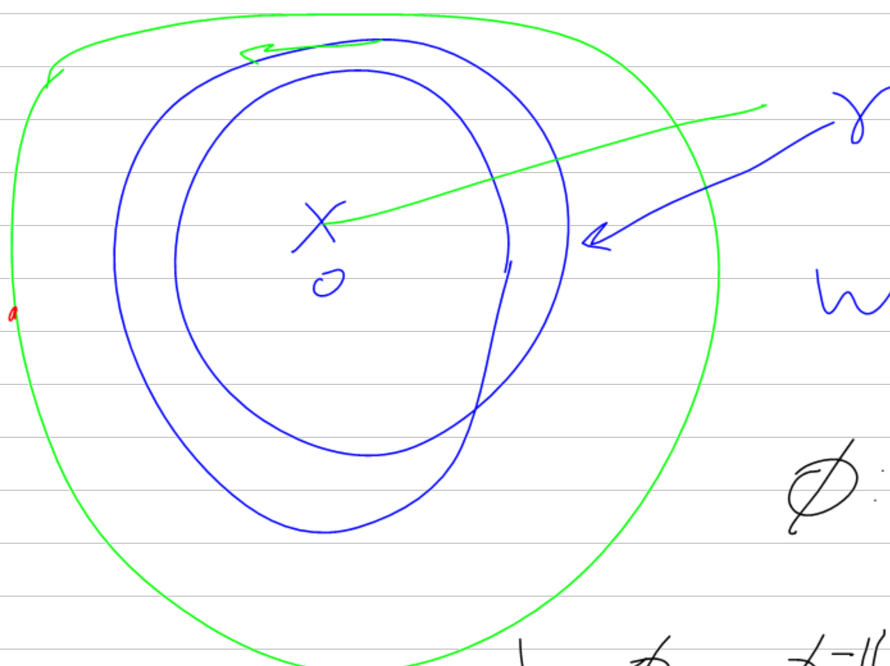
e := # of edges of D

$\mathcal{A}(\circlearrowleft)$:=Span $\left\langle\right.$  $\left.\right\rangle$ / oriented vertices

AS:  +  =0

& more relations





$$w(\gamma) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z}$$

$$\phi : M^n \longrightarrow N^n \ni w$$

$$p \qquad \int_N w = 1$$

$$\deg \phi = \phi^{-1}(p) = \int_M \phi^*(w)$$

$$\langle \Theta, K \rangle_\pi \qquad \phi : \underbrace{S^1 \times S^1 \times S^1 \times \mathbb{R}^3}_{M} \longrightarrow \underset{w \,\times\, w \,\times\, w}{S^2 \times S^2 \times S^2}$$

$$\int_M \phi^*(w \wedge w \wedge w)$$

When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,

(see below)

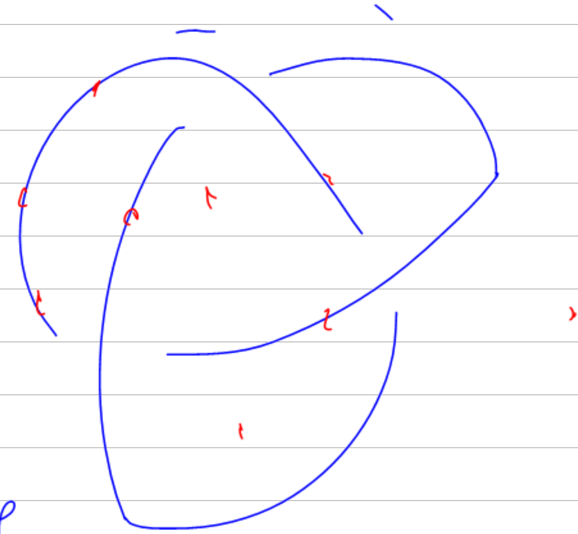
An intersection line cuts through the knot – Solution: Impose STU,

(similar argument)

The Gauss curve slides over a star – Solution: Multiply by a framing–dependent counter–term.

(not shown here)

Theorem. Modulo Relations, $Z(K)$ is a knot invariant!



$$\left\{\begin{matrix}\text{Curves}\\ \text{in } \mathbb{R}^3\end{matrix}\right\} \xrightarrow[\text{Curve}]{\text{Gauss}} \left\{\begin{matrix}\text{Curves}\\ \text{in } S^2\end{matrix}\right\}$$

&

Chern-Simons-Witten theory and Feynman diagrams.

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right]$$

Witten

$$\longrightarrow \sum_{D:\ \text{Feynman diagram}} W_{\mathfrak{g}}(D) \not\sum \mathcal{E}(D) \longrightarrow \sum_{D:\ \text{Feynman diagram}} D \not\sum \mathcal{E}(D)$$

Feynman

$$= \sum_{m=0}^{\infty} \sum_{\text{Feynman diagram}} \iiint_{\mathbb{R}^{N(D)}} e$$

$$\int_{\mathbb{R}^{I}\to\mathbb{R}} e^{Q+P} = \sum_{x_1 \dots x_{17}=1}^{n} e$$

$\mathbb{R}^{\mathbb{R}}$ $\mathbb{R}^{\infty}$

Class of November 2: A Quick Introduction to Feynman Diagrams

We wish to understand

Witten–Chern–Simons:

$$\int_{A\in\Omega^1(\mathbb{R}^3,\mathfrak{g})} \mathcal{D}A\, \underbrace{\mathrm{hol}_\gamma(A)}_{\mathrm{tr}_R} \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \mathrm{tr}\left(\overbrace{A\wedge dA}^{\text{quadratic}} + \frac{2}{3}\overbrace{A\wedge A\wedge A}^{\text{cubic}}\right)\right]$$

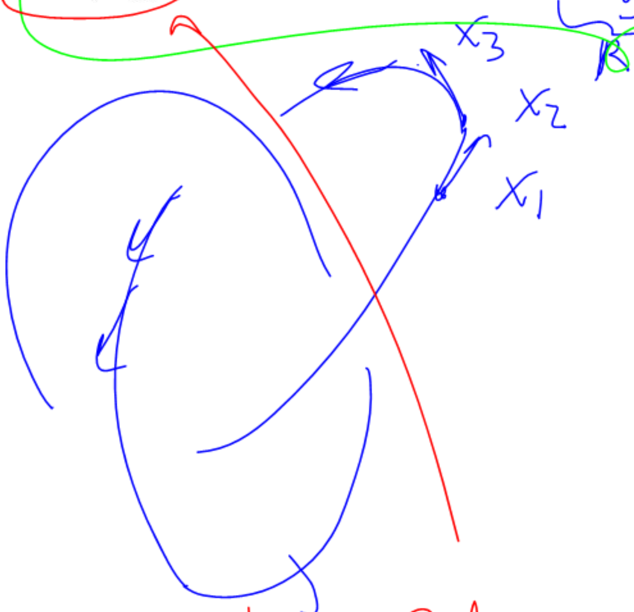
3-form

$\mathfrak{g}\otimes\mathfrak{g}$

$\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \to \mathbb{R}$

$\mathbb{R}$

F^{abc}



does not depend on metric properties

$\Downarrow$

topological invts.

$A = F_1\,dx + F_2\,dy + F_3\,dz$

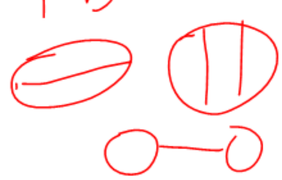
$\mathbb{R}^n \quad \mathbb{R}^{\mathbb{R}}$

$$\int \mathcal{D}A\, e^{\frac{ik}{4\pi}\int_{\mathbb{R}^3}\mathrm{tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)}\ \overline{\mathrm{hol}_\gamma(A)}$$

$\downarrow$

$$\sum_{\text{FD:}} \left(\text{messy combinatorics}\right) \iint_{\mathbb{R}^3} \left(\substack{\text{cubic}\\ \text{combinatorial}}\right)\left(\text{integral kernel}\right)$$

D:



As a warm up, suppose (λ_{ij}) is a symmetric positive definite matrix and (λ^{ij}) is its inverse, and (λ_{ijk}) are the coefficients of some cubic form. Denote by $(x^i)_{i=1}^n$ the coordinates of $\mathbb{R}^n$, let $(t_i)_{i=1}^n$ be a set of "dual" variables, and let ∂^i denote $\frac{\partial}{\partial t_i}$. Also let $C := \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

$$\int dj \int di \int dk\, \mathcal{G}(ijk)$$

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{n} \mathcal{G}(i,N,j)$$

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\lambda_{ij}x^ix^j + \frac{1}{6}\lambda_{ijk}x^ix^jx^k\right)$$

$$= \int_{\mathbb{R}^n} \overbrace{\exp\left(\frac{1}{6}\lambda_{ijk}x^ix^jx^k\right)}^{\sim 1}\exp\left(-\frac{1}{2}\lambda_{ij}x^ix^j\right)$$

$$e^{\frac{1}{2}\lambda_{ij}x^ix^j} \overset{\mathcal{F}}{\sim} e^{\frac{1}{2}\lambda^{ij}t_it_j}$$

The Fourier Transform.

$$(F\colon \overset{\mathbb{R}^n}{V} \to \mathbb{C}) \Rightarrow (\tilde{f}\colon \overset{\mathbb{R}^n}{V^*} \to \mathbb{C})$$

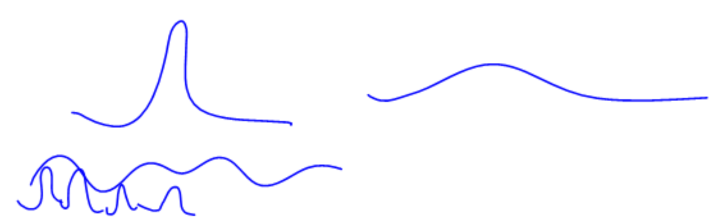
via $\tilde{F}(\varphi) := \int_V f(v)e^{-i\langle\varphi,v\rangle}dv$. Some facts:

- $\tilde{f}(0) = \int_V f(v)dv$.
- $\frac{\partial}{\partial\varphi_i}\tilde{f} \sim \widetilde{v^i f}$.
- $(\widetilde{e^{Q/2}}) \sim e^{Q^{-1}/2}$, where Q is quadratic, $Q(v) = \langle Lv, v\rangle$ for $L\colon V \to V^*$, and $Q^{-1}(\varphi) := \langle\varphi, L^{-1}\varphi\rangle$. (This is the key point in one of the proofs of the Fourier inversion formula!)

$$\tilde{F}(p) = \int F(x)\, e^{-ipx}$$

$$\frac{\partial}{\partial p}\tilde{F} \sim \widetilde{xF}$$

$$\widetilde{e^{-\frac{\lambda x^2}{2}}} \overset{\mathcal{F}}{\sim} e^{-\frac{p^2}{2\lambda}}$$

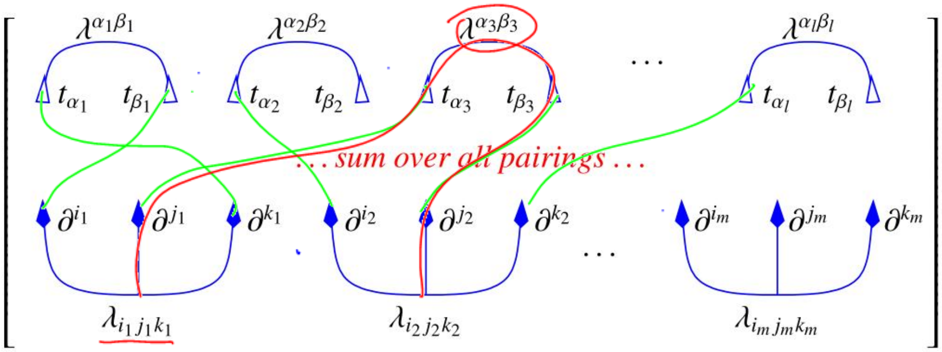


$$= C\exp\left(\frac{1}{6}\lambda_{ijk}\partial^i\partial^j\partial^k\right)\exp\left(\frac{1}{2}\lambda^{\alpha\beta}t_\alpha t_\beta\right)\Bigg|_{t_\alpha=0}$$

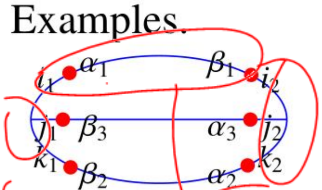
$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}} \frac{C}{6^m m! 2^l l!}\left(\lambda_{ijk}\partial^i\partial^j\partial^k\right)^m\left(\lambda^{\alpha\beta}t_\alpha t_\beta\right)^l$$

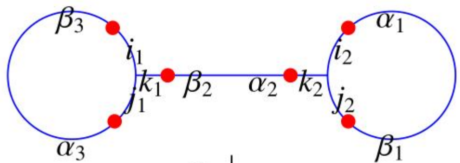
$$\Big(\sum_{\alpha,\beta} -\Big)^l = \sum_{\substack{\alpha_1 \dots \alpha_l\\ \beta_1 \dots \beta_l}} - - - -$$

$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}} \frac{C}{6^m m! 2^l l!}$$



Examples.





etc.

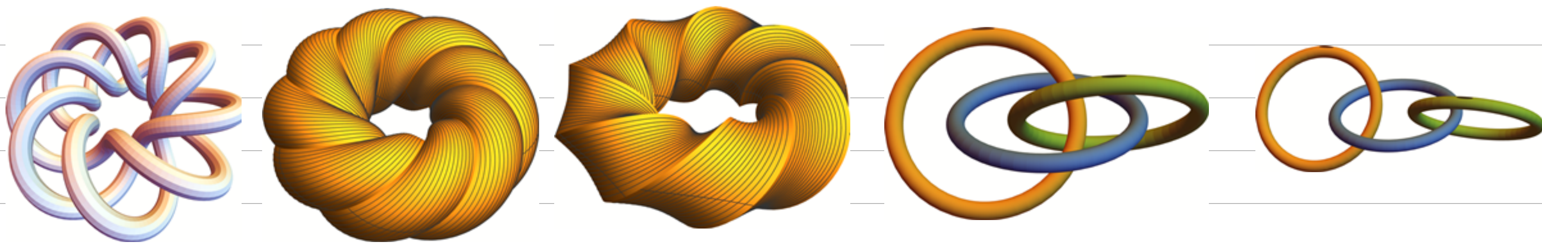
$\lambda_{i_1 j_1 k_1}\lambda_{i_2 j_2 k_2}\lambda^{i_1 i_2}\lambda^{j_1 j_2}\lambda^{k_1 k_2}$ $\qquad$ $\lambda_{i_1 j_1 k_1}\lambda_{i_2 j_2 k_2}\lambda^{i_1 j_1}\lambda^{k_1 k_2}\lambda^{i_2 j_2}$

$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}} \frac{C}{6^m m! 2^l l!} \sum_{\substack{m\text{-vertex fully marked}\\ \text{Feynman diagrams } D}} \mathcal{E}(D)$$

$$\sum_{\substack{i_1 i_2 j_1 j_2\\ k_1 k_2}}$$ Prod of cubics & inverse quadratics

Hour 24, Wednesday November 4: The Fundamental Group / Knot Group.
HW7 on web by midnight! (And I hope to clear my marking backlog soon).
Finite type / Lie Algebras and Reps omissions:
* The KZ proof of the Fundamental Theorem.
* The "Associators" proof of the Fundamental Theorem (also, "Knotted Trivalent Graphs").
* The step-by-step-integration non-proof of the Fundamental Theorem.
* Computing FT Invariants using "Gauss Diagram Formulas".
* Computations of invariants for specific Lie algebras and reps ("Quantum Groups").
* Finite type invariants of other types knotted objects.
* Finite type invariants of 3-manifolds.
* Vogel's work on non-Lie-algebraic weight systems.
* And more....

A Gallery of Pictures from BlownTorus.nb at http://drorbn.net/AcademicPensieve/Classes/20-1350-KnotTheory:



Conjecture 1 Finite type invts separate knots.

Conjecture 2 $\langle W_{g,R} \rangle = \mathcal{A}^*$ (all F.t. invts come from Lie Algebras)

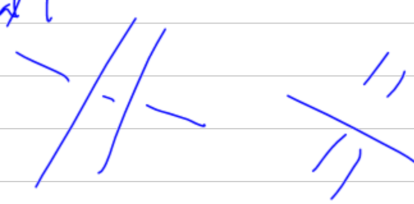
That one is FALSE.

dim $\mathcal{A}_m$ - - - - - match to $m=17$ diverge at $m=18$

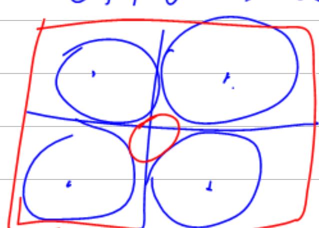
dim $\langle W_{g,R} \rangle_m$ - - -

Alt.

non-alt



Riddle: Let B_n be the largest n-dim ball centered at 0 and bound by the 2^n unit balls centered at $\{\pm 1\}^n$



Let C_n be the smallest ~~box~~ cube containing all of the above

$$\lim_{n\to\infty}\frac{\mathrm{Vol}(B_n)}{\mathrm{Vol}(C_n)}$$

Things get interesting in dim 1206.

Def If K is a knot, $\pi_1(K)$, or the fundamental group of K, the group of K, is

$$\pi_1(K)=\pi_1(\mathbb{R}^3\setminus K)$$

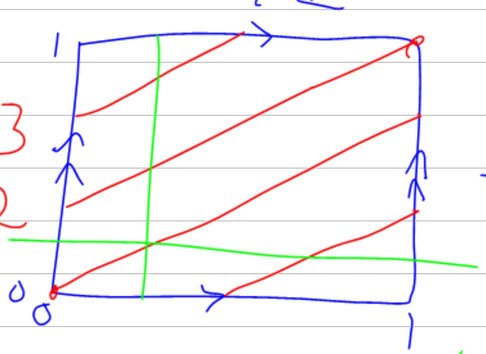
$\pi_1($  $)=\mathbb{Z}$ $\quad\pi_1($  $)$

Example $\pi_1(T_{p,q})$ $\quad T_{p,q}$: (p,q) Torus knots (where (p,q) are rel. prime).

