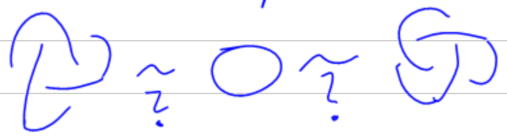
MAT 1350F Topics in Knot Theory
Dror's Open Private Notebook

Class of Wednesday September 9: Introduction to Knots, Knot Colourings, the Jones Polynomial.

1. http://drorbn.net/20-1350, especially ..../About.html
2. The comparisson w/ number theory.



A complicated unknot. ⟸ Picts
The knot table. ⟸

3. Knots, knot colorings, Reid. moves. done line

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4. A word on Jones ⟵ Pict
5. The Kauffman bracket.

$$\langle \times \rangle \to A\underbrace{\langle)(\rangle}_{0\text{-smoothing}} + B\underbrace{\langle \asymp \rangle}_{1\text{-smoothing}}$$

$$\langle D \bigcirc \rangle = d\langle D \rangle$$

$$\rightsquigarrow B = A^{-1},\ d = -A^2 - A^{-2}$$

$$\langle \rho \rangle = -A^3 \langle | \rangle$$

$$J(K) := (-A^3)^{-w(K)}\frac{\langle K \rangle}{d} \Big/ .\ A \to q^{-1/4}$$

6. The Jones skein relation:

$$J(\nwarrow\!\!\!\!\nearrow) = -q^{3/4}\left(q^{-1/4}\langle)(\rangle + q^{1/4}\langle \asymp \rangle\right)$$

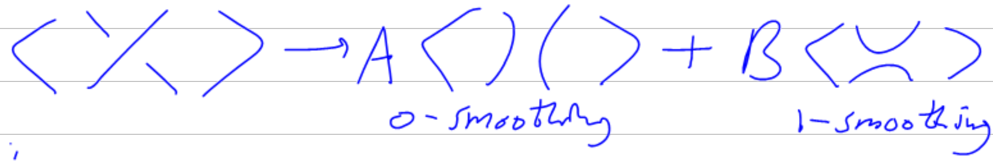
$$J(\nwarrow\!\!\!\!\nearrow) = -q^{-3/4}\left(q^{-1/4}\langle \asymp \rangle + q^{1/4}\langle)(\rangle\right)$$

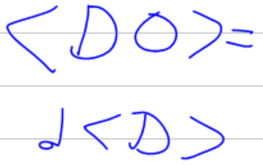
$$\Rightarrow q^{-1}J(\nwarrow\!\!\!\!\nearrow) - q\,J(\nwarrow\!\!\!\!\nearrow) = (q^{1/2} - q^{-1/2})\,J(\uparrow\uparrow)$$

20-1350 Class of Friday September 11: The Jones Polynomial.

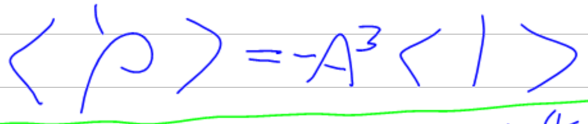
1. A word on Jones ← Pict.

2. The Kauffman bracket.





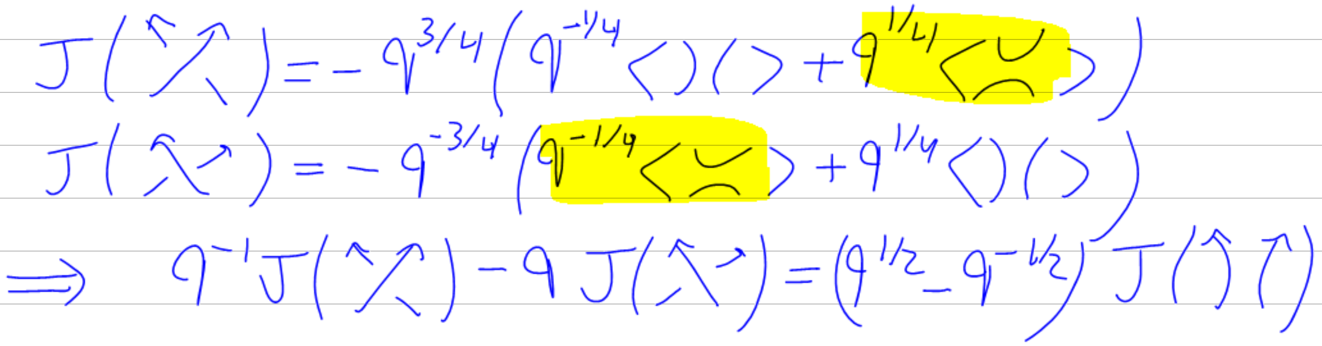
$\longrightarrow B = A^{-1},\ d = -A^2 - A^{-2}$



done line

mon Sep 14: $J(K) := (-A^3)^{-w(K)} \dfrac{\langle K \rangle}{d} \ /.\ A \to q^{-1/4}$

3. The Jones Skein relation:



4. A word about computation, aiming for:

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Knot[3, 1] /. Knots

PD[X[1, 4, 2, 5], X[3, 6, 4, 1], X[5, 2, 6, 3]]