$\mathbb{R}^2/\mathbb{Z}^2$

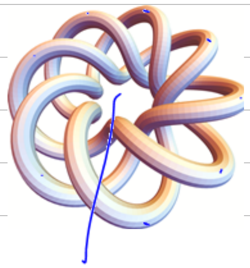
$p=3$

$q=2$



$\xrightarrow{\tau}$

$\gamma(t)=\tau(pt,qt)$ $\quad \gamma:\underbrace{[0,1]/0\sim 1}_{S^1}\xrightarrow{1-1}\mathbb{R}^3$



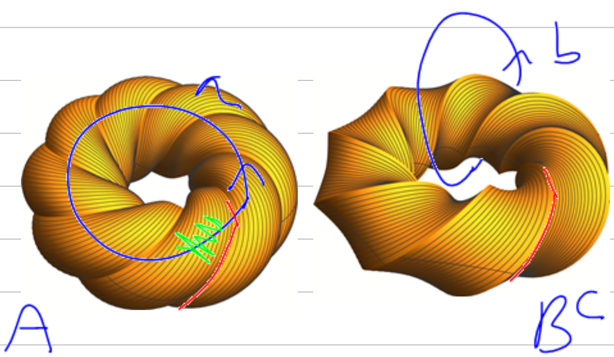
$T_{8,3}$ $\qquad$ $\pi_1(T_{8,3})$

Van Kampen: $A, B, A\cap B$ are connected, $b \in A\cap B$

$$\pi_1(A\cup B) = \pi_1(A) *_{\pi_1(A\cap B)} \pi_1(B)$$

$$\begin{tikzcd} & \pi_1(A\cap B) \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \pi_1(A) & & \pi_1(B) \end{tikzcd}$$

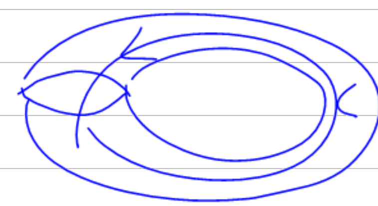
$$= \pi_1(A) * \pi_1(B) \Big/ \forall \gamma \in \pi_1(A\cap B)\ \ \alpha(\gamma) = \beta(\gamma)$$



A = solid torus

$\pi_1(A) = \mathbb{Z} = \langle a \rangle$

$\pi_1(B) = \mathbb{Z} = \langle b \rangle$

$A \cap B =$ 

$\pi_1(A\cap B) = \mathbb{Z}$

$\alpha(c) = a^3$

$\beta(c) = b^8$

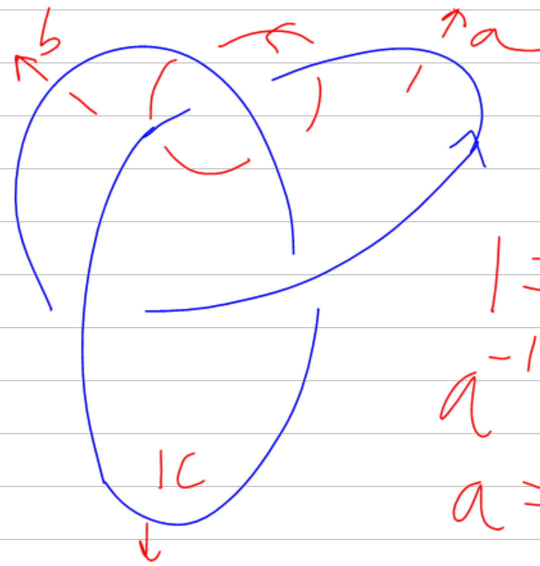
$$\pi_1(T_{8,3}) = \langle a, b \rangle \big/ a^3 = b^8$$

$\pi_1\Big($  $\Big) = \langle a, b \rangle \big/ a^3 = b^2$

$\pi_1\Big($  $\Big)$

$\|$

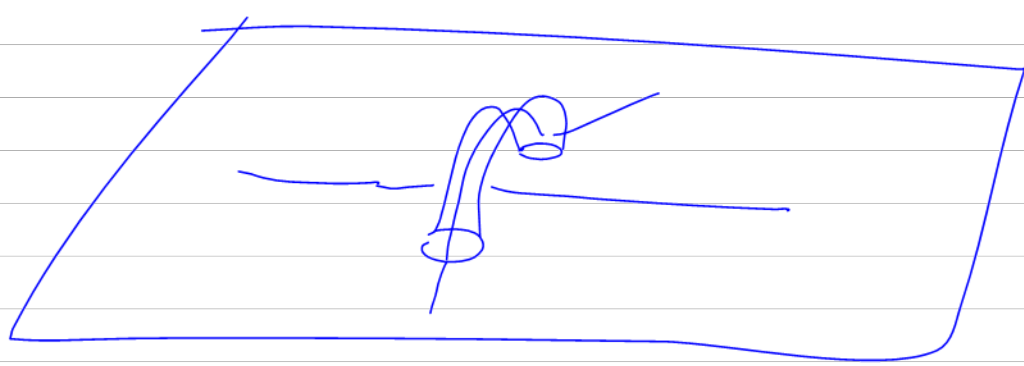
$$\langle a, b, c \rangle \,/\, \begin{array}{l} a = c^b \\ b = a^c \\ c = b^a \end{array}$$



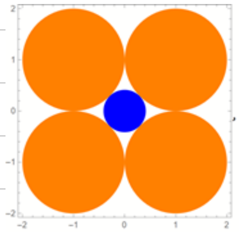
$$1 = a b^{-1} c^{-1} b$$

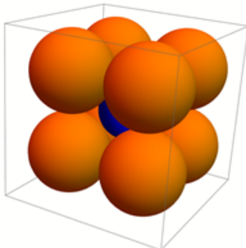
$$a^{-1} = (c^{-1})^b$$

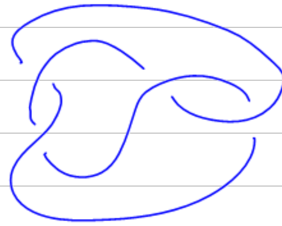
$$a = c^b$$

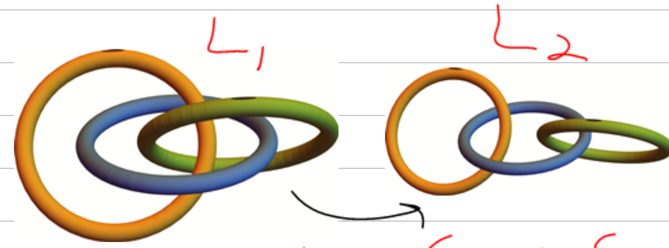


Hour 25, Friday November 6: More on Knot Groups





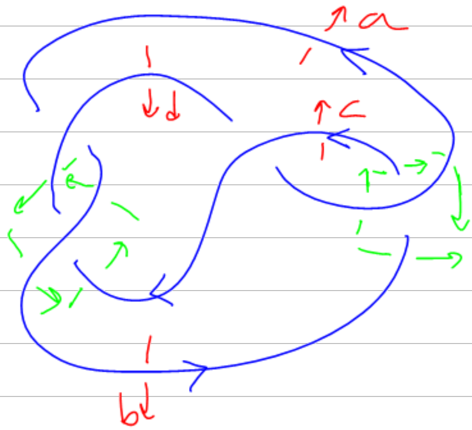




$L_1 \neq L_2$ yet $L_1^c \cong L_2^c$

$$\lim_{n\to\infty} \frac{Vol(B_n)}{Vol(C_n)}$$

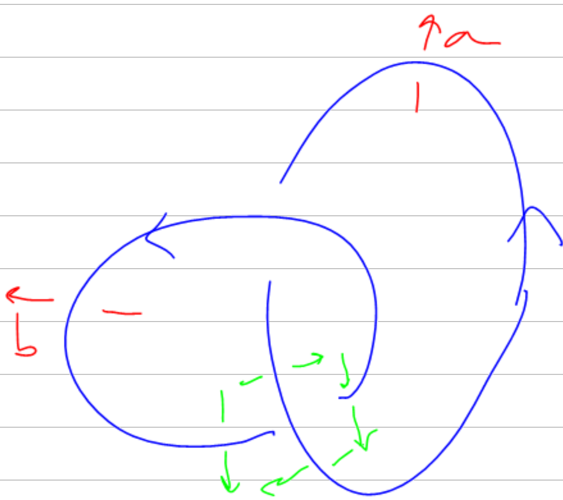
at $n=9$
radius $\sqrt{9}-1=2$



$b = a^{-1} c a = c^a$

$c = bd\,\cancel{a^{-1}}\,b^{-1} = d^{(b^{-1})}$

"Wirtinger presentation"



$b = a^{-1} b a$

$\pi_1(Hopf) = \langle a,b \rangle / ab=ba$

$= \mathbb{Z}^2 = \pi_1(T^2)$, $T^2 = S^1 \times S^1$

Riddle show that
$(S^3 \setminus Hopf) \sim S^1 \times S^1$

π_1 is very strong <u>but</u>

0. "Word problem for groups is insoluble"

$K_1 \longrightarrow \langle g_i^1 \rangle / r_j^1$

$K_2 \longrightarrow \langle g_i^2 \rangle / r_j^2$

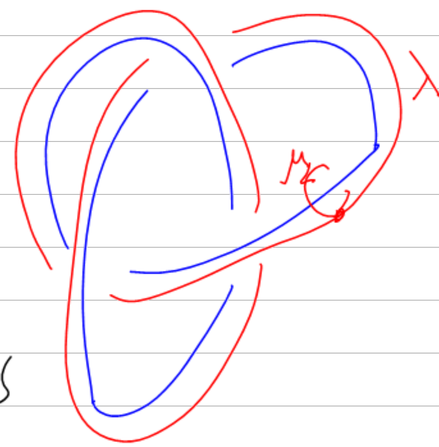
Lickorish's book GTM 175, P115

1. Waldhausen 66':

$$\begin{array}{c} (\pi, \lambda, \mu) \\ \downarrow \ \downarrow \ \downarrow \\ (\pi', \lambda', \mu') \end{array}$$

determines the knot

Also true for links

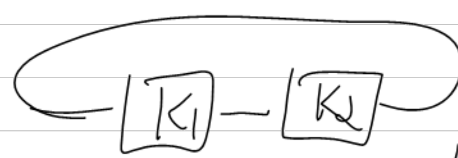


λongtitude
merdian

2. Whitten/Gonzales-Acuña 87'

If K is prime, $\pi_1(K)$ determines K^c (as a manifold)

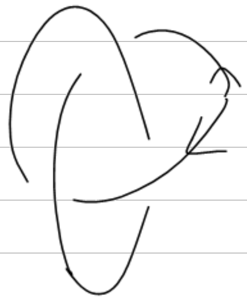
K is prime not

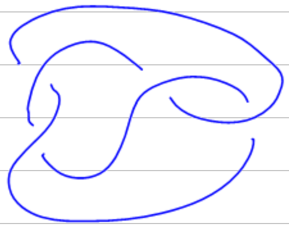


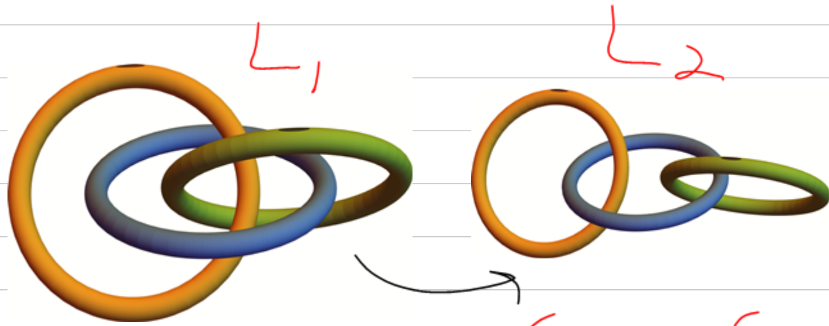
K_1 & K_2 non-trivial.

3. Gordon-Luecke 89'

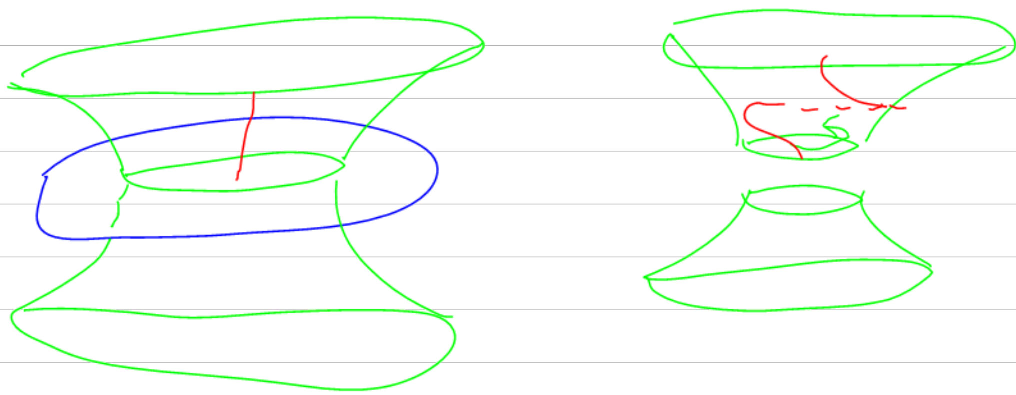
The complement of an <u>unoriented</u> knot determines it.

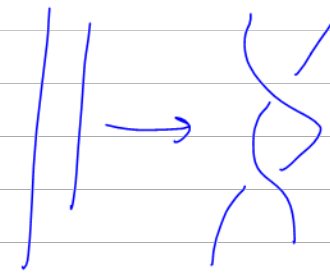






$L_1 \neq L_2$ yet $L_1^c \cong L_2^c$





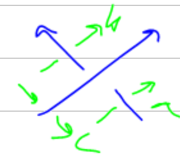
Pick some finite group G (S_5)
$\underline{\underline{D_6}}$

$$\left|\mathrm{Rep}(\pi_1(K) \to G)\right|$$

a knot invariant.

The Wirtinger presentation

$\pi_1(K) = \langle a_i \,;\, a = c^b \dots \rangle$



$\to a = c^{-1} b c$
$= b^c$

Instead pick a finite group G, & colour by

$\mathrm{Hom}(\pi_1, G)$

Rels $a = b^c$

Improve further

$g \in G \quad \mathrm{Hom}(\pi_1, g^G) := \{\phi : \pi_1 \to G :$ 1. Homo... 2. For every generator of π_1, $\phi(a)$ is a conj. of $g\}$

Ex: $\mathbb{C} \to \mathbb{C} \quad z \mapsto e^{\frac{2\pi i}{n}} z$

$D_{2n} \qquad z \mapsto \bar{z}$

$\| $

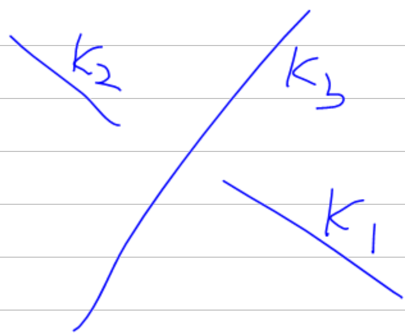
$\left\{ (s,k) : \begin{array}{l} s \in \{\pm 1\} \\ k \in \mathbb{Z}/n \end{array} \right\}$

w/ $(s_1, k_1)(s_2, k_2) = (s_1 s_2, s_2 k_1 + k_2)$

$(s,k)^{-1} = (s, -sk)$

$(-1,0)^G = \{(-1,k)\}$ unless $n=2$

$(-1,k_1)^{(-1,k_3)} = (-1,\ 2k_3-k_1)$



s.t. $k_2 = 2k_3 - k_1$

$k_1 + k_2 = 2k_3$

$k_i \in \mathbb{Z}/n$

If $n=3$, $\Rightarrow$ $k_1+k_2+k_3 = 0$.

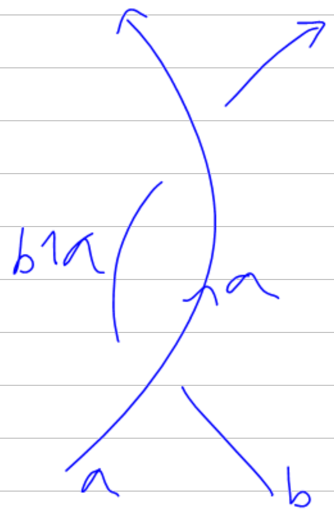
Coloring rules:

A set Q "of colours"

an operation $\wedge : Q \times Q \to Q$, s.t.

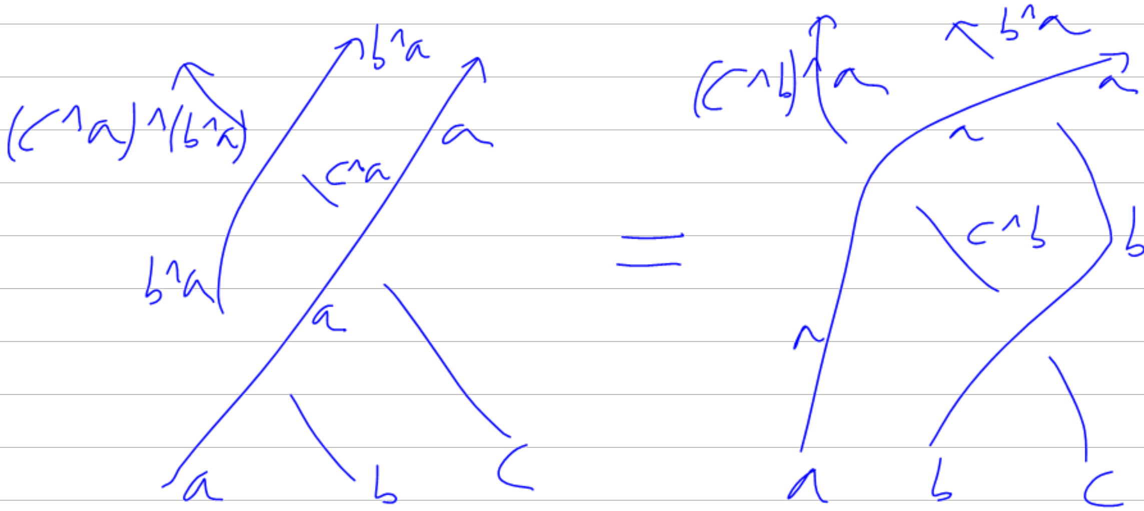
R1

1. $a \wedge a = a$



$(a \bar{\wedge} b)^\wedge b = a$ $\left(\text{think: } a^{(b^{-1})} \right)$

2. $\forall b,\ a \mapsto a^\wedge b$ is an invertible map $Q \to Q$



Easy exercise $a^\wedge b = a^b$ in G, satisfies this

3. $\forall a,b,c \in Q$
$(c^\wedge b)^\wedge a = (c^\wedge a)^\wedge(b^\wedge a)$
$(a^\wedge b)^\wedge c = (a^\wedge c)^\wedge(b^\wedge c)$

<u>Def</u> A quandle is a set Q w/ op
$^\wedge : Q \times Q \to Q$, s.t.

1. $x^\wedge x = x$
2. $\forall y\ x \mapsto x^\wedge y$ is invertible
3. $(x^\wedge y)^\wedge z = (x^\wedge z)^\wedge(y^\wedge z)$

Axiom 3 is sometimes called "$\wedge$ is self-distributive"

$$(x+y)\cdot z = x\cdot z + y\cdot z$$

$$(x\wedge y)\wedge z = (x\wedge z)\wedge(y\wedge z)$$

If $M, *$ is a set w/ binary op. then every $m\in M$ defines

$$T_m : M \longrightarrow M$$

by $T_m(x) = x*m$

<u>Axiom</u> $\forall m$ T_m is an auto.

$$T_m(x*y) = (T_m x)*(T_m y)$$

$$(x*y)*m = (x*m)*(y*m)$$

At the infinitesimal level:

$$T_m : Q\times Q \longrightarrow Q\times Q$$

$$L : Q \otimes Q \longrightarrow Q \otimes Q \qquad L_z x = x * z$$

$$L(x \otimes y) = (Lx \otimes y) + (x \otimes Ly)$$

$$L(x * y) = (Lx) * y + x * (Ly)$$

$$(x * y) * z = (x * z) * y + x * (y * z)$$

$$x * y = [x, y]$$

$$[[x, y], z] = \_\ \_\ \_\ \_ \Leftarrow \text{Jacobi Id.}$$

$$\mathrm{ad}_z : g \longrightarrow g$$

is a ~~homomorphism of Lie alg.~~

Added after class: I was in time pressure, so I said something wrong. The correct statement is "$[\,,\,] : g \otimes g \longrightarrow g$ is an intertwiner of reps, when g is a representation using the adjoint action".

Hour 27, Wednesday November 18: A bit more on quandles, Seifert surfaces.
HW8 on web by midnight tomorrow!

$\mathrm{ad}_z x = [x,z]$

Correction: I said "$\mathrm{ad}_z : L \to L$ is a morphism of Lie Algs". Nonsense!

$Q\times Q \xrightarrow{\wedge} Q$ is equivariant, meaning

$$\begin{tikzcd} Q\times Q \arrow[r, "\wedge"] \arrow[d, "\wedge z"] & Q \arrow[d, "\wedge z"] \\ Q\times Q \arrow[r, "\wedge"] & Q \end{tikzcd} \qquad \begin{tikzcd} L\otimes L \arrow[r, "{[,]}"] \arrow[d, "\mathrm{ad}_z"] & L \arrow[d, "\mathrm{ad}_z"] \\ L\otimes L \arrow[r, "{[,]}"] & L \end{tikzcd}$$

$\mathrm{ad}_z(x\otimes y)$
$\mathrm{ad}_z x\otimes y$
$+\, x\otimes \mathrm{ad}_z y$

Quandles from groups:

1. $x\wedge y := y^{-1}xy$
2. $x\wedge y := y^{-n}xy^{n}$ } (can restrict to a conjugacy class)
3. $x\wedge y := yx^{-1}y$

Vendramin:

TABLE 2. The number of non-isomorphic indecomposable quandles

n	1	2	3	4	5	6	7	8	9	10	11	12
$q(n)$	1	0	1	1	3	2	5	3	8	1	9	10
n	13	14	15	16	17	18	19	20	21	22	23	24
$q(n)$	11	0	7	9	15	12	17	10	9	0	21	42
n	25	26	27	28	29	30	31	32	33	34	35	
$q(n)$	34	0	65	13	27	24	29	17	11	0	15	

Conjeture 3.5. *Let p be an odd prime number and let Q be an indecomposable quandle of $2p$ elements. Then $p\in\{3,5\}$.*

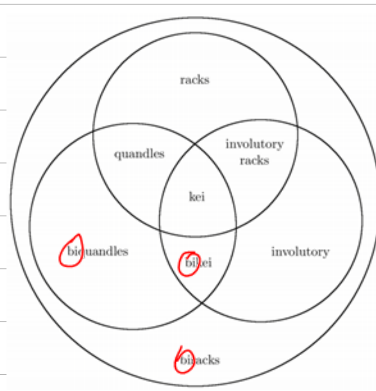
$Q(K) = Q\langle a_i : \underbrace{a = b\wedge c}_{\text{for each xing}}\rangle$

Elhamdadi/Nelson: A whole book

The Fundamental quandle of a knot (Joyce)

On beyond quandles!

Aksoy, Nelson arXiv:1102.1473:

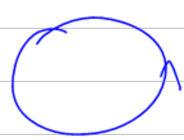


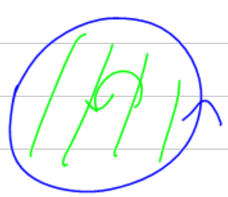
$(B, \wedge, \vee)$

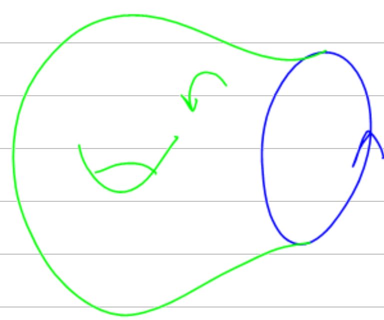
"biquandle"

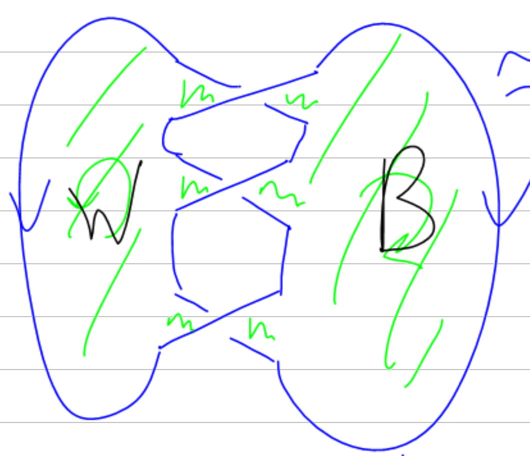
Def A "Seifert surface" for a knot K is a smooth ~~connected~~ oriented 2D surface with bndry Σ, whose oriented bndry is the knot:

$$\partial \Sigma = K$$

$K =$ 

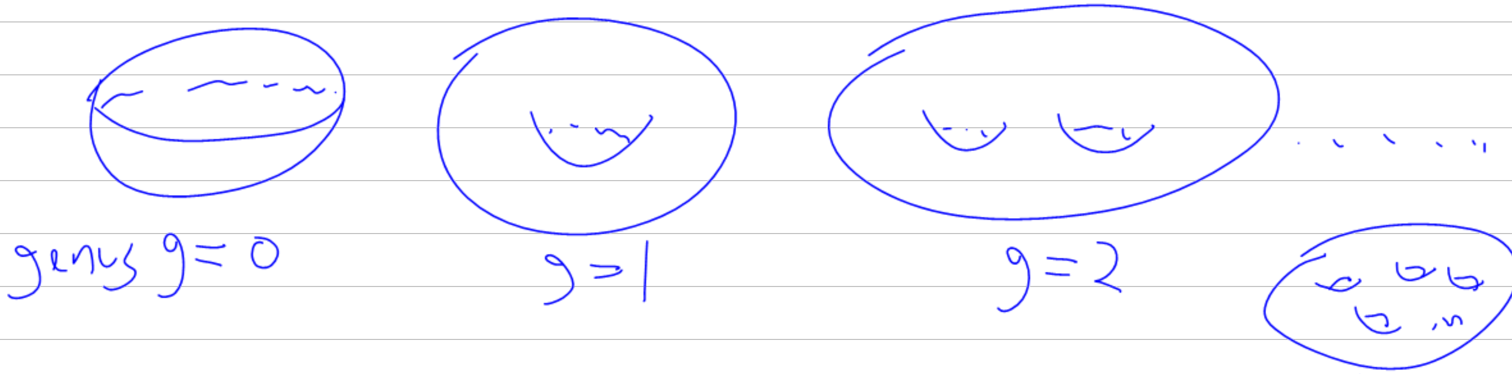






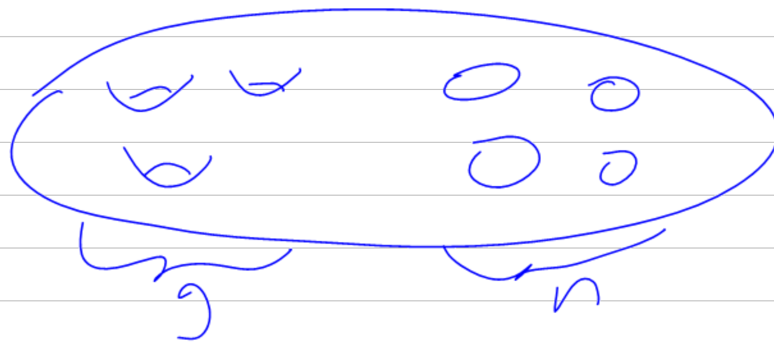
$\simeq \Sigma_{g,n}$

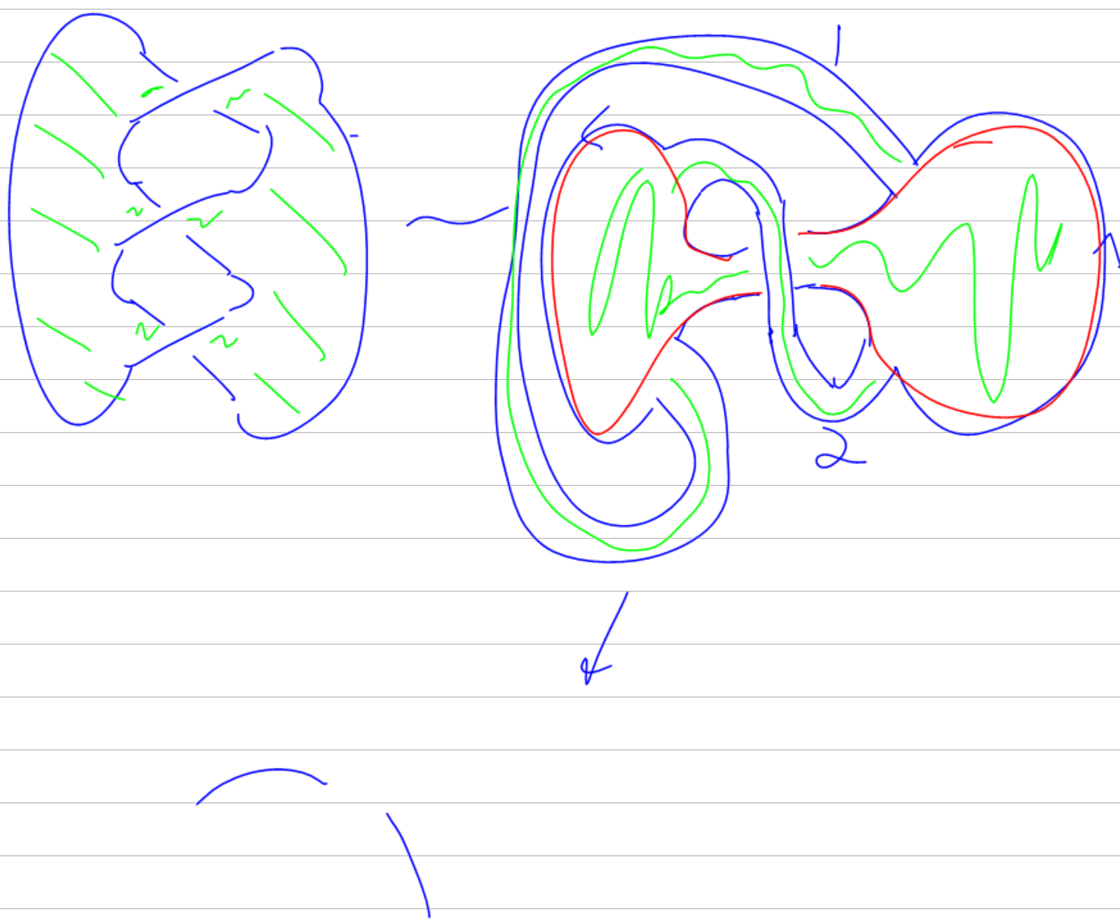
The classification of connected orientable sufaces: Every connected orientable suface is homeomorphic to one of the following:

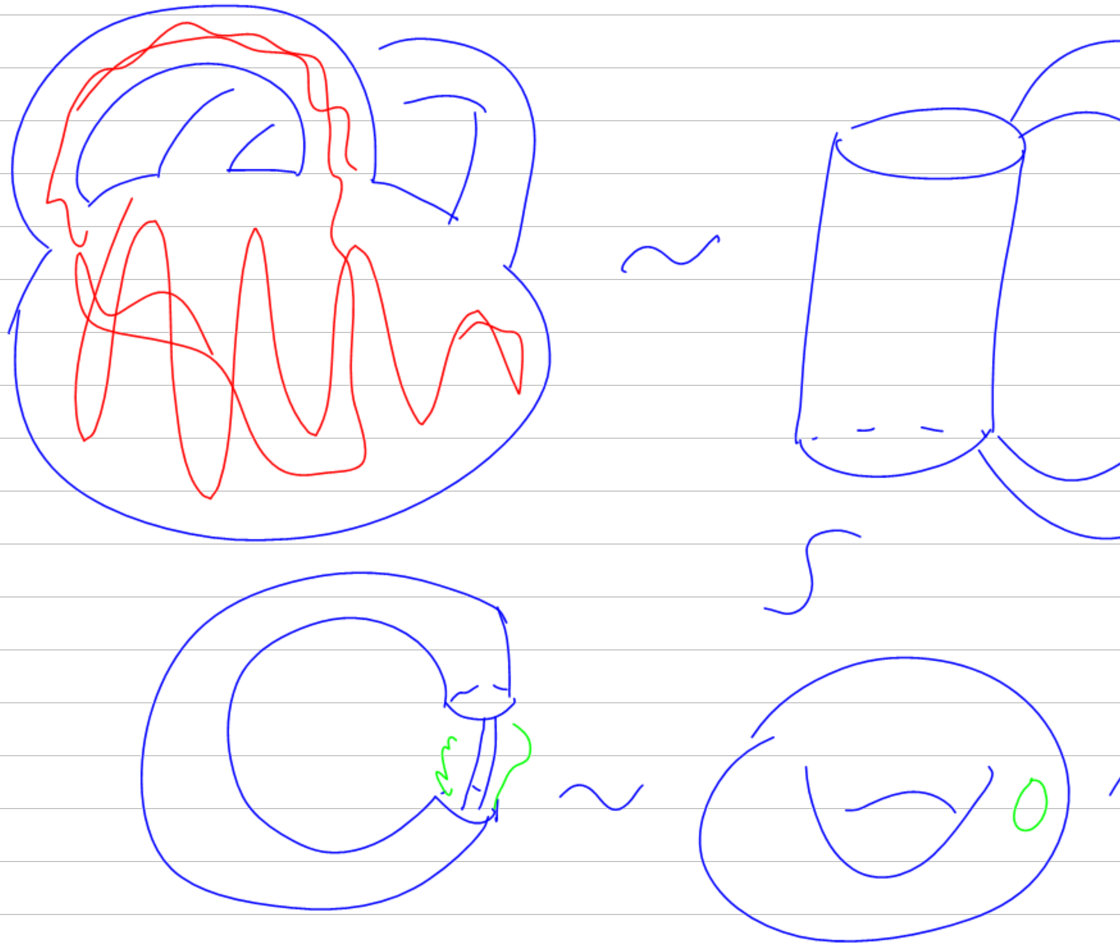


Classification of connected orientable surfaces w/ bndry: Every such surface is homeo. to a $\Sigma_{g,n}$

$\Sigma_{g,n}$







$\sim \Sigma_{1,1}$

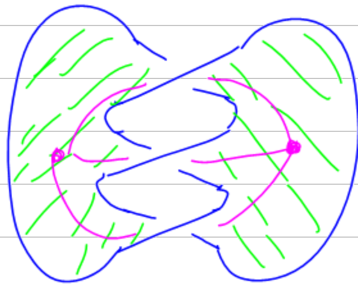
$\boxed{\text{Everything is smooth}}$

Def A Seifert surface for an oriented link K is a connected oriented surface $\Sigma \subset \mathbb{R}^3$ s.t. $\partial\Sigma = K$.

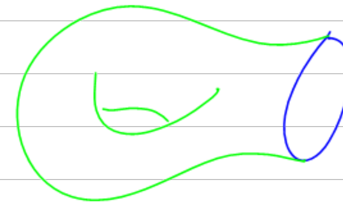
Example

$v=2 \quad e=3$

$=$



topologically this is a $\Sigma_{g,n}$ with $g=1$, $n=1$



$\{\Sigma_{g,n}\}$

$$\chi(X) = \sum (-1)^r \dim H_r(X)$$

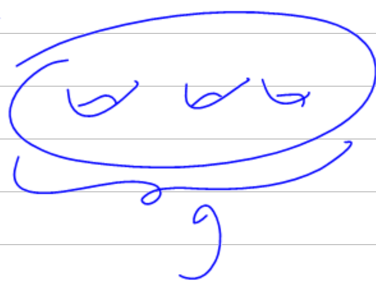
$$= \sum (-1)^r \dim C_r(X)$$

in 2D $= V - e + F$ (verts, edges, faces)

in 1D $= V - e$

homotopy invts.