KB[pd_PD] := Module[{p, t1, t2, t3, t4, B, d},
  SetAttributes[p, Orderless];
  t1 = pd /. X[i_, j_, k_, l_] → A * p[i, j] * p[k, l] + B * p[i, l] × p[j, k];
  t2 = Expand[t1 /. PD → Times];
  t3 = t2 //. {p[i_, j_] × p[j_, k_] → p[i, k]};
  t4 = t3 /. {p[i_, i_] → d, p[i_, j_]^2 → d};
  Expand[t4 /. {B → 1 / A, d → -A^2 - 1 / A^2}]
 ]
```

Wed Sep 11. Continue following "Fast-Jones.nb".

HW1

1. In $\mathbb{Z}/3$, $x+y+z=0$ iff x, y, z are all the same or are all different. Use this to show that $\lambda(D)$ is always a power of 3 and that it can be computed in poly-time.

2. Something about KB at $\sqrt[3]{-1}$.

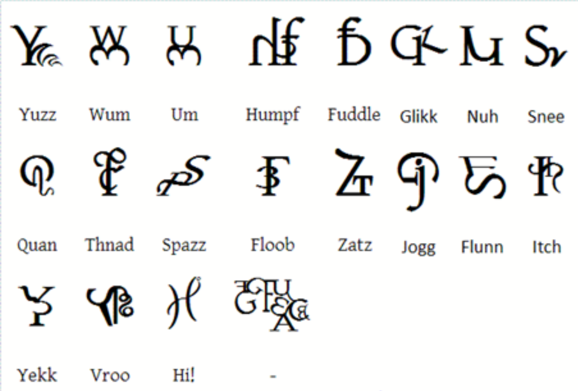
3. Prove that the PD notation of a knot diagrams determines it as a diagram in S^2.

Next few weeks:

1. Khovanov homology.
   - a. For knots.
   - b. For tangles.
2. Finite type invts & Lie Algebras.
3. Back to colouring invts, π_1
4. Prime knots, alternating knots
5. Alexander, Burau.

Class of Friday Sep 18 (hour 5)

Today: Clarify Wednesday's EFKB
Talk about worms in apples.
On beyond Zebra!



"The Kaufman bracket is a morphism from the planar algebra of framed tangles into the the Temperley-Lieb planar algebra".

"$\mathrm{Rank}_{\mathbb{Z}[A,A^{-1}]} TL_{2n} = C_n = \frac{1}{n+1}\binom{2n}{n} \sim 4^n / n^{3/2}\sqrt{\pi}$"

Next: Khovanov homology for knots following my "On Khovanov's categorification of the Jones polynomial", especially the last page.

Hour 6, Monday September 21: Finish Jones and planar algebras, start Khovanov homology

Last time: over $\mathbb{Z}[A^{\pm 1}]$

KB: $\left. \begin{matrix}\text{Planar Alg}\\ \text{of Tangles}\end{matrix} \middle/ R2R3 \right. \longrightarrow TL := \{$  $\} / \bigcirc = d$

$$\mathrm{Rank}_{\mathbb{Z}[A,A^{-1}]} TL_{2n} = C_n = \frac{1}{n+1}\binom{2n}{n} \sim 4^n / n^{3/2}\sqrt{\pi}$$

Some discussion and proof.

Next: Khovanov homology for knots following my "On Khovanov's Categorification of the Jones polynomial", especially the last page.

$\underline{\text{Khovanov}}$: A graded $\overbrace{\text{complex}}^{\text{chain}}$ for each knot diagram, whose Euler characteristic is the Jones poly., and whose homology is invariant (stronger than Jones!) +more...

Define all these notions!

Hour 12, Monday October 5: Examples of finite type invariants.

$$V : \left\{\begin{matrix}\text{oriented knots}\\ \text{in oriented } \mathbb{R}^3\end{matrix}\right\} \longrightarrow A \quad \left(\begin{matrix}\text{an Abilian}\\ \text{group}\end{matrix}\right)$$

$$V^{\cancel{(m)}}\Big(\underbrace{\times \cdots \times}_{m}\Big) := V^{\cancel{(m-1)}}\Big(\nearrow\!\!\!\!\nwarrow \times \cdots \times\Big) - V^{(m-1)}\Big(\nwarrow\!\!\!\!\nearrow \times \cdots \times\Big)$$

"V of type m" means $V^{(m+1)} = 0$ means $V\Big(\underbrace{\times \cdots \times}_{>m}\Big) \equiv 0$

Example 1. Linking numbers.

2. Self-linking numbers (& framed knots)

3. The Conway poly.

4. The Jones poly $q^{-1}L_+ - qL_- = (q^{1/2} - q^{-1/2})L_0$

5. & more!

W.S., CD, FI, 4T, The Fund. Thm.

Hour 13, Wednesday October 7: The basics of finite type invariants.

$$V \in \underset{\substack{\uparrow \\ \text{v.s. of type } m \\ \text{invts}}}{\mathcal{V}_m} \iff V^{(m+1)} = 0 \iff V(\underbrace{X \cdots X}_{>m}) \equiv 0$$

W.S., CD, FI, 4T, The Fund. Thm.

The Fundamental Thm of $\overbrace{\text{FTI}: \forall W \in}^{\text{Kontsevich}+\dots} \mathcal{W}_m = (\mathcal{D}_m/FI, 4T)^*$

$\exists V \in \mathcal{V}_m$ s.t. $W = W_V \quad (\sim V^{(m)})$.

---

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}^r_m$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	184	316	548

$\uparrow$ $\mathcal{A}^r, \mathcal{A}$

Def $\mathcal{A} = \bigoplus_{m=0}^{\infty} \mathcal{A}_m \qquad \hat{\mathcal{A}} = \prod_{m=0}^{\infty} \mathcal{A}_m$ [I won't always keep them separate]

$\mathcal{A} \sim K$ $\quad$ K has a commutative product
$\quad\quad\quad\quad$ K^* has —"—

Thm $\mathcal{A}$ is a connected graded commutative & co-comm. bi-algebra.

* Talk about the product of $\mathcal{A}$.

Prop. If $V_1 \in \mathcal{V}_{m_1}$ & $V_2 \in \mathcal{V}_{m_2}$ then $V_1 \cdot V_2 \in \mathcal{V}_{m_1+m_2}$

Prop $\exists! \; \square: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ s.t. $W_{V_1 \cdot V_2} = \square // W_{V_1} \otimes W_{V_2}$

Thm $(\mathcal{A}, m, \square, \epsilon, \eta)$ is as stated above.

Hour 15, Wednesday October 14: \calA

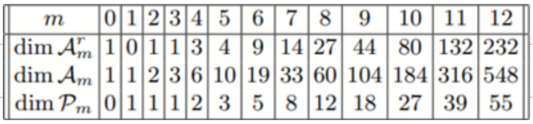
Prop. If $V_1 \in \mathcal{V}_{m_1}$ & $V_2 \in \mathcal{V}_{m_2}$ then $V_1 \cdot V_2 \in \mathcal{V}_{m_1+m_2}$

Prop $\exists!$ $\Box : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ s.t. $W_{V_1 \cdot V_2} = \Box /\!/ W_{V_1} \otimes W_{V_2}$

W_{conway}.

Thm $(\mathcal{A}, m, \Box, \epsilon, \eta)$ is as stated above.

Milnor-Moore.



m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}^r_m$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	184	316	548
$\dim \mathcal{P}_m$	0	1	1	1	2	3	5	8	12	18	27	39	55

$\mathcal{A}^t$

Lie algebras & weight systems.

Hour 16, Friday October 16: \calA, \calA^t

Thm $(\mathcal{A}, m, \Box, \epsilon, \eta)$ is as stated above.

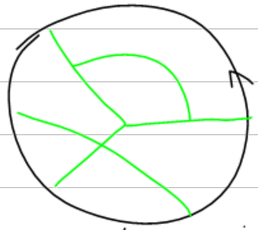
Milnor-Moore.

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}^r_m$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	184	316	548
$\dim \mathcal{P}_m$	0	1	1	1	2	3	5	8	12	18	27	39	55

$\mathcal{A}^t$

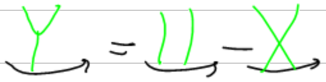
Lie algebras & weight systems.

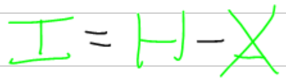
Hour 17, Monday October 19: \calA^t and Lie Algebras

Thm $\mathcal{A} \cong \mathcal{A}^t := \Big\langle$  $\Big\rangle \Big/$

* connected, oriented vertices, deg $= \frac{1}{2}$(verts)

AS: 

STU: 

IHX: 

Pf. . . .

Lie algebras, metrized Lie algebras, reps in R

F, t, r, $W_{L,R}(D)$.

Hour 18, Wednesday October 21: Lie Algebraic Weight Systems, gl(N).
By midnight: HW5 due, HW6 assigned.

Last time: on board

$/4T \cong$ $/\begin{smallmatrix}AS\\IHX\\STU\end{smallmatrix}$

NTS: STU ⇒ AS, IHX.

Reminders. $\mathfrak{g}$: metrized f.d. Lie Algebra

R: Representation thereof.

Thm $\exists\, W_{\mathfrak{g},R}: \mathcal{A} \longrightarrow \mathbb{F}$

$\mathfrak{g} = \langle X_a \rangle_{a=1}^{\dim \mathfrak{g}}$ $\quad R = \langle e_\alpha \rangle_{\alpha=1}^{\dim R}$

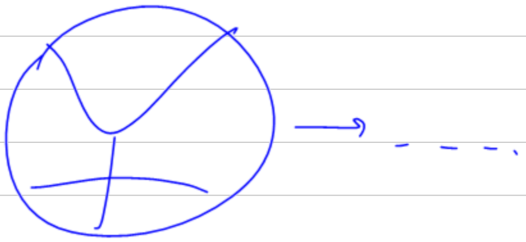
sym sym

$[X_a, X_b] = f_{ab}^{c} X_c \quad \langle X_a, X_b \rangle = t_{ab} \quad t_{ab} t^{bc} = \delta_a^c$

$\nearrow a\leftrightarrow b.$

$F_{abc} = \langle [X_a, X_b], X_c \rangle = f_{ab}^{d} t_{dc} \quad X_a e_\alpha = \rho_{a\alpha}^{\beta} e_\beta$

↖ totally as.



Lemma 1 This is well defined on diagrams.

Pf 1. Phys. way.
2. Tens. calc way.
3. hands on.

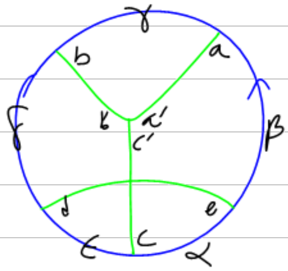
Lemma 2 $W_{\mathfrak{g},R}$ satisfies AS, STU, IHX.

Hour 19, Friday October 23: gl(N).

$\mathfrak{g} = \langle X_a \rangle_{a=1}^{\dim \mathfrak{g}} \qquad R = \langle v_\alpha \rangle_{\alpha=1}^{\dim R}$

$F_{abc} = \langle [X_a, X_b], X_c \rangle \quad \langle X_a, X_b \rangle = t_{ab} \quad t_{ab} t^{bc} = \delta_a^c \quad X_a v_\alpha = r_{a\alpha}^\beta v_\beta$

$W_{\mathfrak{g},R}:$