$C_0 =$, $C_1 =$, $C_2 =$

$\chi($  $) = 2-2g$

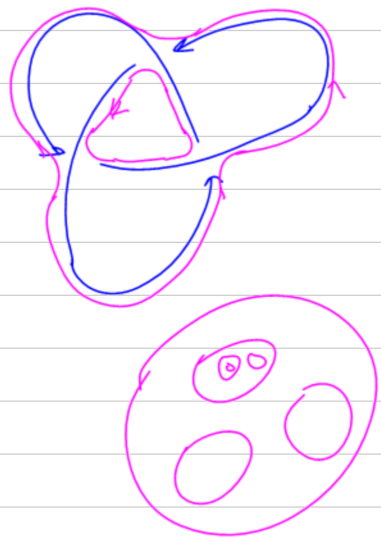
$$\chi(\Sigma_{g,n}) = 2-2g-n$$

$$\chi = 2-3 = -1 = 2-2g-1$$

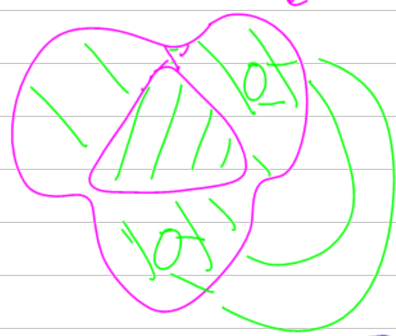
$$g=1$$

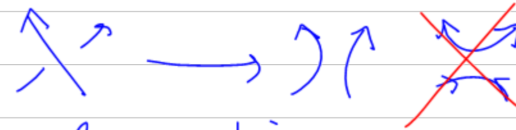
Thm Every K has a Σ.

Pf 1 "use Seifert cycles"

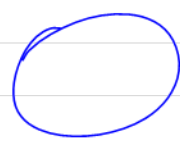
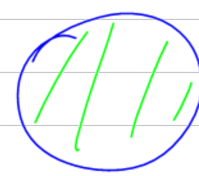


Seifert cycles



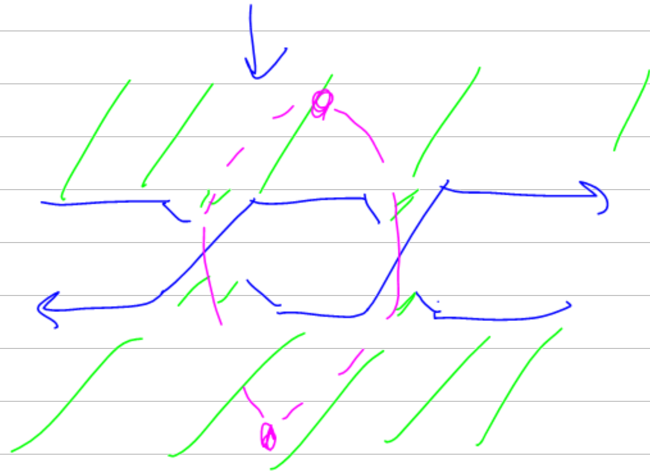
1. 
2. Place disks in $\mathbb{R}^3$ w/ boundaries the Seifert cycles
3. Wherever there was a xing, put a twisted band

Pf 2 By induction on unknotting number = # of xing you have to flip to get to the unknot.

$u = 0$ $K =$  $\Sigma =$ 

If $K = \partial\Sigma$ and K' is obtained from K by flipping one xing, then $\exists\, \Sigma'$ s.t. $\partial\Sigma' = K'$.

K: 

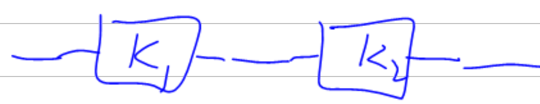


$\square$

Def $g(K)$ = "the genus of K" is the minimal genus of a s.s. for K.

Obs. $K = O \iff g(K) = 0$

$g(K) \le u(K)$ $\quad O + K = K$



Thm $g(K_1 + K_2) = g(K_1) + g(K_2)$

Cor $\mathcal{K}$ isn't a group as if $K_1 + K_2 = O$ then $K_1 = K_2 = O$

Pf $\Downarrow$ $g(K_1+K_2)=0$ $= g(K_1)+g(K_2) \Rightarrow g(K_1)=g(K_2)=0$ $\Uparrow$

Def K is "prime" if $K_1 + K_2 = K$

$\Rightarrow$ either $K_1 = 0$ $K_2 = K$
or $K_2 = 0$ $K_1 = K$.

Cor 2 A knot of genus 1 is prime. In particular the trefoil is prime.

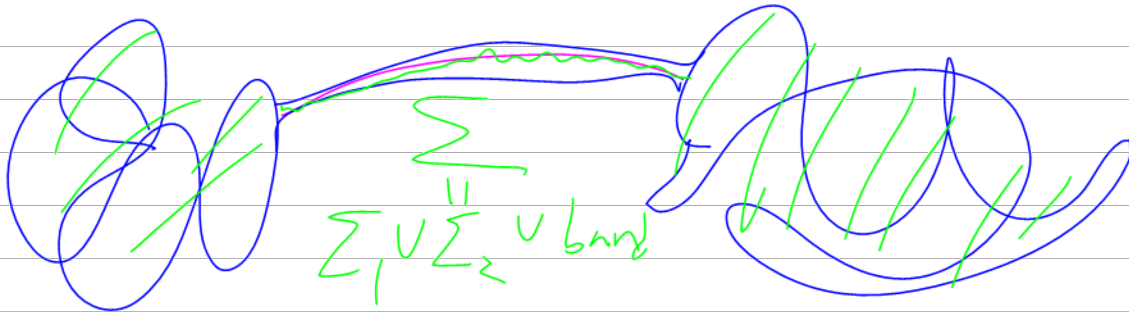
Cor 3 Every K can be decomposed as a sum of primes. [Later: uniquely]

$$K = K_1 + K_2$$

Pf of Thm (modulo ~~all~~ lots of diff geom & top of $\mathbb{R}^2$.

$$g(K_1 + K_2) \leq g(K_1) + g(K_2)$$

Suppose $\partial \Sigma_i = K_i$ $g(\Sigma_i) = g(K_i)$



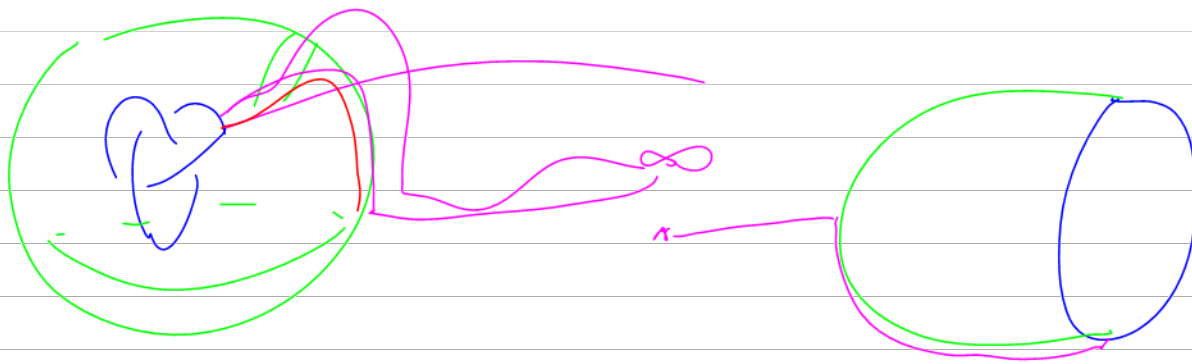
$$\Sigma = \Sigma_1 \cup \Sigma_2 \cup \text{band}$$

?

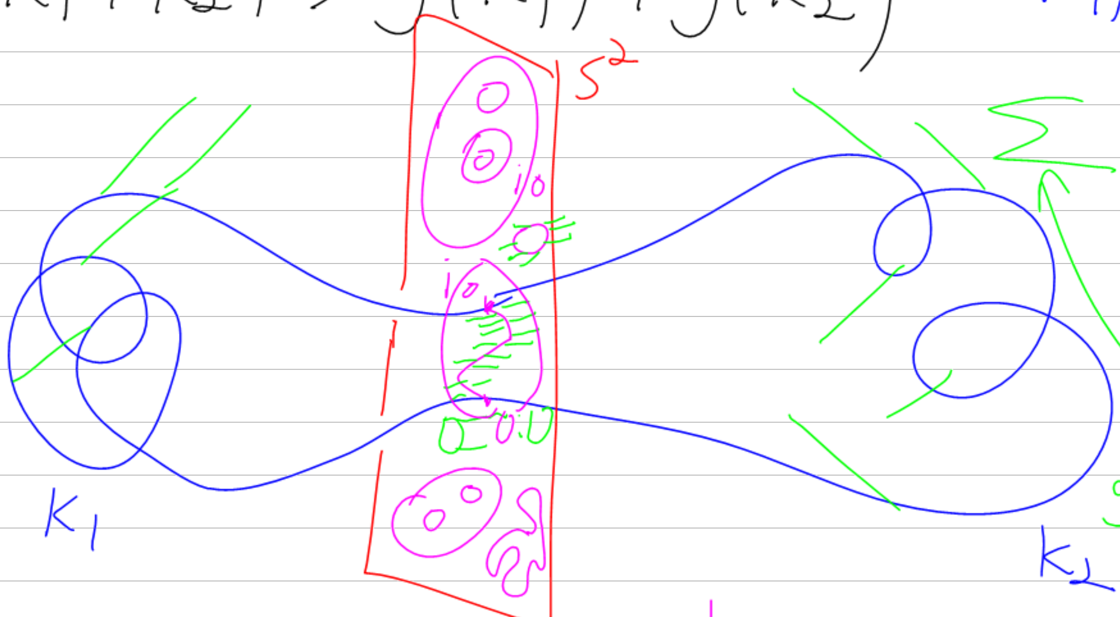
Thm $g(K_1+K_2)=g(K_1)+g(K_2)$

Pf of thm (Modulo all about diff geom & the topology of $\mathbb{R}^2$)

$g(K_1+K_2)\leq g(K_1)+g(K_2)$ [Easy, yet...]



$g(K_1+K_2)\geq g(K_1)+g(K_2)$ $\quad K_{1,2}\subset S^3$

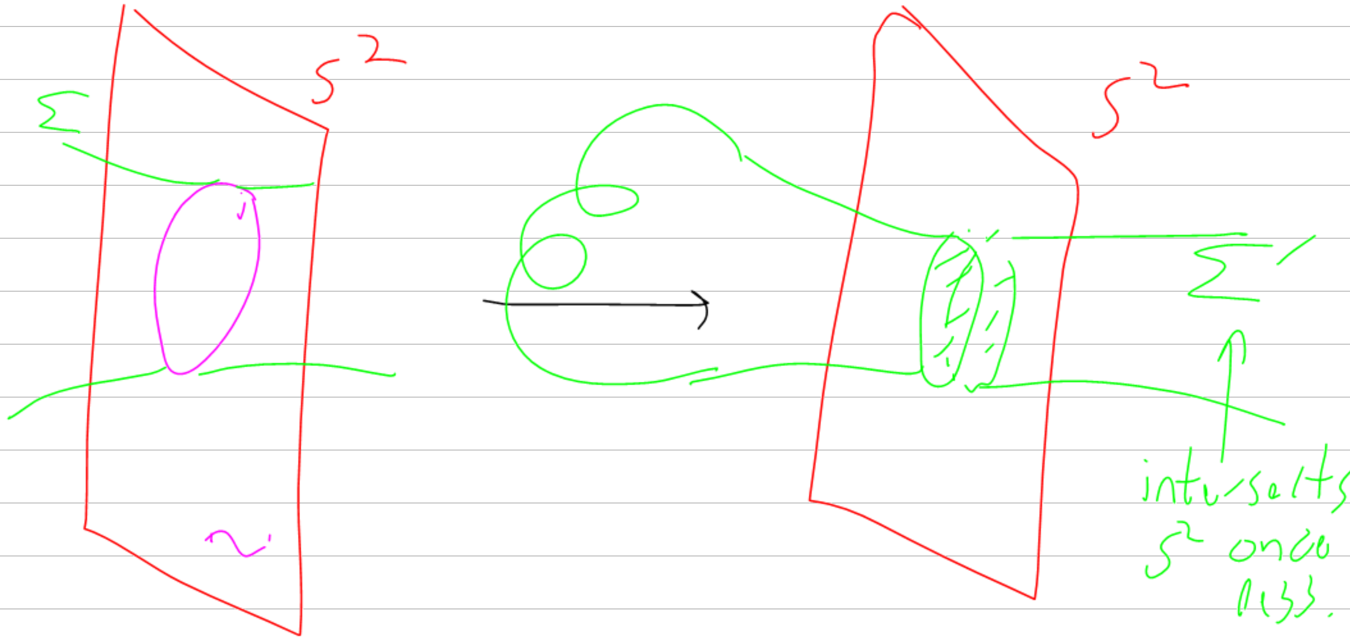


Σ a s.s. for K_1+K_2

WLOG, $\Sigma \pitchfork S^2$

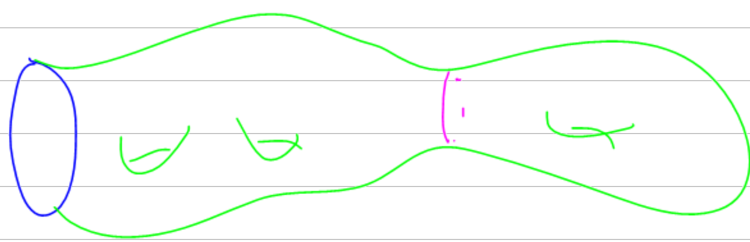
genus minimal $g(\Sigma)=g(K_1+K_2)$

Pick an inner most C in $\Sigma\cap S^2$



Case 1 Σ' is no longer connected

$\to$ Keep only the component of Σ' that touches $K_1 + K_2$, call it Σ''



$g(\Sigma'') \leq g(\Sigma)$

So as Σ was gen-min,

$g(\Sigma'') = g(\Sigma)$

Case 2 Σ' is connected

$g(\Sigma') < g(\Sigma) \Rightarrow\Leftarrow$

Iterate until there are no circles in $\Sigma \cap S^2$.

Found Σ_1 & Σ_2 w/ $\partial \Sigma_i = K_i$

$$g(K_1)+g(K_2) \leq g(\Sigma_1)+g(\Sigma_2) = g(\Sigma) = g(K_1+K_2)$$

__Thm__ Suppose $P+Q = \underbrace{K_1}_{P+L} + K_2$ where P is prime. Then,
either $\exists L$ s.t. $P+L = K_1$ & $Q = L+K_2$
or $\exists L$ s.t. $P+L = K_2$ & $Q = L + K_1$

__Cor 1__ If $P+Q_1 = \underbrace{P}_{K_1}+\underbrace{Q_2}_{K_2}$ then $Q_1 = Q_2$

__pf__ By Thm either

$$\exists L \quad \underbrace{P+L=P}_{L=0} \quad \& \quad Q_1 = L+Q_2 \Rightarrow Q_1 = 0+Q_2 = Q_2$$

or $\exists L \quad P+L=Q_2 \;\&\; Q_1 = L+P$

$$\Rightarrow Q_1 = Q_2 \qquad \square.$$

<u>Cor 2</u> If $P_1 + \dots + P_n = P_1' + \dots + P_{n'}'$ where P_i & P_i' are primes, $n \leq n'$ then $n = n'$ & (P_i) are a perm of the (P_i').

<u>Pf</u> By induction on n.

If $n=0$,

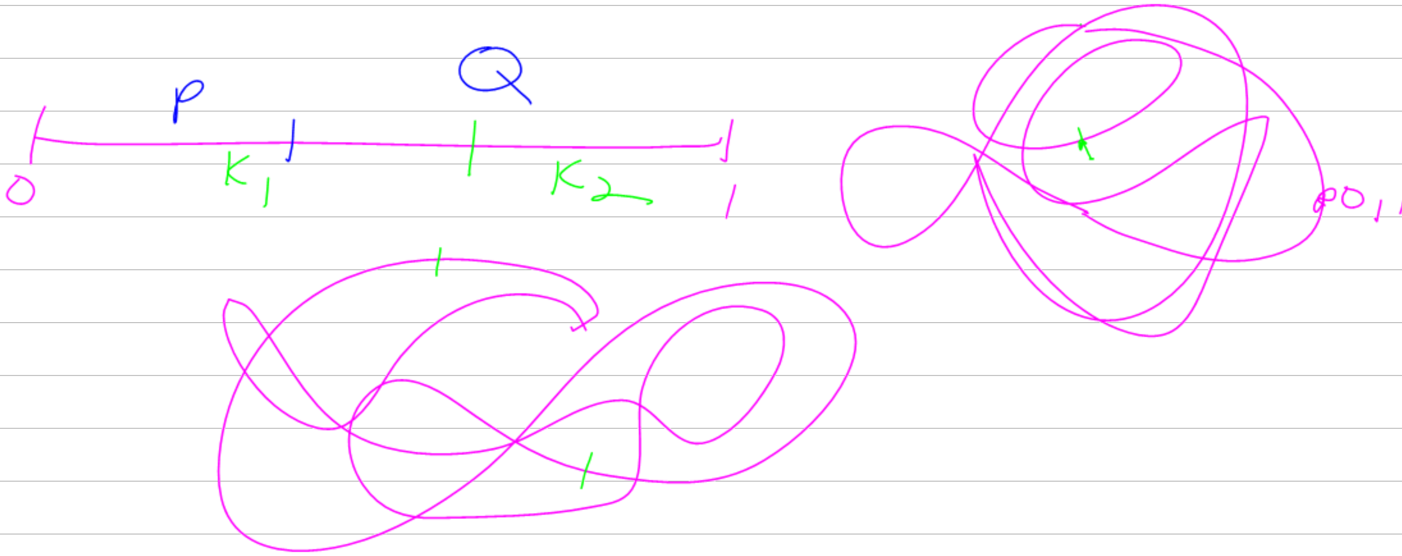
$$O = P_1' + \dots P_{n'}' \Rightarrow n' = 0,$$

otherwise

$$\underbrace{P_1}+ \dots + P_n = P_1' + \dots + P_{n'}'$$

P_1 divides P_i' for some i, so $P_1 = P_i'$,

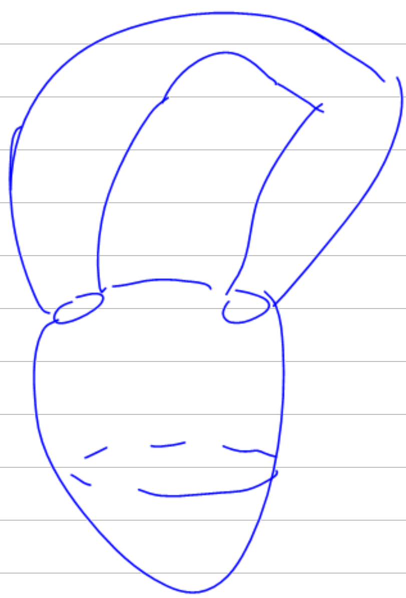
Cancel P_1 & P_i' from the two sides & use induction. $\square$.


P
Q
0
K1
K2
1

Thm If $P+Q = K_1 + K_2$ w/ P prime then
either $K_1 = P+L$ & $Q = L+K_2$
or $K_2 = P+L$ & $Q = L+K_1$

We started class all wrong to prove this!

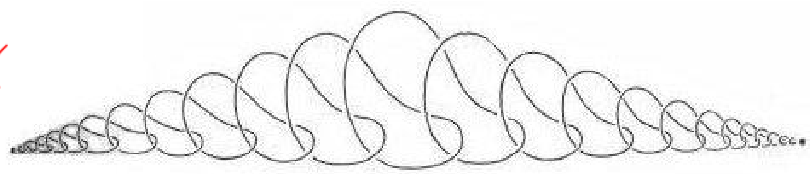
Get a feel for funny $\mathbb{R}^3$.



Topological Pathologies in R^3

An embedding of an interval in R^3 whose complement is not simply connected:

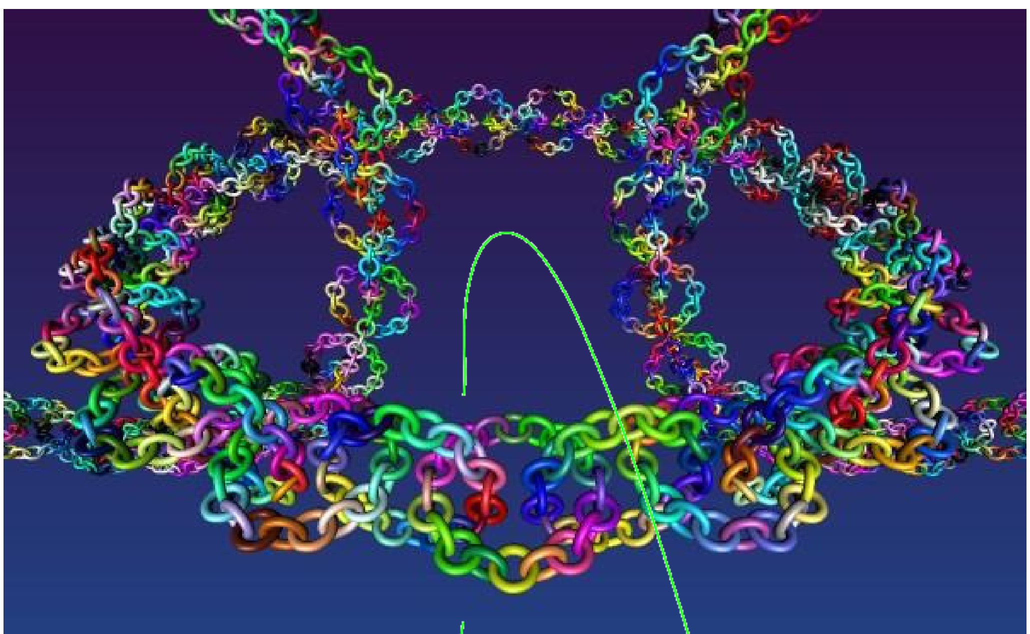
$\subset \mathbb{R}^3$ $\not\simeq$



See Hocking and Young's *Topology* pp. 176-177.

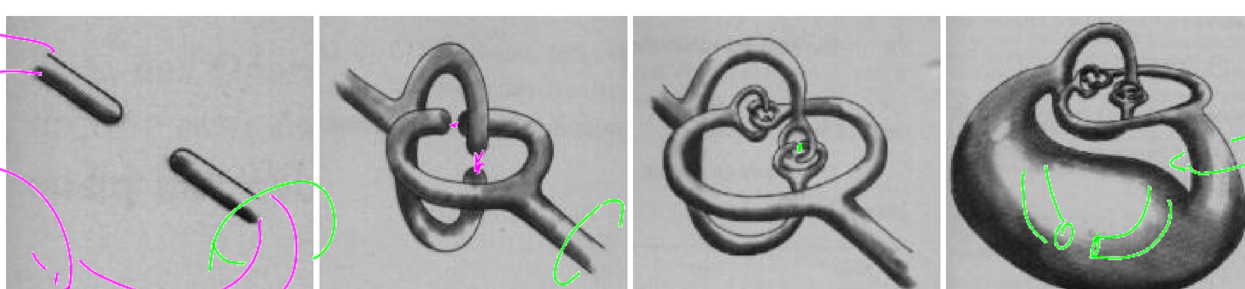
See http://www.math.ohio-state.edu/~fiedorow/math655/Jordan.html.

Antoine's necklace - an embedding of a Cantor set in R^3 whose complement is not simply connected:



See http://www.cs.ubc.ca/nest/imager/contributions/scharein/various/AntoinesNecklace.html.

The Alexander horned sphere - a continuous embedding of a ball in R^3 whose complement is not simply connected:



See http://users.math.uni-potsdam.de/~oeitner/EIGENES/RAEUME/hornsph.htm.

Yet in $\mathbb{R}^2$ if γ is a closed cont. simple loop, then $S^2 \setminus \gamma$ has two connected comps both homeomorphic B^2

<u>Alexander-Schöflies thm</u> a smooth S^2 in S^3 divides S^3 into two components each diffeomorphic to a ball.

pf: 2. Allan Hatcher: "3 manifolds ..."
1. DBN → students → Morton-Ferguson

<u>Cor</u> There are no knotted smooth S^2 in S^3, i.e. if S is an S^2 in S^3 and if S' is equatorial S^2 of $\mathbb{R}^3$ then

$$(S^3, S) \sim (S^3, S')$$

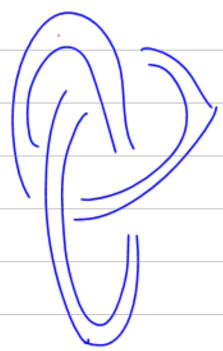
ϕ

$S^1 \times S^1 = T^2$

Yet there are knotted tori in S^3

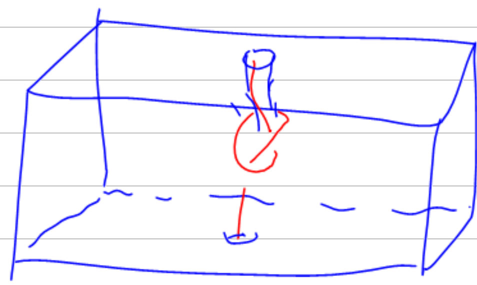
A T^2 in $\mathbb{R}^3$ still divides $\mathbb{R}^3$ (smooth)

into two connected comps one containing ∞ "~~out~~ outside" & one that doesn't "inside"



inside is ~~a std~~ homeo to $B^2 \times S^1$

outside is not homeo to $B^2 \times S^1$



If T is a smooth T^2 in S^3 then either the inside or the outside is $B^2 \times S^1$.

---

4 equiv. defn of knot in $\mathbb{R}^3$.

A smooth simple curve in $\mathbb{R}^3$ modulo

a. smooth homotopies of such.

b. "Ambient isotopies"

$\gamma_0 \sim \gamma_1$ if $\exists\, H_t : \mathbb{R}^3 \to \mathbb{R}^3$ of diffeomorphisms s.t. $H_0 = Id$

$$H_1 \circ \gamma_1 = \gamma_0$$

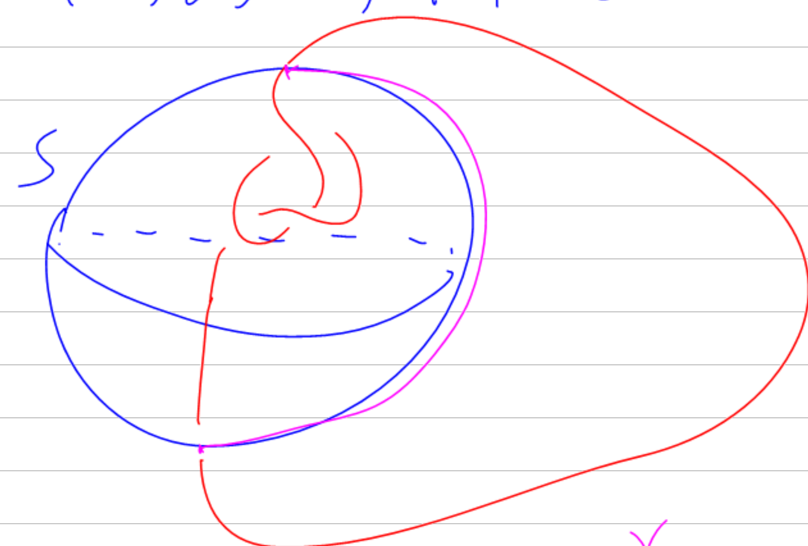
c. $\gamma_0 \sim \gamma_1$ if $\exists\, H: \mathbb{R}^3 \circlearrowleft$ $\underbrace{\text{s.t.}}_{\text{orientation pres. diffeomorphism}}$

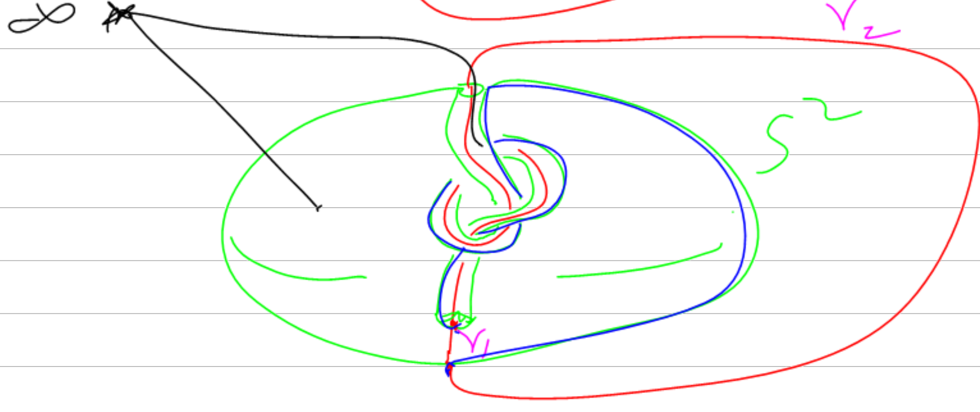
$$H \circ \gamma_1 = \gamma_0$$

4. Planar curves / R123.

Thm These are all equiv.

Def γ is prime if whenever an S^2 S intersect γ exactly twice, then ~~one of the sides is trivial~~ it is non-trivial

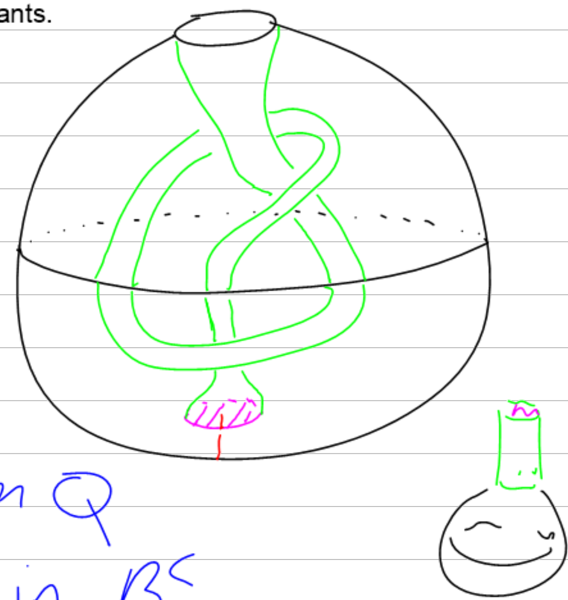




(B_{inside}, γ_1) non-triv.

$(B_{outside}, \gamma_2)$ trivial

Hour 31, Friday November 27: Decomposition into primes, end. Next: Braids and variants.
Should we have a "HW discussion" class?
HW9 is online!



Thm If $P+Q=K_1+K_2$ w/ P prime then

either $K_1=P+L$ & $Q=L+K_2$

or $K_2=P+L$ & $Q=L+K_1$

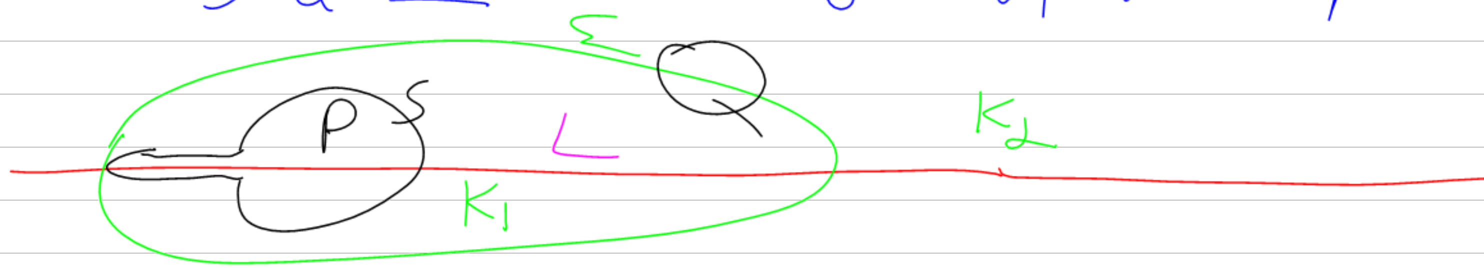
Pf γ The curve

B a ball separating P from Q

$S=\partial B$ P is in B & Q in B^c

Σ a sphere separating K_1 & K_2

If S & Σ are disjoint, thm is proven

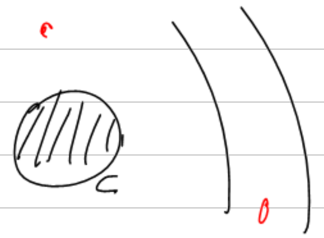


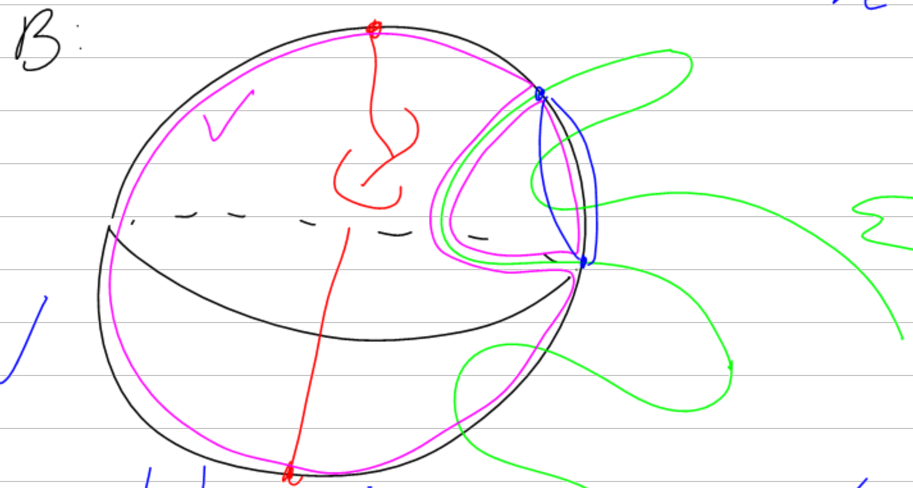
otherwise Σ wears a pajama:

Color Σ black where it is inside B & white otherwise. Bndries are $\Sigma\cap S$

$\Sigma\cap S$ is a finite collection of smooth closed curves on Σ.

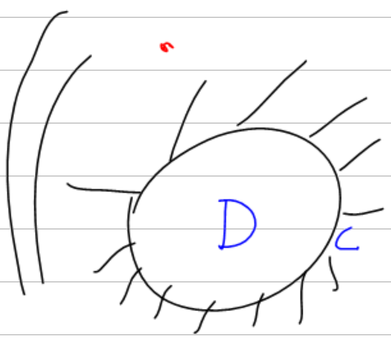
If there's a black island on Σ, w/o red dot.

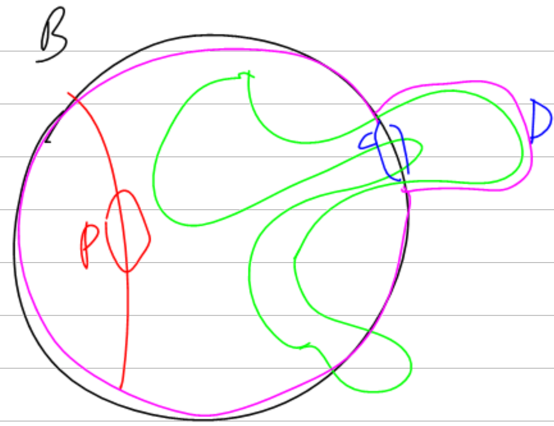


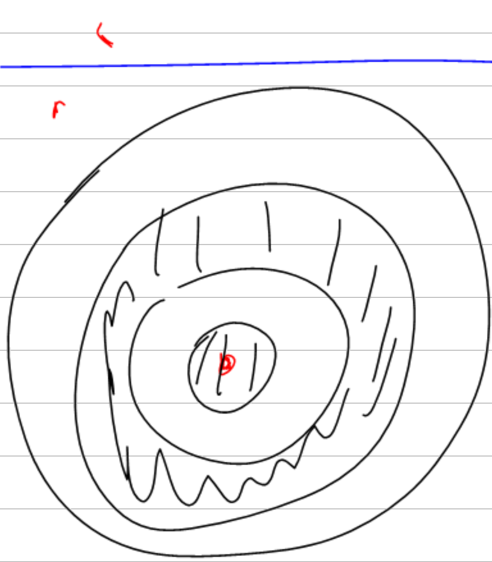


Replace B by V

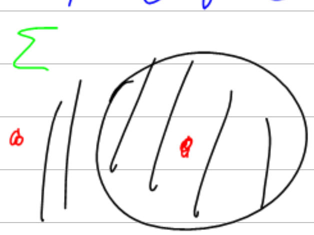
If there's a white disk on Σ, w/o red.