$$\longrightarrow \sum_{\substack{a'b'c' \\ abcde \\ \alpha\beta\gamma\delta\epsilon}} f_{a'b'c'} t^{aa'} t^{bb'} t^{cc'} t^{ed} \cdot r_{a\beta}^{\gamma} r_{b\gamma}^{\delta} r_{d\delta}^{\epsilon} r_{c\epsilon}^{\alpha} r_{e\alpha}^{\beta} \in \mathbb{Q}$$

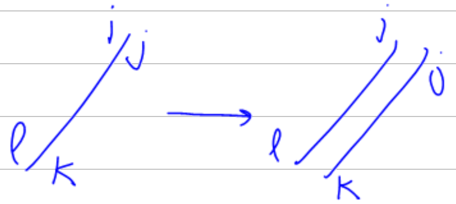
The gl_N calculation [All About δ_{ij}]

$$X_{ij} = {}_i\begin{pmatrix} & & \\ & 1 & \\ & & \end{pmatrix}_{\!\!j} \qquad a \leftrightarrow (ij) \qquad X_{ij} X_{kl} = \delta_{jk} X_{il}$$

$$[X_{ij}, X_{kl}] = \delta_{jk} X_{il} - \delta_{il} X_{kj}$$

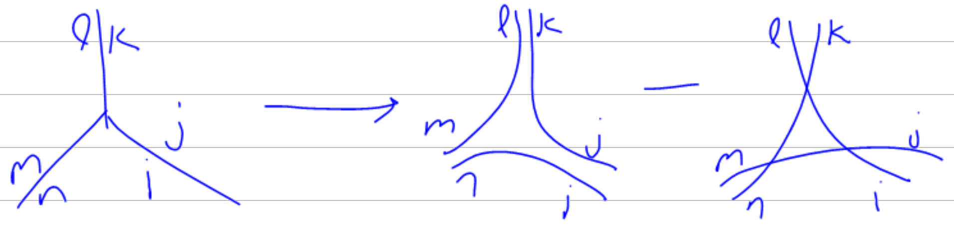
$t_{(ij)(kl)} = tr(X_{ij} X_{kl}) = \delta_{jk} \delta_{il}$

$t^{(ij)(kl)} =$ same.

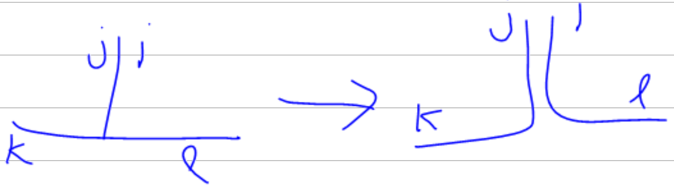


Indeed, $t_{(ij)(kl)} t^{(kl)(mn)} = \delta_{jn} \delta_{im}$

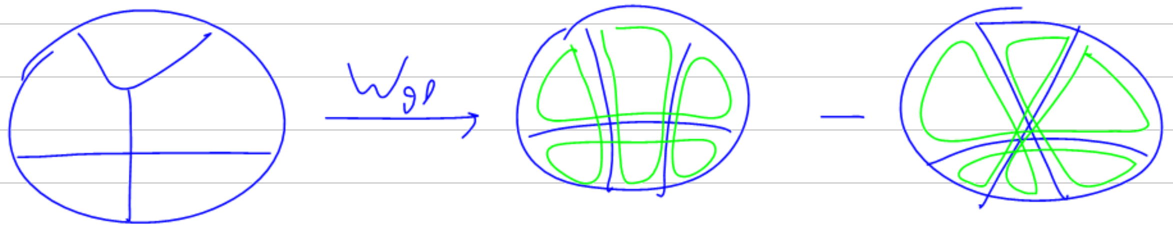
$F_{ij,kl,mn} = \langle \delta_{jk} X_{il} - \delta_{il} X_{kj}, X_{mn} \rangle = \delta_{jk} \delta_{lm} \delta_{ni} - \delta_{il} \delta_{kn} \delta_{jm}$



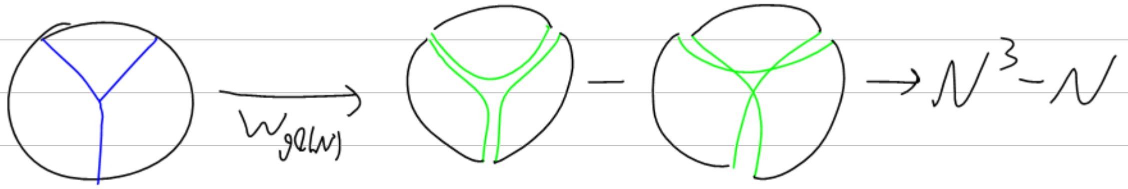
$r_{(ij)k}^{l} = \delta_{jk} \delta_i^l$



Now compute



Hour 20, Monday October 26: The 4CT, Expansions.



HW6 Q2 If D is blue, $|W^{top}_{gl(N)}(D)|$ = # planar embeddings of D

Last bit of H19: If D is blue & planar, $|W_{sl(2)}(D)|$ = # edge 3-colorings of D.

Thm If D is b&p,

$$4(\#\text{ edge 3-colorings of } D) = \#(\text{map 4-colorings of } D)$$

So 4CT $\iff$ (# planar embeddings of D $\neq 0$ $\Rightarrow$ # edge 3-colorings of D $\neq 0$)

$$\iff W_{sl(2)}(D)=0 \Rightarrow W^{top}_{gl(N)}(D)=0$$

Super reasonable?

Note on HW6 Q1: $\exists$ a good $P: \mathcal{A}^\wedge \to \mathcal{A} \xrightarrow{W_{g,R}} F \ldots$

Prop (Fund Thm (framed knots)) $\iff$ $\exists\, Z: \{\text{Knots}\} \to \hat{\mathcal{A}}$ s.t.

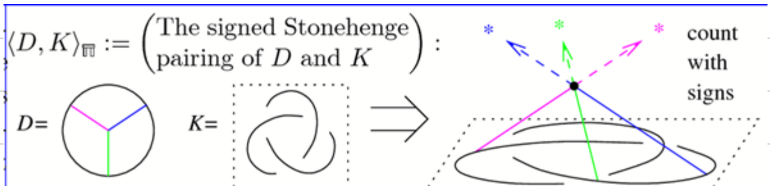
$$Z(\overset{\text{sing}}{K}) = D_K + h.o.$$

Such Z is "an expansion" or a UFTI.

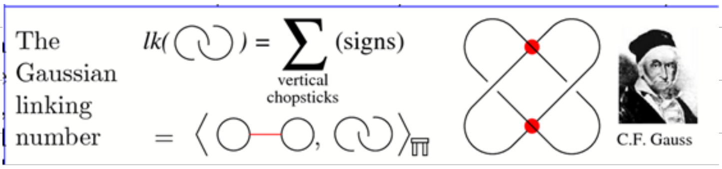
Pf Given Z, . . .

Given F.T., take a basis $a_{m,i}$ of $\mathcal{A}_m$, let $w_{m,i}$ be the dual basis, let $V_{m,i}$ be s.t. $W(V_{m,i})=w_{m,i}$ & set $Z(K)=\sum a_{m,i}V_{m,i}(K)$

Hour 21, Wednesday October 28: Expansions and Cosmic Coincidences.



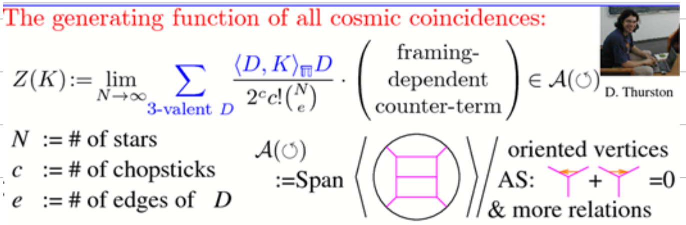
$\langle D, K\rangle_{\sqcap\!\sqcap} := \begin{pmatrix}\text{The signed Stonehenge}\\ \text{pairing of } D \text{ and } K\end{pmatrix}$:



The Gaussian linking number

$lk(\text{◯◯}) = \sum_{\substack{\text{vertical}\\ \text{chopsticks}}} (\text{signs})$

$= \langle \text{○—○}, \text{◯◯} \rangle_{\sqcap\!\sqcap}$



The generating function of all cosmic coincidences:

$$Z(K) := \lim_{N\to\infty} \sum_{\text{3-valent } D} \frac{\langle D,K\rangle_{\sqcap\!\sqcap} D}{2^c c! \binom{N}{e}} \cdot \begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix} \in \mathcal{A}(\circlearrowleft)$$

N := # of stars

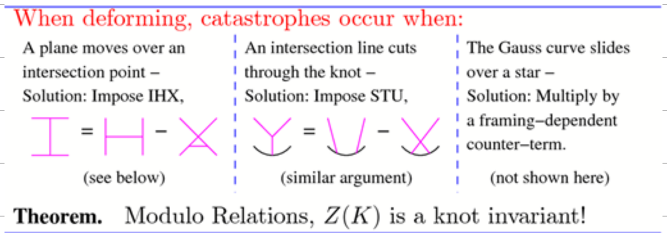
c := # of chopsticks

e := # of edges of D

$\mathcal{A}(\circlearrowleft) := \text{Span}\langle \cdot \rangle \Big/$ oriented vertices, AS: Y + Y = 0, & more relations

Done line

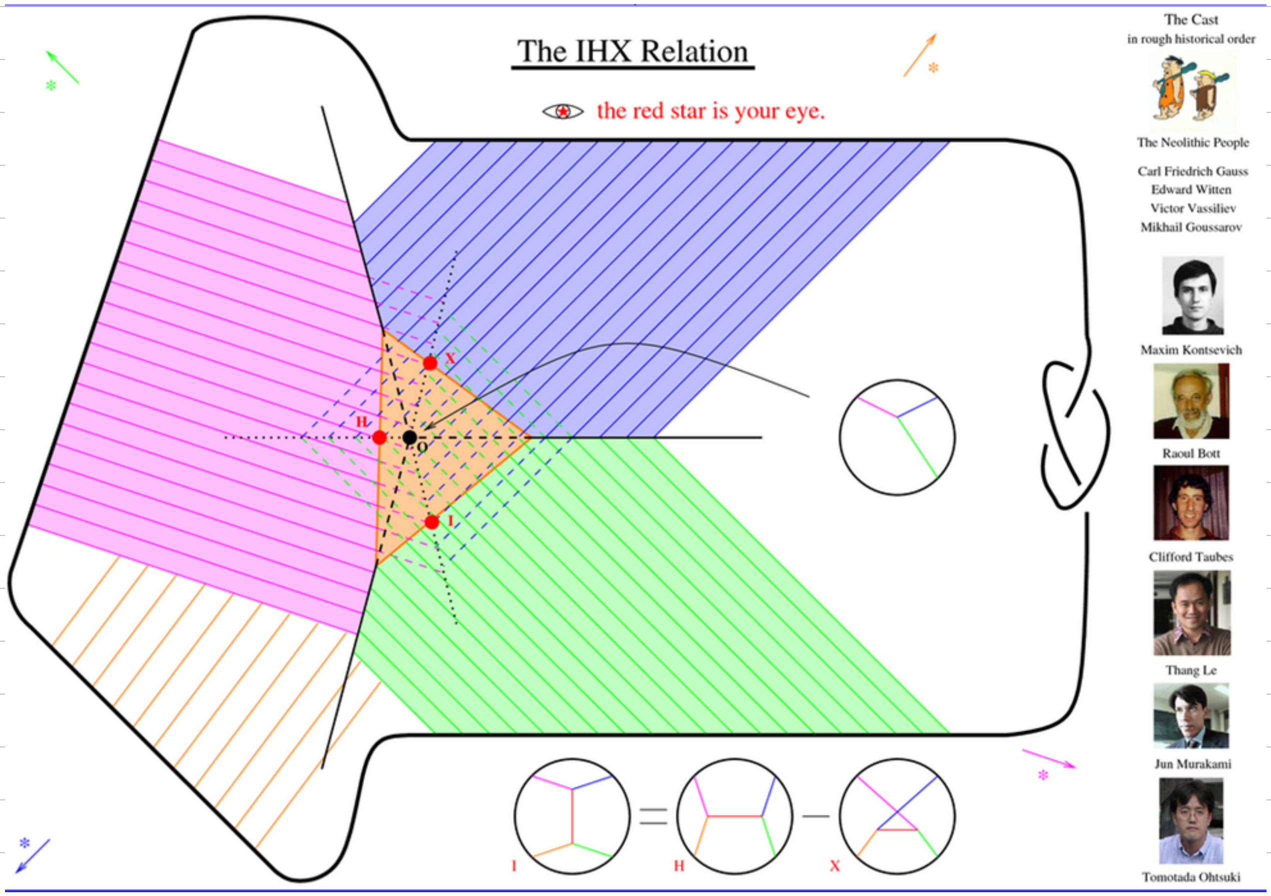
Friday Oct 30, hour 22



When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,	An intersection line cuts through the knot – Solution: Impose STU,	The Gauss curve slides over a star – Solution: Multiply by a framing–dependent counter–term.
I = H − X	Y = U − X	
(see below)	(similar argument)	(not shown here)

Theorem. Modulo Relations, $Z(K)$ is a knot invariant!



The IHX Relation

the red star is your eye.

The Cast
in rough historical order

The Neolithic People

Carl Friedrich Gauss
Edward Witten
Victor Vassiliev
Mikhail Goussarov

Maxim Kontsevich

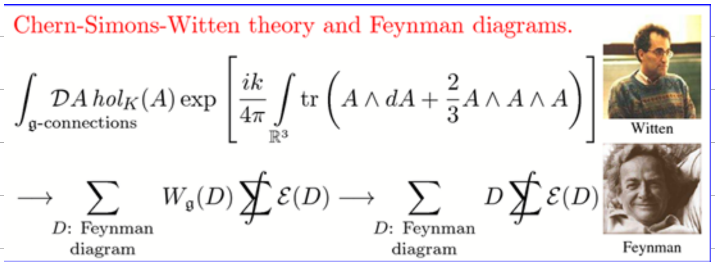
Raoul Bott

Clifford Taubes

Thang Le

Jun Murakami

Tomotada Ohtsuki



Chern-Simons-Witten theory and Feynman diagrams.

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right]$$

$$\longrightarrow \sum_{\substack{D:\ \text{Feynman}\\ \text{diagram}}} W_{\mathfrak{g}}(D) \sum\!\!\!\!\!\!\int \mathcal{E}(D) \longrightarrow \sum_{\substack{D:\ \text{Feynman}\\ \text{diagram}}} D \sum\!\!\!\!\!\!\int \mathcal{E}(D)$$

Monday Nov 2: A quick intro to Hour 23

Fyyman Diagrams. See notes for

Week 9.

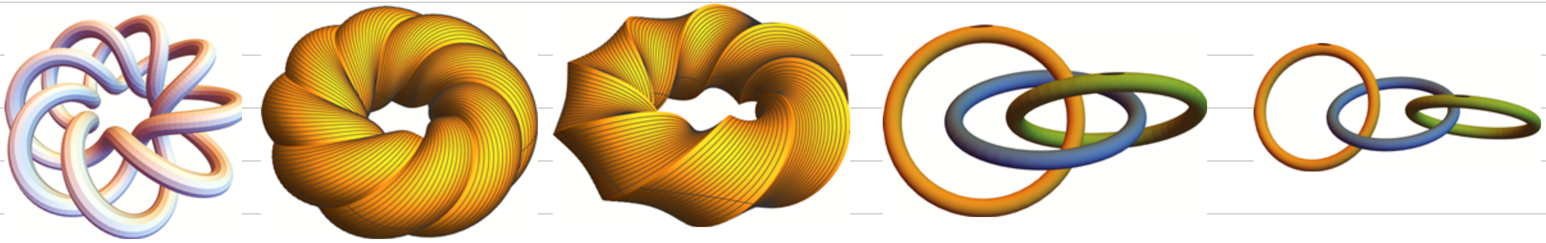
Hour 24, Wednesday November 4: The Fundamental Group

HW7 on web by midnight! (And I hope to clear my marking backlog soon).

Finite type / Lie Algebras and Reps omissions:

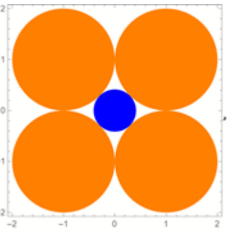
- The KZ proof of the Fundamental Theorem.
- The "Associators" proof of the Fundamental Theorem (also, "Knotted Trivalent Graphs").
- The step-by-step-integration non-proof of the Fundamental Theorem.
- Computing FT Invariants using "Gauss Diagram Formulas".
- Computations of invariants for specific Lie algebras and reps ("Quantum Groups").
- Finite type invariants of other types knotted objects.
- Finite type invariants of 3-manifolds.
- Vogel's work on non-Lie-algebraic weight systems.
- And more....

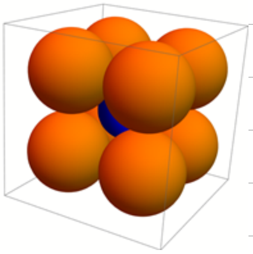
A Gallery of Pictures from BlownTorus.nb at http://drorbn.net/AcademicPensieve/Classes/20-1350-KnotTheory:

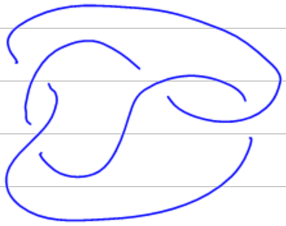


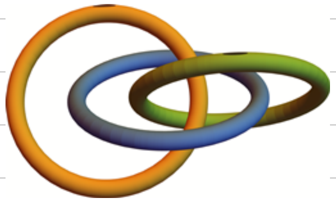
π_1 ; $\pi_1(T_{8,3})$, Thm svK, Wirtinger, $\pi_1(\text{Hopf})$

Hour 25, Friday November 6: More on Knot Groups









The 1206 riddle

$\pi_1(4_1)$

$\pi_1(\text{Hopf})$ why so easy?

π_1 nearly separates!

$\text{Rep}(\pi, G)$

$\text{Rep}(\pi, g^G)$

Lickorish GTM 175, P115.

0. The word problem is insoluble. [also LI!]
1. Waldhausen 66': (π_1, λ, μ) determines K.
2. Whitten/Gonzales-Acuña 87': If K is prime, $\pi_1(K)$ determines K.
3. Gordon-Luecke 89': The compliment of an unoriented K determines it.
4. The links above have homeomorphic compliments.

Example $D_{2n}=\{(\overset{\{\pm1\}}{\overset{\rotatebox{90}{$\in$}}{s}},\overset{\mathbb{Z}/k}{\overset{\rotatebox{90}{$\in$}}{k}})\}$ w/ $(s_1,k_1)(s_2,k_2)=(s_1s_2, s_2k_1+k_2)$

$(s,k)^{-1}=(s,-sk)$

$(-1,k_1)^{(-1,k_2)}=(-1,2k_2-k_1)$

Quandles.

Post-reading week plans:

Week 10: Colouring, quandles
prime knots, decomposition
into primes.

Week 11: (u,v,w)-braids.
Combing u/v/w-braids
OU-stuff.

Week 12: Expansions for groups

Week 13 (2 classes): Burau & Alexander.

But first, ctrl-C ctrl-V from Oct 14 and from Sep 14:

$C:\{\text{links}\}\longrightarrow\mathbb{Z}[z]$

$C(\underbrace{\bigcirc\cdots\bigcirc}_{k}) = \delta_{k1} \longrightarrow C_0\begin{pmatrix}k\text{ comp}\\ \text{link}\end{pmatrix}=\delta_{k1}$

$C(\times) = C(\slashoverback) - C(\backoverslash) = zC(\upuparrows)$

Oct 14

$C(K)=\sum_{m=0}^{\infty} C_m(K)\cdot z^m$

is of type m.

$W_{C_m}: \mathcal{A}_m \longrightarrow \mathbb{Z}$

m-singular knot whose underlying c.o. is D.

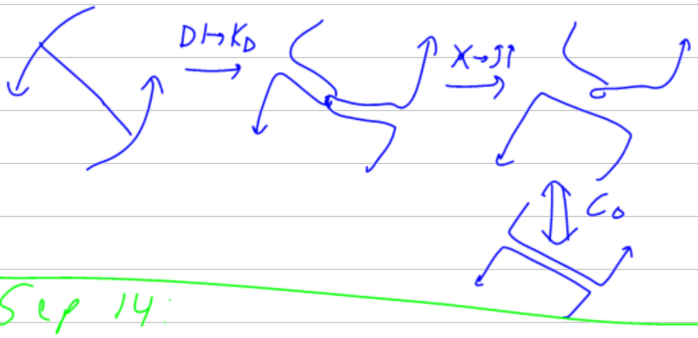
$W_{C_m}\Big(\underbrace{\bigoplus}_{m\text{ chords}} = D\Big) = C_m(K_D)$

$= \text{Coeff}_{z^m}\big(\overset{z^m}{C}(K_D/\times\to\upuparrows)\big)$

$= \text{Coeff}_{z^0}\big(C(K_D/\times\to\upuparrows)\big)$

$= C_0(K_D/\times\to\upuparrows)$

$$W_{C_m}(D) = \begin{cases} 1 & \text{if } K_D/\times\longrightarrow\upuparrows \text{ is connected.}\\ 0 & \text{otherwise}\end{cases}$$



Sep 14:

The Jones skein relation:

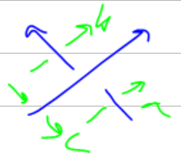
$q^{-1}J(\slashoverback) - qJ(\backoverslash) = (q^{1/2}-q^{-1/2})J(\upuparrows)$