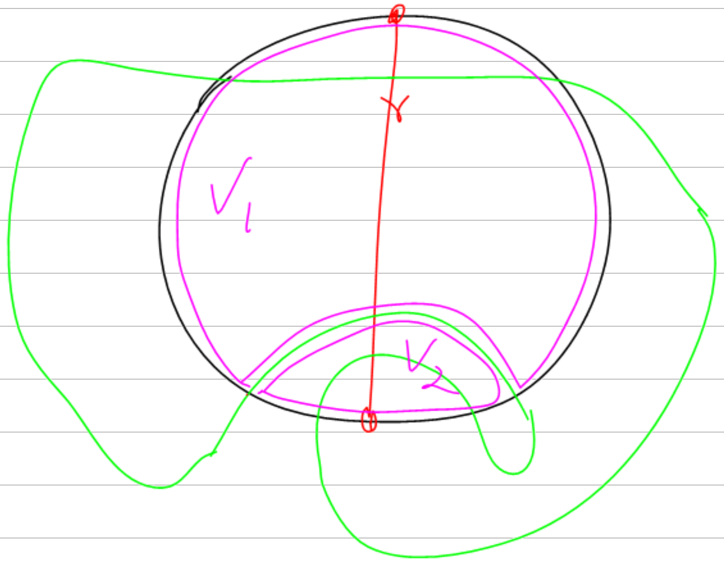




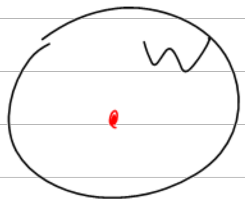
Next case: A black disk w/ a red dot inside:



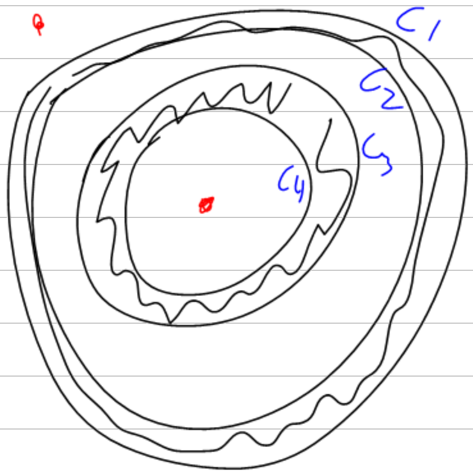
B:



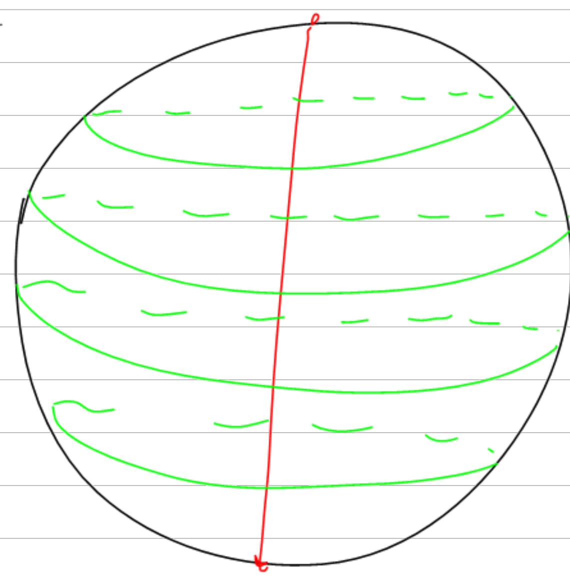
p lies in V_1 or V_2
If p is in V_j replace B by V_j

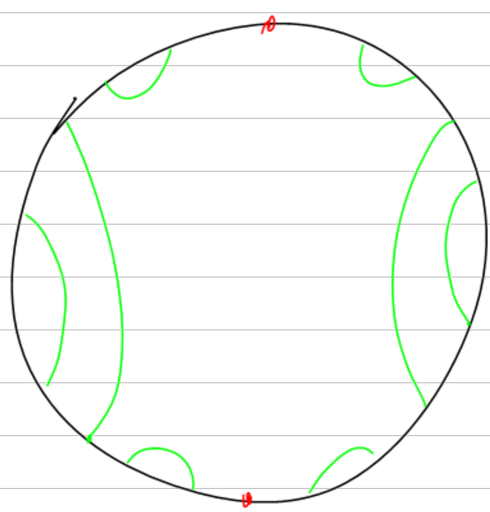


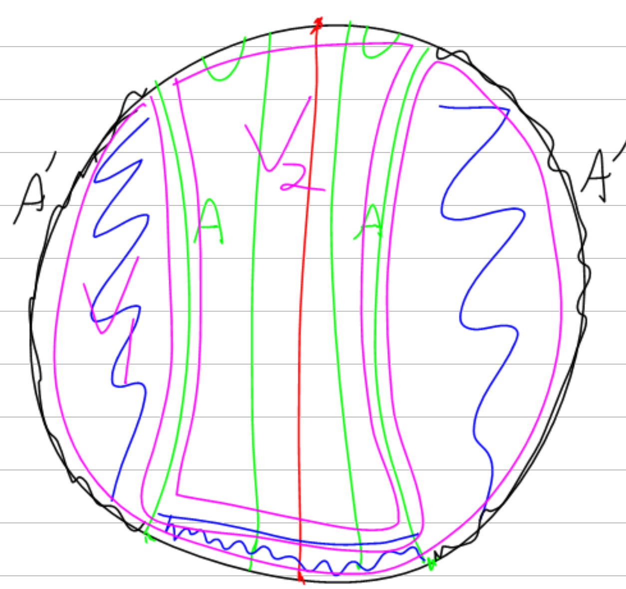
I dunno.



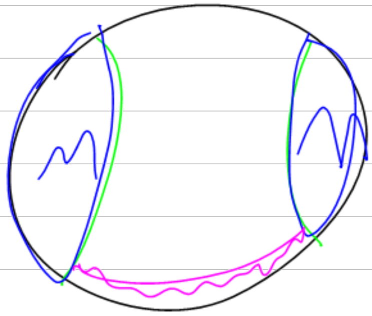
B:



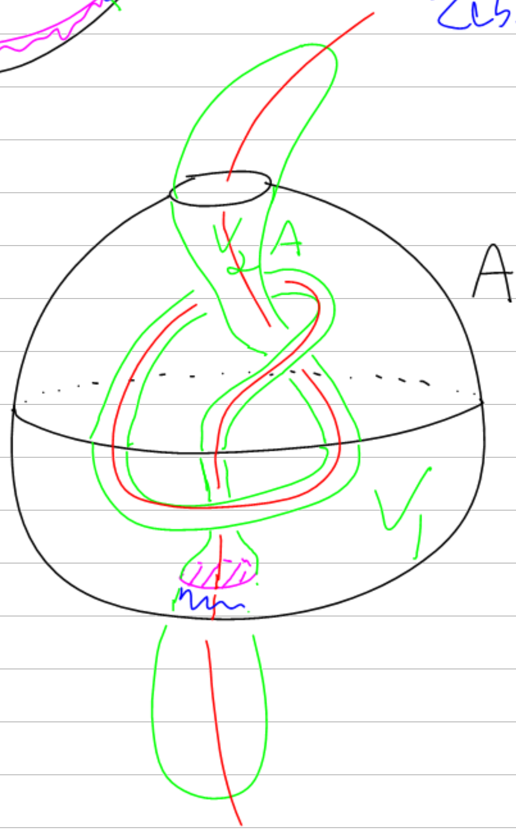




One of the V_j contains p replace

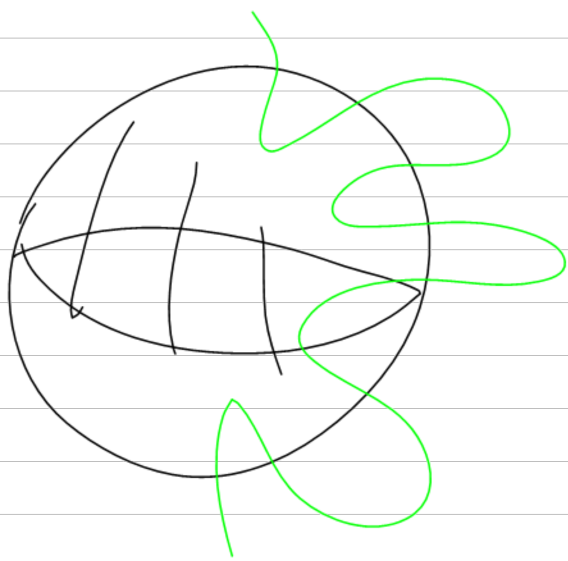


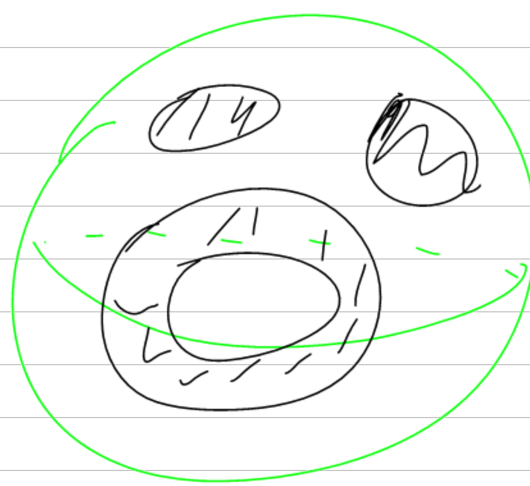
B by that V_1 and we're down one zebra stripe.

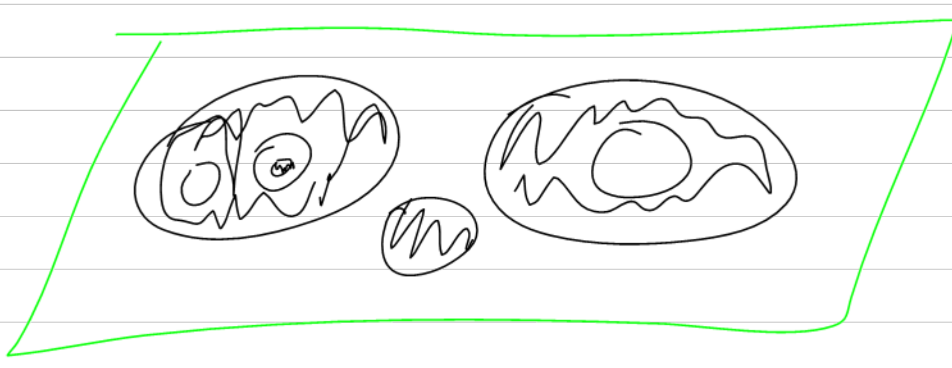


See Lickorish p 19–21

$\square$



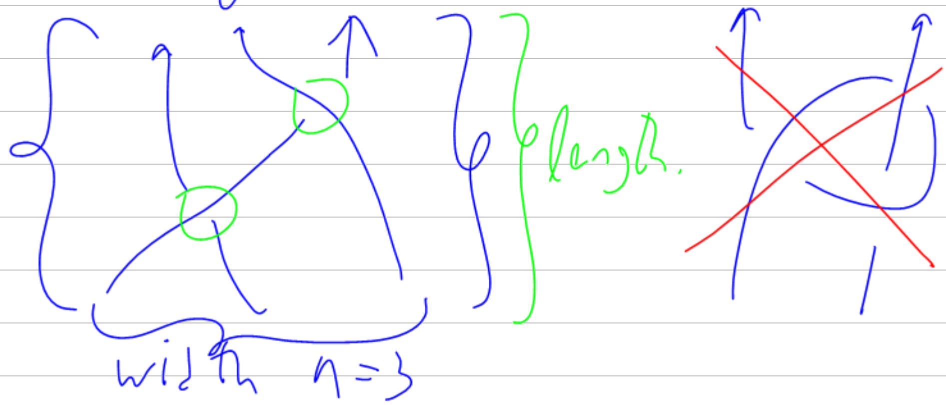


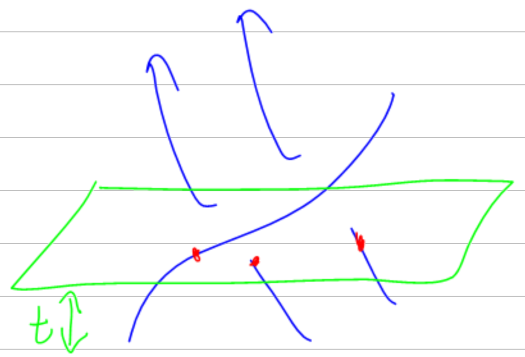


Hour 32, Monday November 30: Braids and knots, combing.
Should we have a "HW discussion" class?

B_n = The braid group on n strands:

$$B_n = \left\{ \quad \right\} / \text{length.}$$

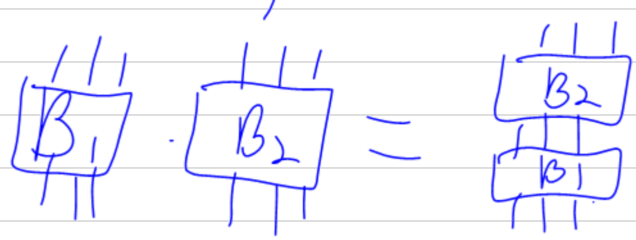


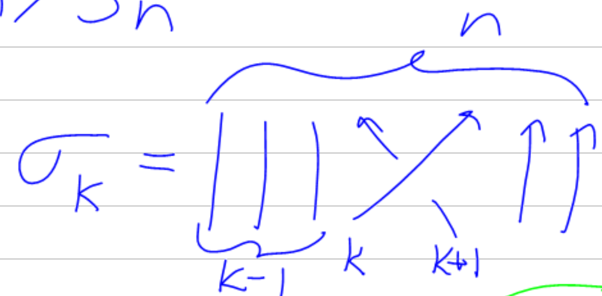


Braid $\mapsto \beta : [0,1] \to \tilde{C}_n$

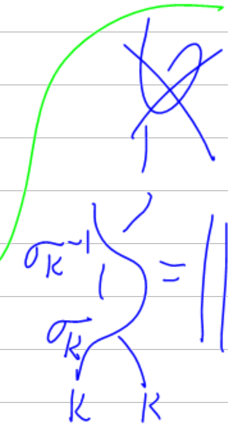
$$C_n = \left\{ (z_1 \ldots z_n) \in \mathbb{C}^n : i \neq j \Rightarrow z_i \neq z_j \right\}$$

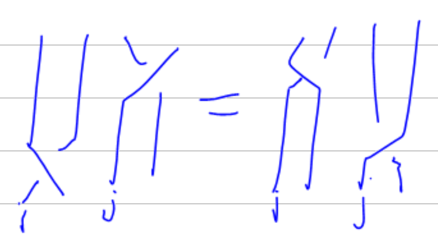
$$B_n = \pi_1(\tilde{C}_n) \qquad \tilde{C}_n = C_n / S_n$$





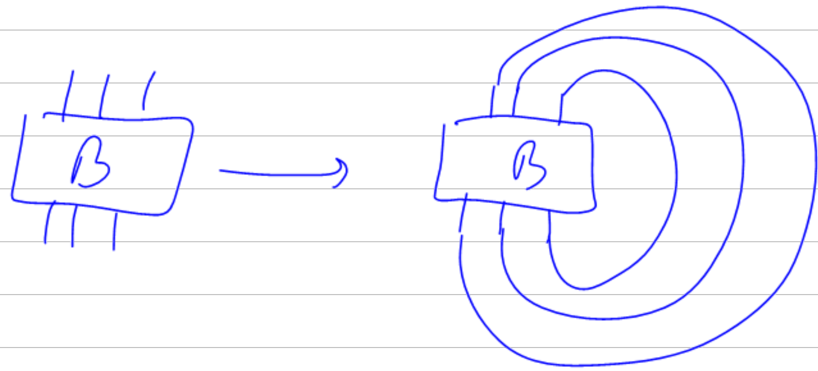
$$B_n = \left\langle \sigma_1 \ldots \sigma_{n-1} : \begin{array}{l} \sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1 \end{array} \right\rangle$$



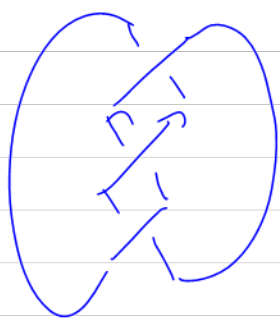


Alexander

Map $A : B_n \longrightarrow \left\{ \begin{array}{l} \text{knots} \\ \text{\& links} \end{array} \right\}$

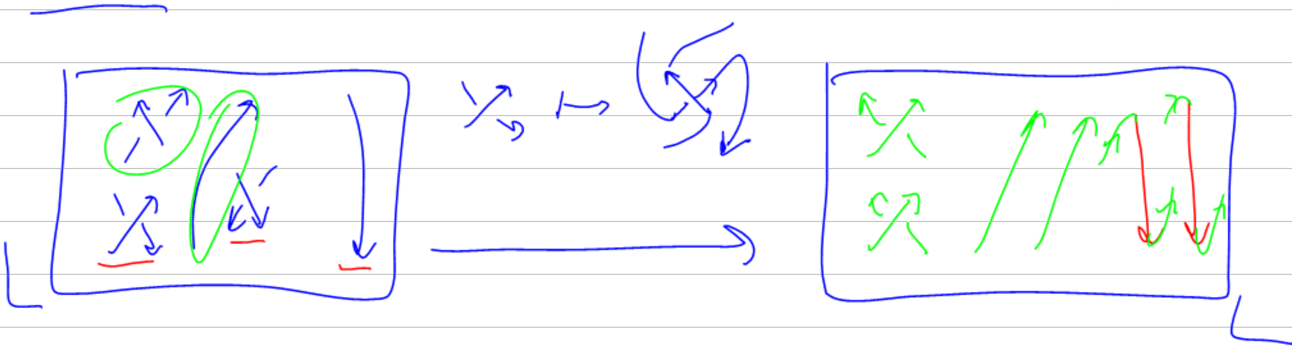


$\sigma_1^{\,3} \in B_2 \overset{A}{\longmapsto}$ Trefoil

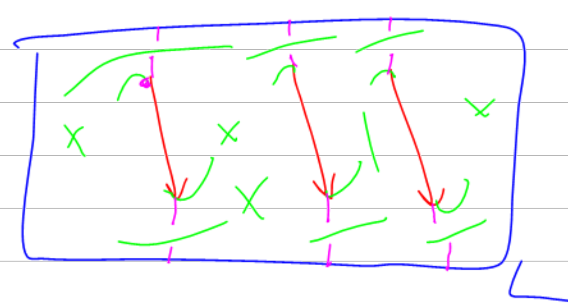


Alexander's thm:
A is onto.
Any knot or link is a closure of a braid.

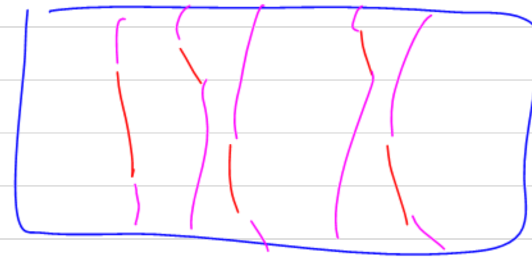
Pf schematic pictorial pf



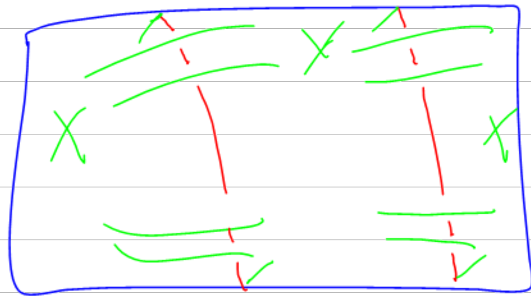
Connect all mins to bottom & all maxes to top with purple lines that run under everything, &

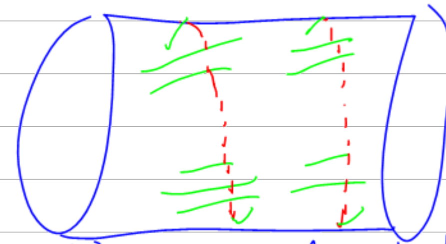


s.t. red & purple never cross.



pull along purple





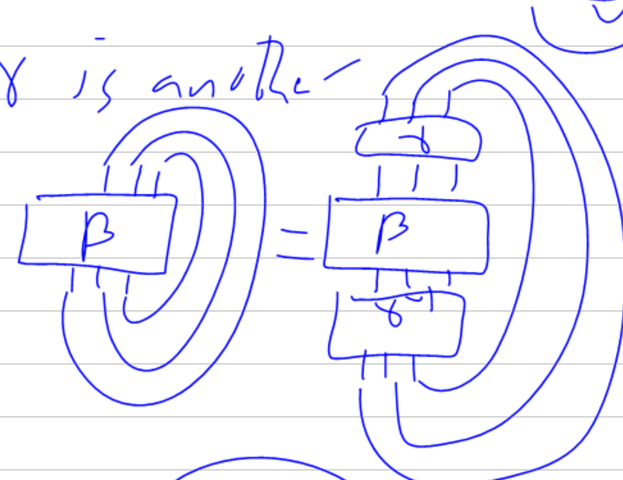
This is a closed braid

<u>Markov's Thm</u> β_1 & β_2 are braids, then

$A(\beta_1) = A(\beta_2)$ iff β_1 & β_2 differ by a sequence of moves of the following kinds:

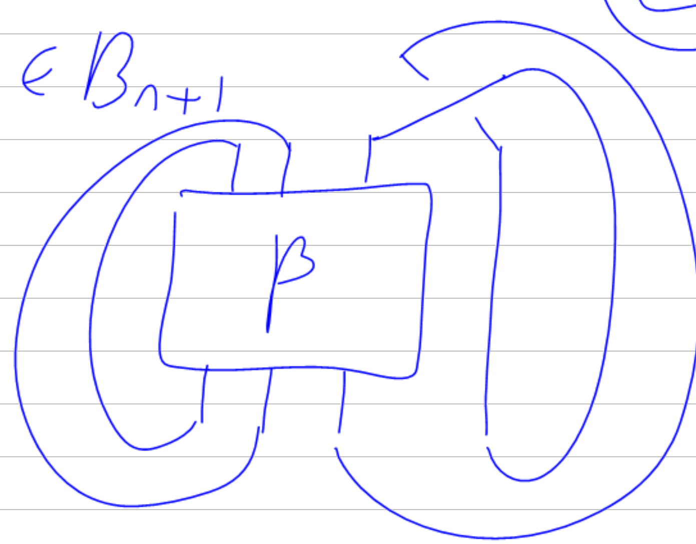
0. R-moves on braids

1. $\beta \mapsto \gamma^{-1}\beta\gamma$ where γ is another braid.



2. if $\beta \in B_n$ then

$B_n \ni \beta \sim \beta \cdot \sigma_n^{\pm 1} \in B_{n+1}$



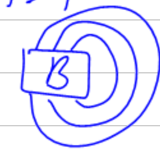
Harder if $A(\beta_1) = A(\beta_2)$

$\Rightarrow \beta_1 \sim \beta_2$

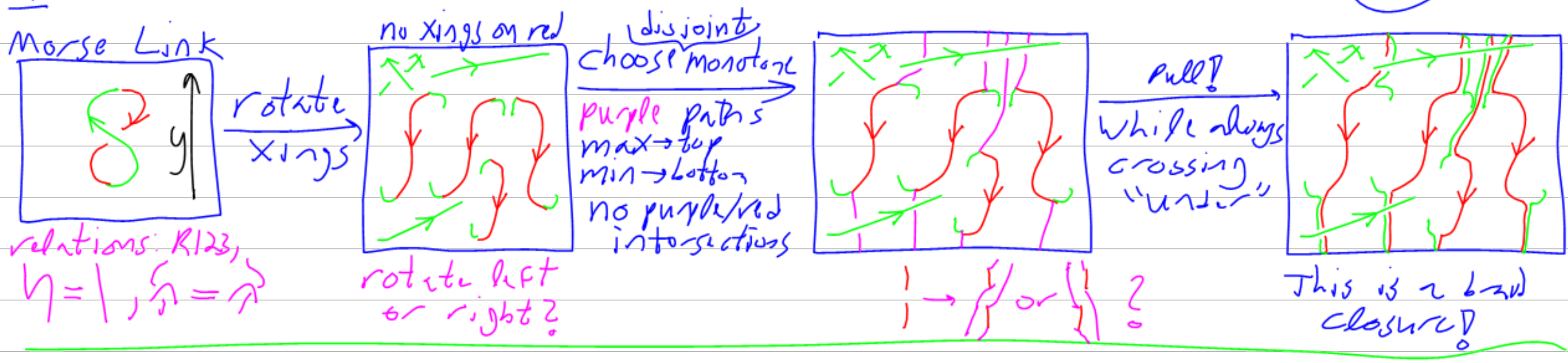
Hour 33, Wednesday December 2: Combing braids.
IOU a correction for the last bit of the proof of unique factorization.
HW10 on web by Thu midnight!

$C_n = \{z \in \mathbb{C}^n : z_i \neq z_j \text{ for } i \neq j\}$ $\quad \tilde{C}_n = C_n / S_n$

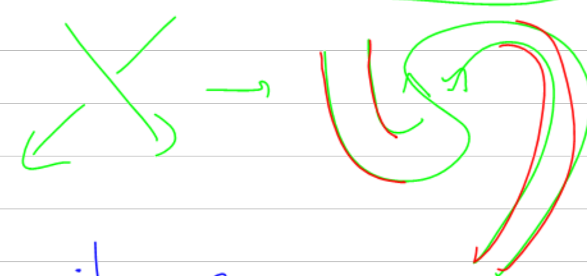
$$B_n := \pi_1(\tilde{C}_n) = \left\langle \sigma_i,\ 1 \le i \le n-1 : \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j|>1 \end{array} \right\rangle$$

Alexander's Thm. Every knot/Link is a braid closure: 

Pf



Comments 1. Computationally inefficient.

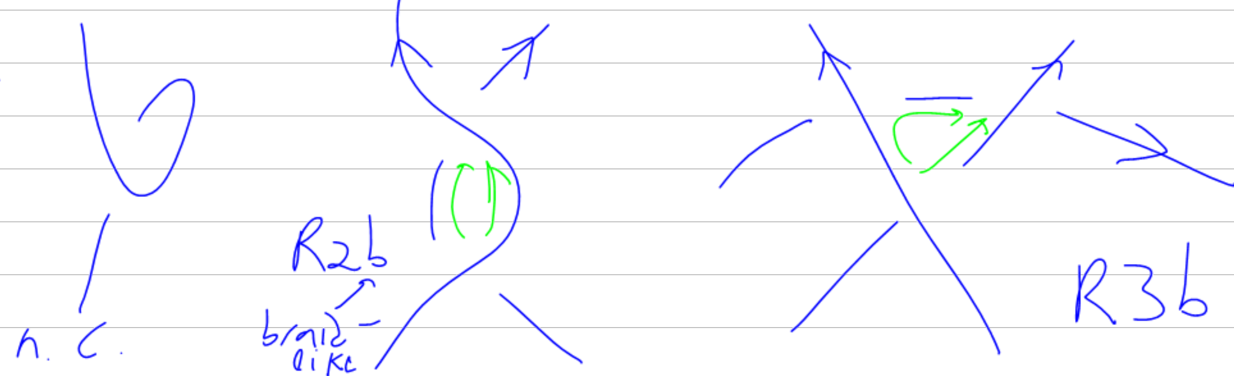


knot of complexity n $\longleftarrow$ braid of complexity $\cancel{n^2}$ $n^{3/2}$

locally ✓

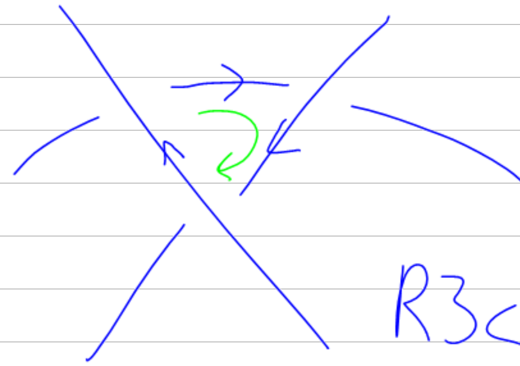
Open problem. Find a sequence K_n of n-xing knots such that the minimal length $l(K_n)$ of a braid whose closure is K_n grows faster than Cn.

2.



R2c

cyclic



R3c

knots $\qquad$ diagrams

$$K = \mathcal{D} \big/ R1 \quad R2_{b\&c} \quad R3_{b\&c}$$

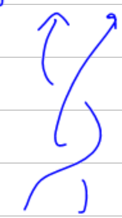
$$K' = \mathcal{D} \big/ R1, R2b, R3b$$

Markov's thm implies that K' contains a copy of K.

* There are elements of K' that are not braid closures!
* If you regard knots as braid closures, there are fewer moves to check.

  I know just one example where this is useful!

$$(*) \quad 1 \to PB_n \to B_n \xrightarrow{\pi_1(\tilde{C}_n)} S_n \to 1$$

("Pure braids": braids that induce the identity perm. E.g.  $=\sigma_1^2$

$PB_n = \pi_1(C_n)$

1. studying B_n & PB_n is more or less the same. $\beta_1 \cdot \beta_2^{-1}$
2. Yet (*) does not "split".

Aside on split exact sequence:

what means where $p\circ s = \mathrm{Id} = s // p$

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] & C \arrow[r] \arrow[l, bend left, "s"] & 1 \end{tikzcd} \quad (\text{Grps})$$

$b \qquad c = pb$

0. $B \supset A$ $(B \supset iA)$ $\quad B \supset C$ $(B \supset sC)$

1. $B = A \cdot C$ (not $B = A \times C$)

Also need $A \cap B = \{1\}$?!

$b = \underbrace{\alpha}\, s(p(b)) \quad \alpha = b\cdot(spb)^{-1} \quad p\alpha = 1 \quad$ So $\alpha \in iA$.

2. $A \triangleleft B$ as $A = \ker p$.

So $B = A \rtimes C$. $\quad B_n = \cancel{PB_n \rtimes S_n}$

Hour 34, Friday December 4: Combing braids.
IOU a correction for the last bit of the proof of unique factorization.
HW10 on web!

$$1 \to PB_n \to B_n \xrightarrow{\sigma} S_n \to 1$$

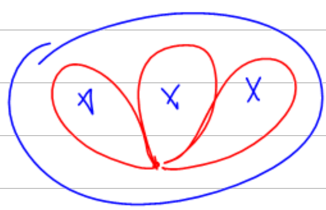
1. studying B_n & PB_n is more or less the same.
2. Not split?

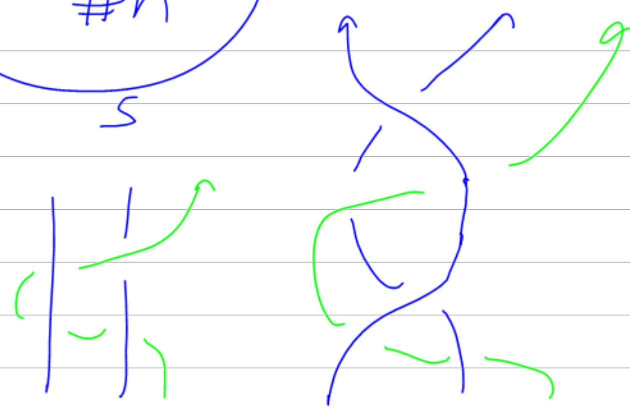
Aside on split exact: $1 \to A \xrightarrow{i} B \underset{s}{\overset{p}{\rightleftarrows}} C \to 1$ $\quad s/p = Id_C$

1. $B = AC$ and $A \cap C = \{1\}$ $\quad$ So
2. $A \triangleleft B$ hence C acts on A. $\quad B = A \rtimes C$

$\pi_1(D \setminus \text{n-1 pts})$

$$\begin{tikzcd} 1 \arrow[r] & F_{n-1} \arrow[r] & PB_n \arrow[r, "d", "\text{drop strand } \#n"'] & PB_{n-1} \arrow[r] \arrow[l, bend left, "s"] & 1 \end{tikzcd}$$





$$PB_n = PB_{n-1} \rtimes F_{n-1}$$

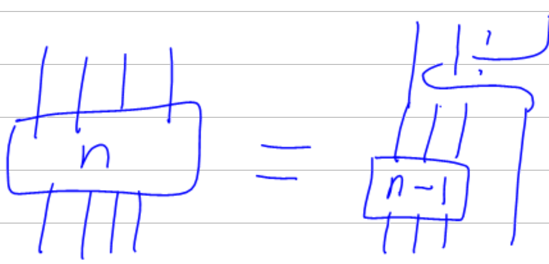
$$= (PB_{n-2} \rtimes F_{n-2}) \rtimes F_{n-1}$$

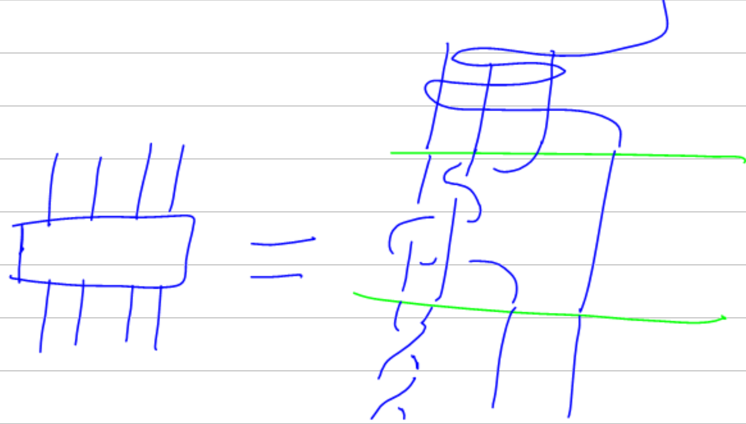
$$= ((F_1 \rtimes F_2) \rtimes F_3 \ldots) \rtimes F_{n-1}$$

$PB_1 = \{1\}$

$PB_2 = PB_1 \rtimes F_1$

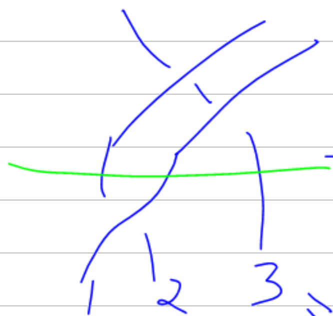
$= \mathbb{Z}$



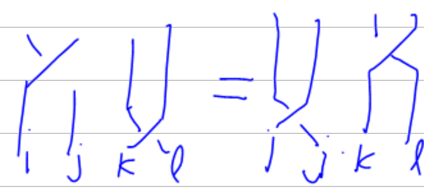


"combing the braid"

$$B_n = \left\langle \sigma_i \ \ 1\le i\le n-1 \ : \ \begin{matrix} \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j|>1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \end{matrix} \right\rangle$$



$\longrightarrow \sigma_{12}\sigma_{13}\sigma_{23}$



$=$

$\longrightarrow \sigma_{23}\sigma_{13}\sigma_{12}$

$$\left\langle \sigma_{ij} \ \ 1\le i\ne j\le n \ : \ \begin{matrix} \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} & \text{if } |\{i,j,k,l\}|=4 \\ \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} & \text{if } |\{i,j,k\}|=3 \end{matrix} \right\rangle$$

$\shortparallel$

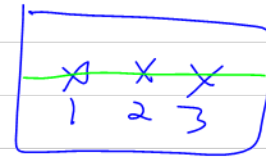
PvB_n "Pure virtual Braids"

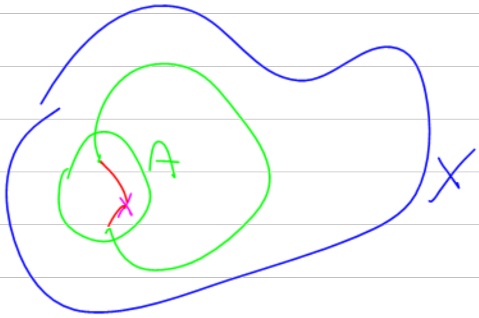
$$PwB_n = PvB_n \Big/ \underbrace{\sigma_{ij}\sigma_{ik} = \sigma_{ik}\sigma_{ij}}_{\text{"overcrossings commute" } OC} \qquad |\{i,j,k\}|=3$$

$\uparrow$ weakly-virtual
welded
warmup

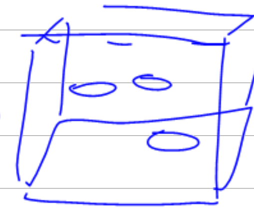
$PB_n = P_u B_n$ = "motion group of n pts in $\mathbb{R}^2$"

↑ usual





"motion group of n horiz rings in $\mathbb{R}^3$"



"group of horizontal flying rings"

$\pi_1(X, A)$ — A simply connected

$P_wB_n = \pi_1\left(\text{horiz. flying rings, external to each other}\right)$

↑ claim

σ_{ij} = j flies through i in the positive dir.