$J(\bigcirc^k) = (-q^{1/2}-q^{-1/2})^k$

Hour 26, Monday November 16: More on Knot Colourings.

The Wirtinger presentation

$\pi_1(K) = \langle a_i ; a = c^b \ldots \rangle$



$\to a = c^{-1}bc$
$= c^b$

$\mathrm{Rep}(\pi, G)$

$\mathrm{Rep}(\pi, g^G)$

Example $D_{2n} = \{(\overset{\{\pm 1\}}{\overset{\Downarrow}{s}}, \overset{\mathbb{Z}/k}{\overset{\Downarrow}{k}})\}$ w/ $(s_1,k_1)(s_2,k_2) = (s_1s_2, s_2k_1+k_2)$

$(s,k)^{-1} = (s, -sk)$

Quandles. $a,b \mapsto a \wedge b$ $\quad (-1,k_1)^{(-1,k_2)} = (-1, 2k_2-k_1)$

1. Axiomatize conjugation
2. Derivation from R1, R2, R3 "self-distributivity"
3. Examples.
4. A binary op that induces an action by automorphisms
5. Lie algebras.

Seifert surfaces

Construction using Seifert cycles.

Example: The Trefoil [compute genus using χ]

Construction using unknotting

Def $g(K)$

Thm $g(K_1 \# K_2) = g(K_1) + g(K_2)$

Hour 27, Wednesday November 18: More on quandles, Seifert surfaces.
HW8 on web by midnight tomorrow!

Correction I said "$\mathrm{ad}_z : L \to L$ is a morphism of LieAlg's"
Nonsense!

$Q\times Q \xrightarrow{\wedge} Q$ is equivariant, meaning

$$\begin{tikzcd}
Q\times Q \arrow[r, "\wedge"] \arrow[d, "*z"] \arrow[dr, phantom, "\circlearrowright"] & Q \arrow[d, "*z"] \\
Q\times Q \arrow[r, "\wedge"] & Q
\end{tikzcd}
\qquad
\begin{tikzcd}
L\otimes L \arrow[r, "{[,]}"] \arrow[d, "\mathrm{ad}_z"] \arrow[dr, phantom, "\circlearrowright"] & L \arrow[d, "\mathrm{ad}_z"] \\
L\otimes L \arrow[r] & L
\end{tikzcd}$$

Quandles from groups:

1. $x\wedge y := y^{-1}xy$ } (can restrict to a conjugacy class)
2. $x\wedge y := y^{-n}xy^{n}$ }
3. $x\wedge y := yx^{-1}y$

Vendramin:

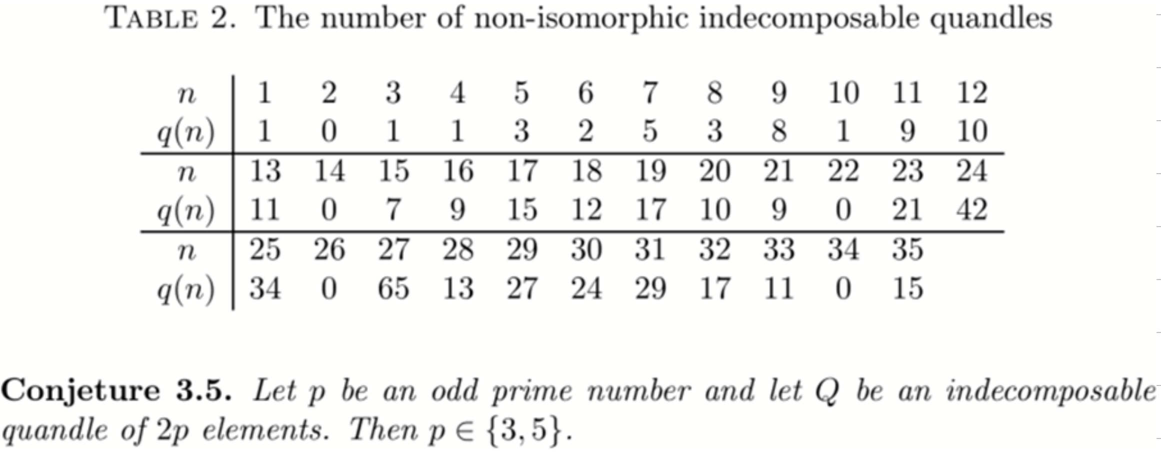


TABLE 2. The number of non-isomorphic indecomposable quandles

n	1	2	3	4	5	6	7	8	9	10	11	12
$q(n)$	1	0	1	1	3	2	5	3	8	1	9	10
n	13	14	15	16	17	18	19	20	21	22	23	24
$q(n)$	11	0	7	9	15	12	17	10	9	0	21	42
n	25	26	27	28	29	30	31	32	33	34	35	
$q(n)$	34	0	65	13	27	24	29	17	11	0	15	

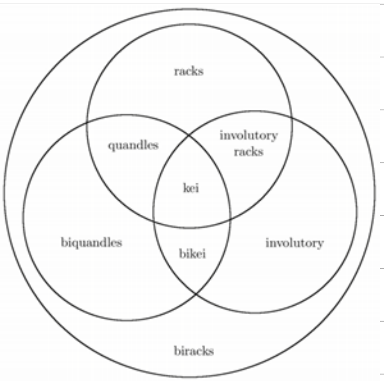
Conjeture 3.5. *Let p be an odd prime number and let Q be an indecomposable quandle of $2p$ elements. Then $p \in \{3,5\}$.*

Elhamdadi/Nelson: A whole book

The Fundamental quandle of a knot (Joyce)

On beyond quandles!

Aksoy, Nelson arXiv:1102.1473:



Seifert surfaces

Construction using Seifert cycles.

Example: The Trefoil [compute genus using χ]

Construction using unknotting

<u>Def</u> $g(K)$

<u>Thm</u> $g(K_1 \# K_2) = g(K_1) + g(K_2)$

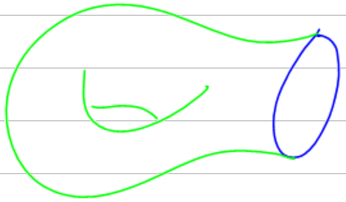
Hour 28, Friday November 20: More on Seifert surfaces.

[Everything is smooth]

Def A Seifert surface for an oriented link K is a connected oriented surface $\Sigma \subset \mathbb{R}^3$ s.t. $\partial\Sigma = K$.

Example

topologically this is a $\Sigma_{\underset{g\ \ n}{1,1}}$



Computing genus using χ

Construction using Seifert cycles.

Example: The Trefoil

Construction using unknotting

Def $g(K)$ **Note** $g(K)=0 \iff K=0$.

Thm $g(K_1+K_2)=g(K_1)+g(K_2)$

Cor1 Knots don't make a group! If $K_1+K_2=0$, then $K_1=K_2=0$.

Cor2 A knot of genus 1 is prime. 3_1 is prime.

Cor3 Existence of prime decomposition (not yet uniqueness)

Pf of thm (Modulo all about diff geom & the topology of $\mathbb{R}^2$)

$g(K_1+K_2) \le g(K_1)+g(K_2)$ [Easy, yet ...]

$g(K_1+K_2) \ge g(K_1)+g(K_2)$:

Hour 29, Monday November 23: Addivity of genus, decomposition into primes.

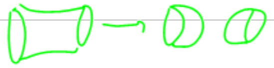
Thm $g(K_1+K_2)=g(K_1)+g(K_2)$

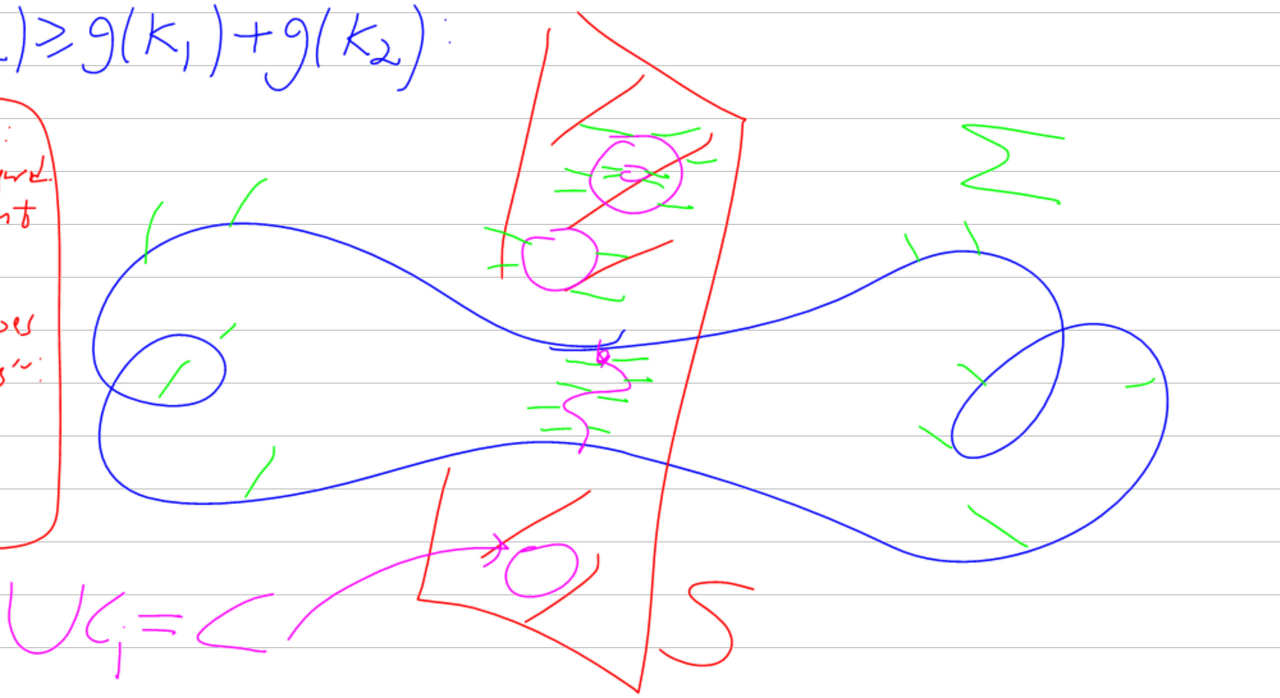
Pf of thm (Modulo all about diff geom & the topology of $\mathbb{R}^3$)

$g(K_1+K_2)\le g(K_1)+g(K_2)$ [Easy, yet...]

$g(K_1+K_2)\ge g(K_1)+g(K_2)$:

Added after class: I should have prepared a clear statement of "what does neck cutting does to to the genus":





Thm If $P+Q=K_1+K_2$ w/ P prime then

either $K_1=P+L$ & $Q=L+K_2$

or $K_2=P+L$ & $Q=L+K_1$ **Pf later!**

Cor If $P+Q_1=P+Q_2$, P prime, then $Q_1=Q_2$

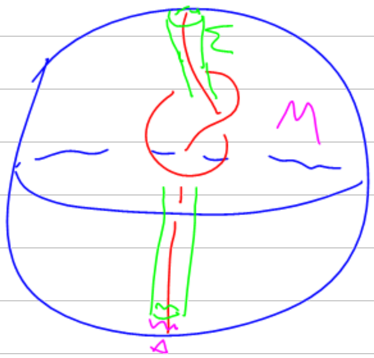
Cor If $P_1+\dots+P_n=P_1'+\dots+P_{n'}'$, $n<n'$ and all are prime, then $n=n'$ & (P_i') is a perm of (P_i).

Pf By induction on n. $n=0$ ...
$n>0$ ...

The $PQ = K_1K_2$ Theorem Following Lickorish.

γ: the curve.
B: a ball w/ P inside & Q outside.
Σ: a sphere separating K_1 & K_2

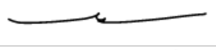
If $\Sigma \cap B = \emptyset$, we're done



Plan for Nov 25.

1 restate PQ=KK thm

We started class all wrong to prove it!

2. Pathologies in $\mathbb{R}^3$: , knotted intervals,
knotted cantor sets, knotted S^2's

3. Alex-Scho thm, ref to M-F.

4. No knotted smooth S^2's; Every (S^3,S) is (S^3,S^2)

5. But yes knotted T^2's! w/ two examples.

6. 3 def's of knots.

7. Reformulation of prime in terms of spheres, an example:

8. start the PQ=KK proof.

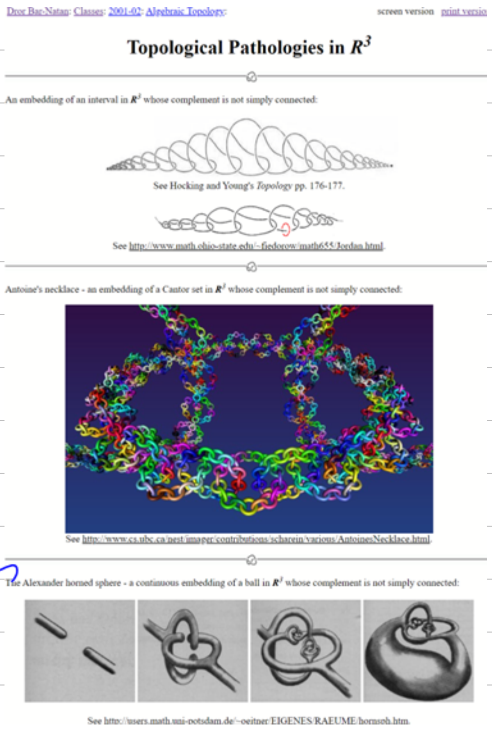
Hour 30, Wednesday November 25: Funny things in R^3, decomposition into primes.

Thm If $P+Q=K_1+K_2$ w/ P prime then

either $K_1=P+L$ & $Q=L+K_2$

or $K_2=P+L$ & $Q=L+K_1$

We started class all wrong to prove this!



Dror Bar-Natan: Classes: 2001-02: Algebraic Topology: screen version print versio

Topological Pathologies in R^3

An embedding of an interval in R^3 whose complement is not simply connected:

See Hocking and Young's *Topology* pp. 176-177.

See http://www.math.ohio-state.edu/~fiedorow/math655/Jordan.html.

Antoine's necklace - an embedding of a Cantor set in R^3 whose complement is not simply connected:

See http://www.cs.ubc.ca/nest/imager/contributions/scharein/various/AntoinesNecklace.html.

The Alexander horned sphere - a continuous embedding of a ball in R^3 whose complement is not simply connected:

See http://users.math.uni-potsdam.de/~oeitner/EIGENES/RAEUME/hornsph.htm.

My hope is to get some better "feel" for $\mathbb{R}^3$ today....

Yet there's the Jordan curve thm, & Alexander-Schönflies. A smooth S^2 in S^3 divides S^3 into two smooth balls. Best proof: Calder Morton-Ferguson

http://katlas.math.toronto.edu/caldermf/3manifolds/3manifolds.pdf

Following Hatcher's

http://pi.math.cornell.edu/~hatcher/3M/3Mdownloads.html

Cor 1. No knotted S^2 in S^3.

2. Every (S^3, S) is (S^3, S^2)

3. But there are knotted T^2 in $\mathbb{R}^3$!

1. nbds of knots (inside is std)
   balls w/ knot removed (outside is std) (never both!)

4. 4 def'ns of "knots": smooth curves.

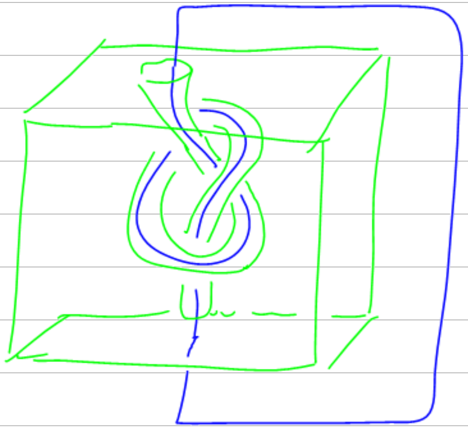
a. Smooth deformations b. Ambient isotopy

c. diffeo d. diags/R-moves.

5. Prime: If an S^2

intersects γ twice, ...

Post Factum: 5.5 below.



6. Beginning of the $PQ = KK$ pf:

γ: the curve.

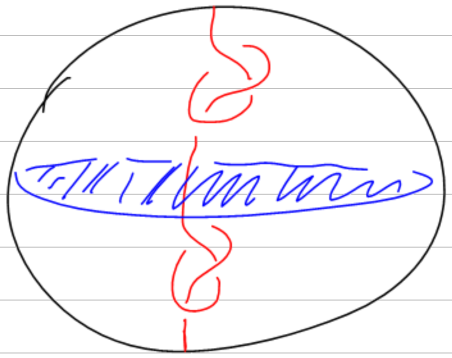
B: a ball w/ P inside & Q outside.

$S := \partial B$

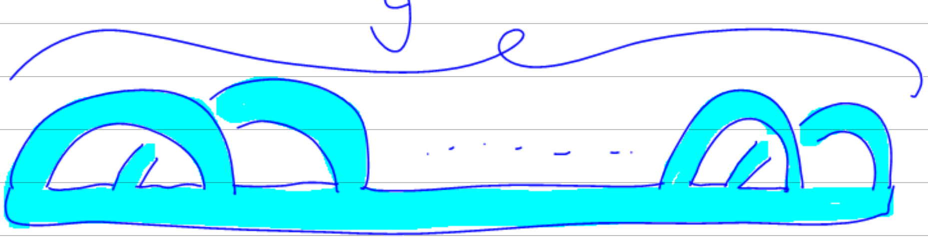
Σ: a sphere separating K_1 & K_2

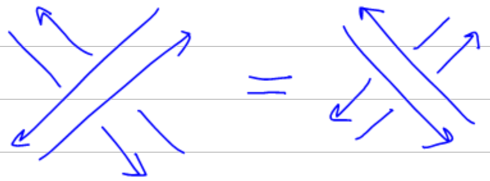
If $S \cap \Sigma = \emptyset$, we are done. Otherwise . . . .

5.5 I should have added words about a knot in a ball & cutting it in half by an equatorial plane that intersects it once:

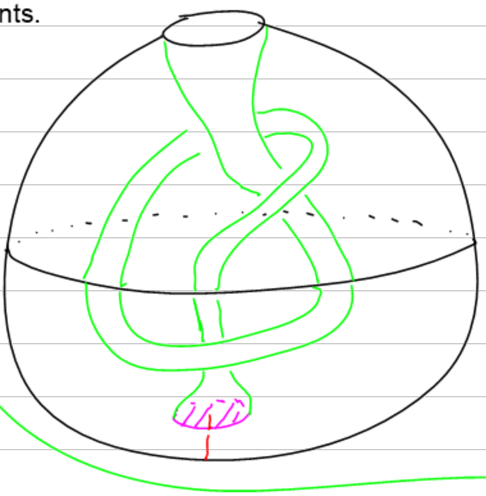


Figures for HW9:





Hour 31, Friday November 27: Decomposition into primes, end. Next: Braids and variants.
Should we have a "HW discussion" class?
HW9 is online!



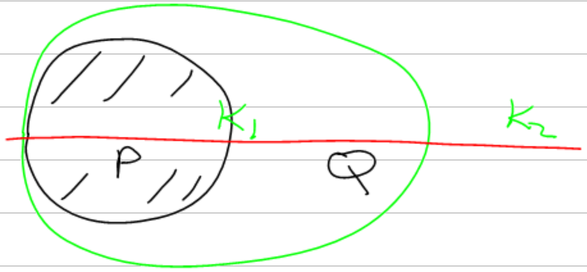
Thm If $P+Q=K_1+K_2$ w/ P prime then