Hour 35, Monday December 7.
Goal for the remaining two classes: More on (uvw)B. Prove that PwB has a "Taylor Expansion".
But first an apology regarding unique factorization, following
http://drorbn.net/AcademicPensieve/Classes/20-1350-KnotTheory/LickorishOnUniqueFactorization.pdf

"silly braids"

$PuB := \ker(uB \longrightarrow S)$ $\quad$ $PvB = \langle \sigma_{ij} : \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij},\ \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \rangle$

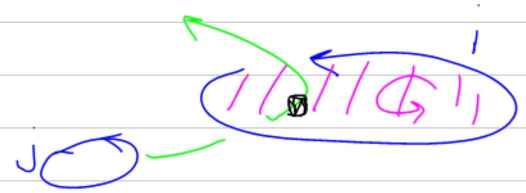
"i over j, +"

"pure virtual braids"

$PwB := PvB / \sigma_{ij}\sigma_{ik} = \sigma_{ik}\sigma_{ij}$

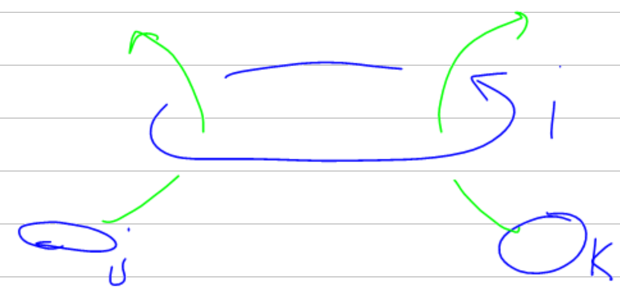
"pure w-braids" $\quad$ "Overcrossings commute (OC)".

σ_{ij}

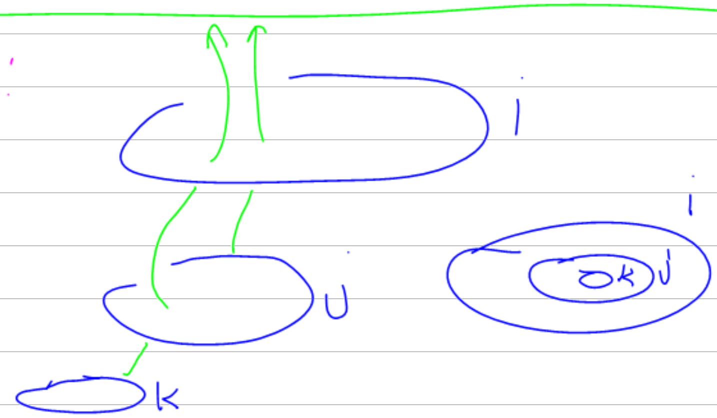


"The group of (horizontal) flying rings"

3:

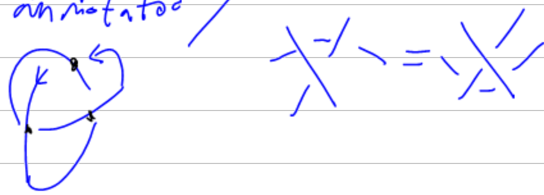


1:

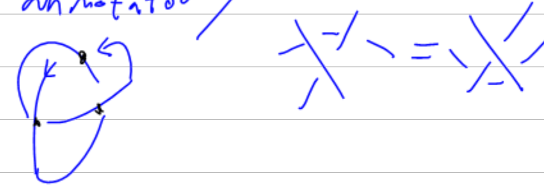


"Virtual Knots"

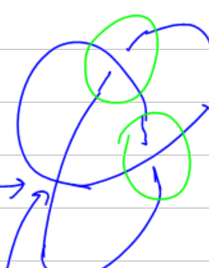
Knots = planar quadrivalent graphs, appropriately annotated / R123



V-Knots = quadrivalent graphs, appropriately annotated / R123



Morally-wrong equiv. def

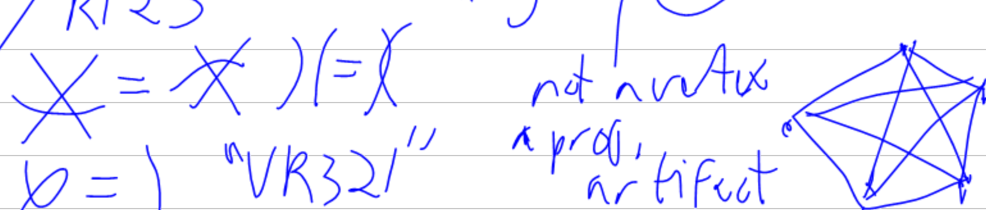


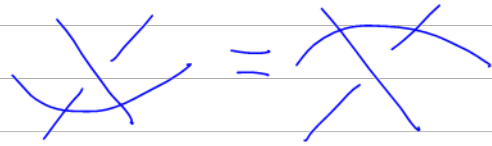
2-xing v-knots

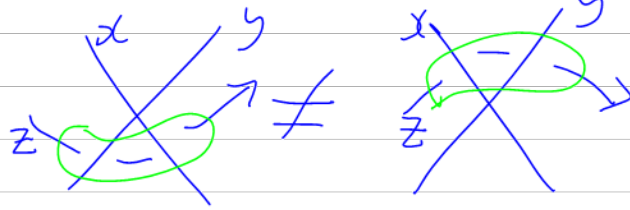
"virtual xing"

vK = {PD w/ two types of xings, regular & virtual} / R123, "VR321"

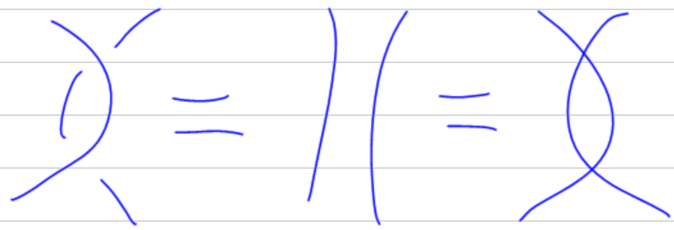
"not a vertex, a proof artifact"



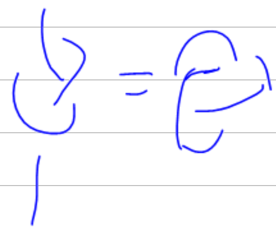
M: 

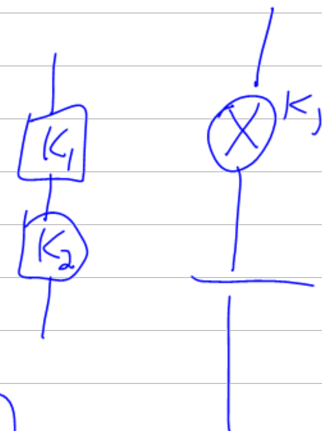
F "Forbidden" 

Story as z tells it
left: "I go under y then under x"
right: "I go under x then under y"



Some differences:

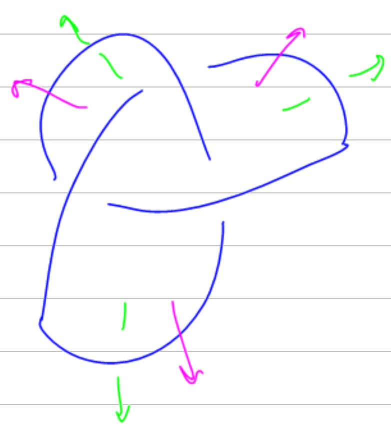
*  long v-knot $\longrightarrow$ S^1-v-knots

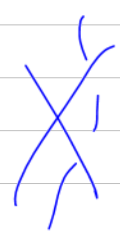


* v-K not an Abelian monoid.

* Two mirrors!

* Two π_1's.





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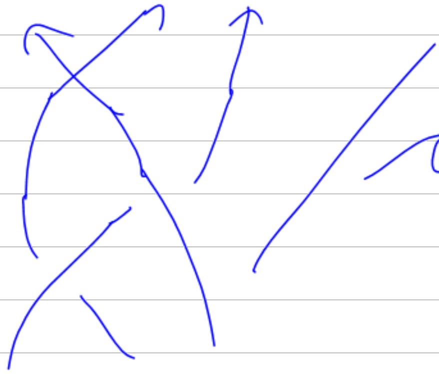
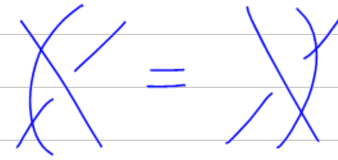
virtual braids:

 cls  

$$vB_n = \left\langle \begin{array}{l} \gamma_i : \text{ordinary} \\ \tau_i : \text{virtual} \end{array} \right\rangle \Big/ \begin{array}{l} \gamma_i \gamma_{i+1} \gamma_i = \gamma_{i+1} \gamma_i \gamma_{i+1} \\ \gamma_i \gamma_j = \gamma_j \gamma_i \quad |i-j|>1 \\ \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \\ \tau_i \tau_j = \ldots \\ \tau_i^2 = 1 \\ \tau_i \gamma_{i+1} \tau_i = \tau_{i+1} \gamma_i \tau_{i+1} \\ \tau_i \gamma_j = \gamma_j \tau_i \quad |i-j|>1 \end{array}$$

$$\|$$

$$PvB_n \rtimes S_n$$

Hour 36 and last, Wednesday December 9.
No time for the full story of "Taylor Expansion for PwB".
Goal: Explain what is a "Taylor Expansion" and mumble about why, for PwB, it is useful.

$$S=\langle \tau_i : \tau_i^2=1,\ \tau_i\tau_{i+1}\tau_i=\tau_{i+1}\tau_i\tau_{i+1},\ \tau_i\tau_j=\tau_j\tau_i\ \ |i-j|>1\rangle$$

$$uB=\langle \gamma_i : \gamma_i\gamma_{i+1}\gamma_i=\gamma_{i+1}\gamma_i\gamma_{i+1},\ \gamma_i\gamma_j=\gamma_j\gamma_i\ \ |i-j|>1\rangle \longleftrightarrow PuB=\ker\left(B \xrightarrow{\gamma_i\to\tau_i} S\right)$$

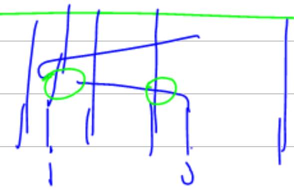
$PuB = G/R$?

$$vB=B*S\Big/\left\langle \begin{matrix}\tau_i\gamma_{i+1}\tau_i=\tau_{i+1}\gamma_i\tau_{i+1}\\ \tau_i\gamma_j=\gamma_j\tau_i\ \ |i-j|>1\end{matrix}\right\rangle \overset{?}{\longleftrightarrow} PvB=\left\langle\sigma_{ij} : \begin{matrix}\sigma_{ij}\sigma_{ik}\sigma_{jk}=\sigma_{jk}\sigma_{ik}\sigma_{ij}\\ \sigma_{ij}\sigma_{kl}=\sigma_{kl}\sigma_{ij}\end{matrix}\right\rangle$$

$$\boxed{wB=vB/\langle\gamma_i\gamma_{i+1}\tau_i=\tau_{i+1}\gamma_i\gamma_{i+1}\rangle} \overset{?}{\longleftrightarrow} \boxed{PwB=PvB/\sigma_{ij}\sigma_{ik}=\sigma_{ik}\sigma_{ij}}$$

In fact, $vB=PvB\rtimes S$ and $wB=PwB\rtimes S$ [so a "good" invariant of PwB may lead to an invt of links]

σ_{ij}



"Taylor expansions" for groups.

Let G be a group

$\mathbb{Q}G$: the group ring of $G=\left\{\sum a_i g_i : \begin{matrix}a_i\in\mathbb{Q}\\ g_i\in G\end{matrix}\right\}$

I: the augmentation ideal in $\mathbb{Q}G$:

$$I=\left\{\sum a_i g_i : \sum a_i=0\right\}=vs\langle g_1-g_2 : g_1,g_2\in G\rangle$$

$$=\mathrm{ideal}\langle g-1 : g\in G\rangle$$

$$\mathbb{Q}G=I^0\supset I^1\supset I\cdot I=I^2\supset I^3\supset I^4\supset \dots$$

"Filtration"

$$\mathcal{A}(G) = \prod_{m=0}^{\infty} I^m / I^{m+1}$$

a graded ring: $\underset{\substack{\cup \\ I^k}}{[a]}_{I^{k+1}} \cdot \underset{\substack{\cup \\ I^l}}{[b]}_{I^{l+1}} = [\underset{\substack{\cap \\ I^{k+l}}}{a \cdot b}]_{I^{k+l+1}}$

Def An expansion for a group G is a linear map

$$Z : G \longrightarrow \mathcal{A}(G)$$ associated graded of G.

s.t. if $a \in I^m$ then

$$Z(a) = (0, 0, \ldots, \underset{m-1}{0}, [a], *, *, *) .$$

Aside take $R = C^\infty(\mathbb{R}^d)$

$$I = \{ f \in R : f(0) = 0 \}$$

I^m : Functions w/ zero of order m at 0.

I^m / I^{m+1} : homogeneous polynomials of deg m.

$\mathcal{A}(R) =$ Power series around 0.

$Z : R \to \mathcal{A}(R)$: $Z : C^\infty \longrightarrow$ Power series

HW: Verify that the Taylor expansion

is an expansion.

1. Expansion is "homomorphic" if
$Z: G \to \mathcal{A}(G)$ is multiplicative.

2. $G \to H \Rightarrow \mathcal{A}(G) \longrightarrow \mathcal{A}(H)$

$\Delta: G \to G \times G \Rightarrow \mathcal{A}(G) \xrightarrow{\Delta} \mathcal{A}(G \times G) = \mathcal{A}(G) \otimes \mathcal{A}(G)$

$\Delta: g \mapsto (g,g)$

Expansion is called co-homo if

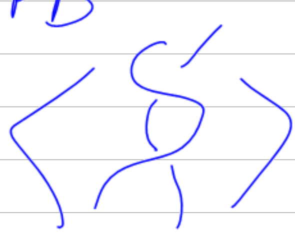
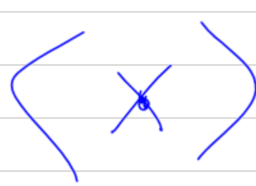
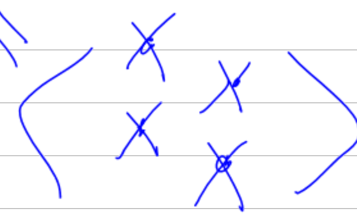
$$\begin{tikzcd} G \arrow[r, "Z"] \arrow[d, "\Delta"] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{A}(G) \arrow[d, "\Delta"] \\ G \times G \arrow[r, "Z \otimes Z"] & \mathcal{A}(G) \otimes \mathcal{A}(G) \end{tikzcd}$$

HW 2. The Taylor exp. is homo & co-homo.

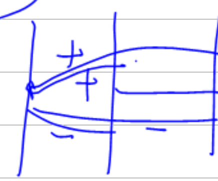
3. Any expansion $C^\infty \to$ p.s. which is both homo & co-homo is Taylor.

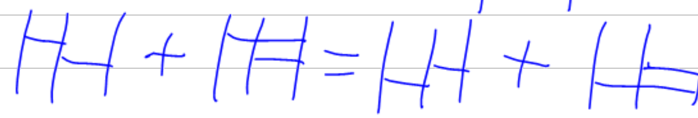
Hence one is curious which groups have Taylor expansions . .

$G = PB$ $\qquad I_n$ $\qquad I^m =$

$QG =$    

$A(G) = \Big\langle$  $\Big\rangle \Big/ \text{rels}$

4T: 



Is there a homo-to-homo expansion for ~~braids~~ PB: Yes, but it's hard.

$$PB = \Big\langle \sigma_{ij} : \begin{array}{ll} \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} & \\ \sigma_{ij}\sigma_{ik}\sigma_{jk} = \underline{\qquad\qquad} & R3 \\ \sigma_{ij}\sigma_{ik} = \sigma_{ik}\sigma_{ij} & OC \end{array}$$

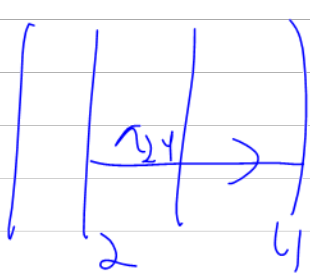
$\parallel$

G

$I : \langle \sigma_{ij} - 1 \rangle$ in $A(G) \ni [\sigma_{ij} - 1]_{I^2} = a_{ij}$

$$A(G) = \Big\langle a_{ij} : \begin{array}{l} [a_{ij}, a_{kl}] = 0 \\ [a_{ij}, a_{ik}] = 0 \end{array} \Big\rangle$$

4T: $[a_{ij} + a_{ik}, a_{jk}] = 0$



$Z : G \longrightarrow A(G)$

$Z(\sigma_{ij}) = e^{a_{ij}}$ $\rightsquigarrow$ Alexander poly.