either $K_1=P+L$ & $Q=L+K_2$
or $K_2=P+L$ & $Q=L+K_1$

Proof γ: the curve.

B: a ball w/ P inside & Q outside.

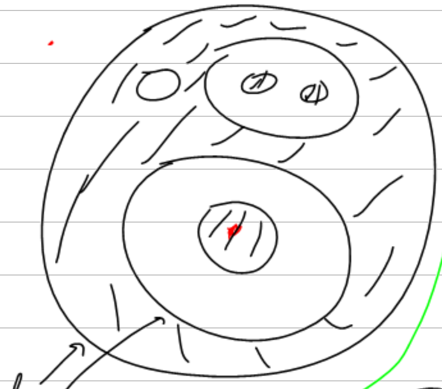
$S:=\partial B$ For drawing, B is standard.

Σ: a sphere separating K_1 & K_2

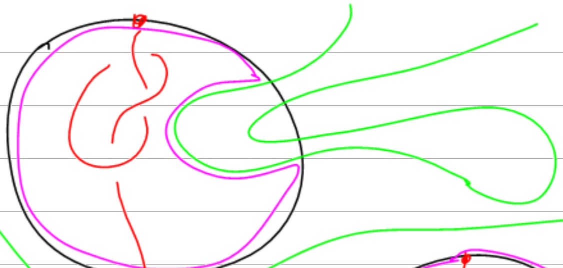


If $S\cap\Sigma=\emptyset$, we are done.

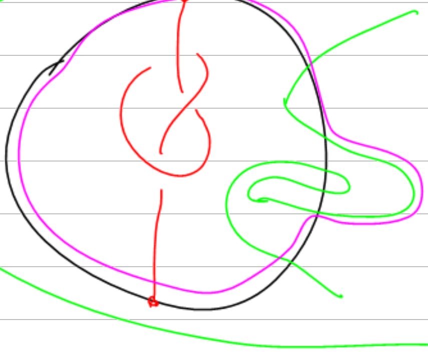
Otherwise, Σ wears a pajama:

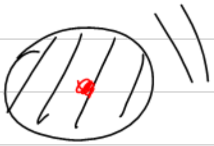


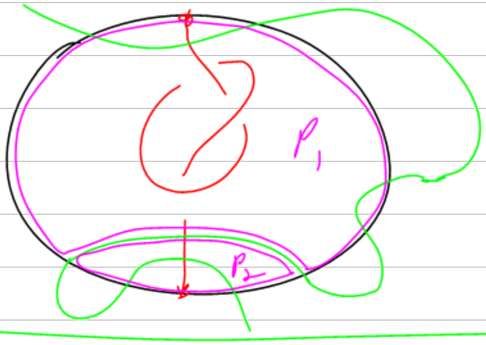
Cases:



Let P be the new B!

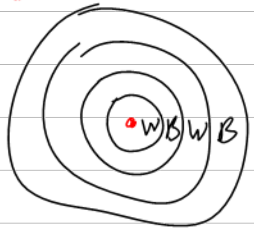


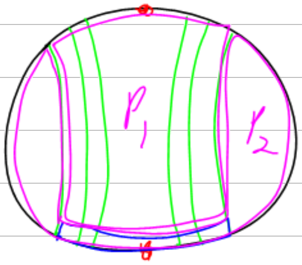




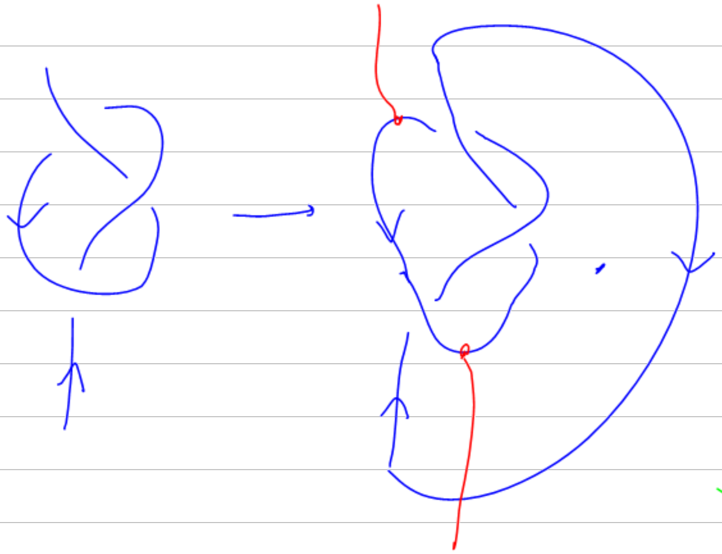
one of P_1 or P_2 is the new B

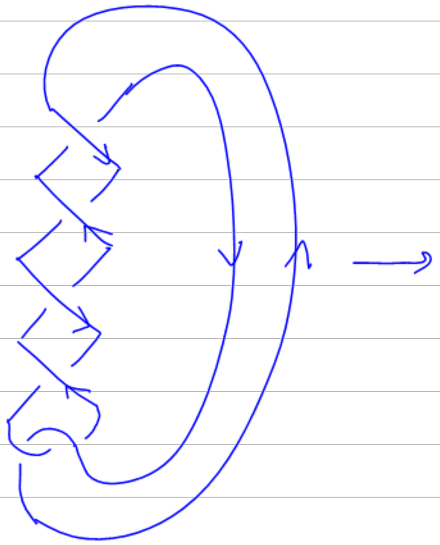
Σ sees

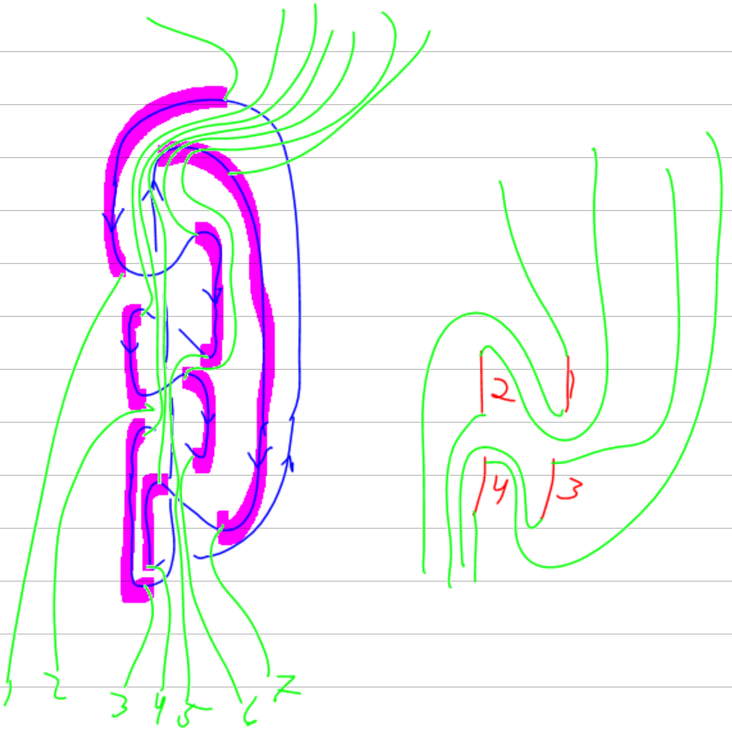




one of P_1 or P_2 is the new B



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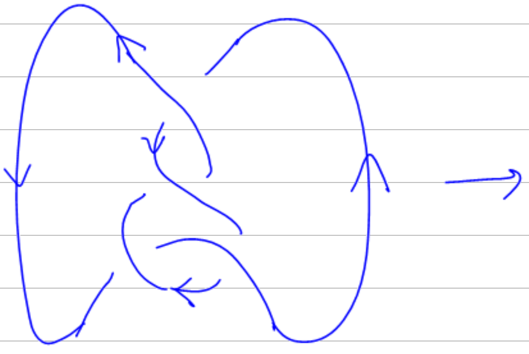
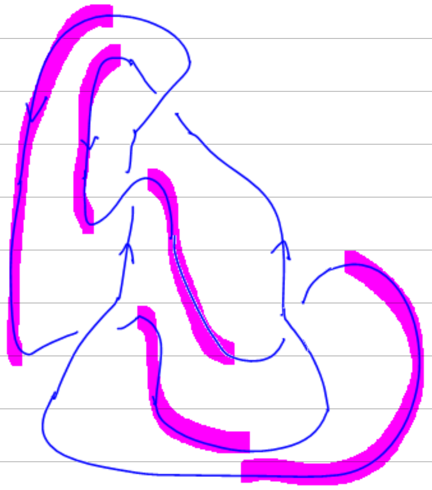
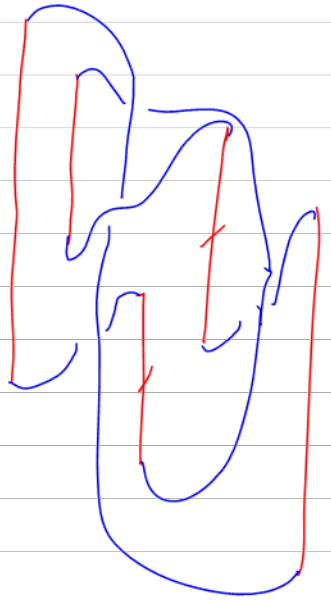






Hour 32, Monday November 30: Braids and knots, combing.
Should we have a "HW discussion" class?

Braids: B_n
Composition
C_n, $\tilde{C}_n$, $\pi_1(\tilde{C}_n)$
gens & rels.

Relevance to knot theory:
1. Alexander's Thm
2. Markov's Thm
3. The n^2 issue
4. An aside on b-type & c-type R-moves.

$$\begin{tikzcd} 1 \arrow[r] & PB_n \arrow[r] \arrow[d, equal] & B_n \arrow[r, "\sigma"] & S_n \arrow[r] & 1 \\ & \pi_1(C_n) & & & \end{tikzcd}$$

$\Rightarrow$ 1. Studying B_n & PB_n is more or less the same.
2. Not split?

Aside on split exact:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "\iota"] & B \arrow[r, bend left, "p"] & C \arrow[l, bend left, "s"] \arrow[r] & 1 \end{tikzcd}$$

$s/\!/p = Id_C$

1. $B = A \times C$ as sets?
2. $A \triangleleft B$
3. C acts on A.

So $B = A \rtimes C$

$$1 \longrightarrow F_{n-1} \longrightarrow PB_n \longrightarrow PB_{n-1} \longrightarrow 1$$

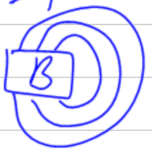
1. split?
2. Combing.

Hour 33, Wednesday December 2: Combing braids.
IOU a correction for the last bit of the proof of unique factorization.
HW10 on web by Thu midnight!

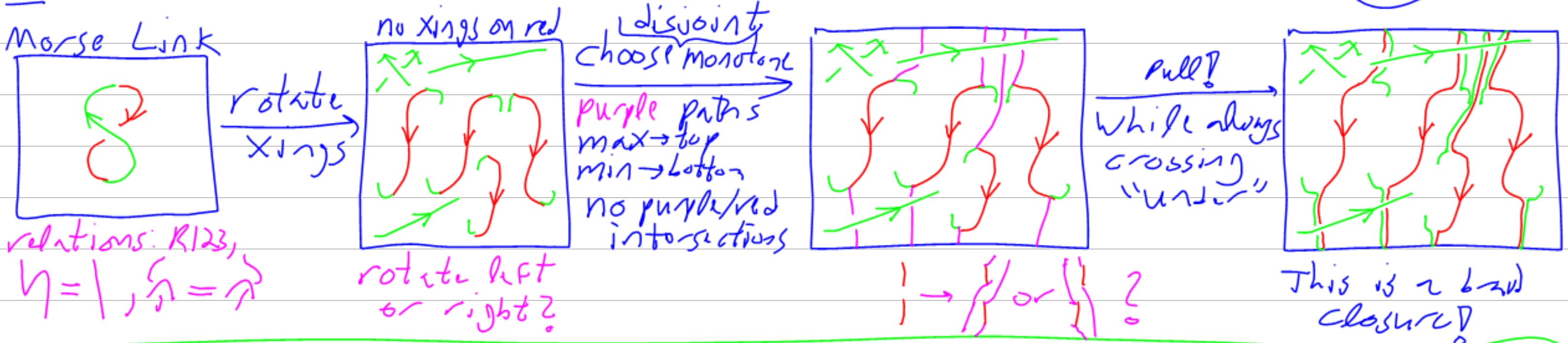
$C_n = \{z \in \mathbb{C}^n : z_i \neq z_j \text{ for } i \neq j\}$ $\quad \tilde{C}_n = C_n / S_n$

$$B_n := \pi_1(\tilde{C}_n) = \left\langle \sigma_i \ \ 1 \le i \le n-1 : \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j|>1 \end{array} \right\rangle$$

Alexander's Thm. Every Knot/Link is a braid closure:



Pf



Describes knots? Comments 1. The n^2 issue

2. The Rb Rc issue.

$$\begin{tikzcd} 1 \arrow[r] & PB_n \arrow[r] \arrow[d, equal] & B_n \arrow[r, "\sigma"] & S_n \arrow[r] & 1 \\ & \pi_1(C_n) & & & \end{tikzcd}$$

⇒ 1. studying B_n & PB_n is more or less the same.

2. Not split!

Aside on split exact:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] & C \arrow[l, bend left, "s"] \arrow[r] & 1 \end{tikzcd}$$

$S/P = Id_C$

1. $B = A \times C$ as sets!

2. $A \triangleleft B$ 3. C acts on A.

So $B = A \rtimes C$

$$1 \longrightarrow F_{n-1} \longrightarrow PB_n \longrightarrow PB_{n-1} \longrightarrow 1$$

1. split!

2. Combing.

Hour 34, Friday December 4: Combing braids.
IOU a correction for the last bit of the proof of unique factorization.
HW10 on web!

$$1 \to PB_n \longrightarrow B_n \xrightarrow{\sigma} S_n \longrightarrow 1$$

1. Studying B_n & PB_n is more or less the same.
2. Not split?

Aside on split exact:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, shift left, "p"] & C \arrow[l, shift left, "s"] \arrow[r] & 1 \end{tikzcd}$$

$s/\!/p = Id_C$

1. $B = AC$ and $A \cap C = \{1\}$
2. $A \lhd B$ hence C acts on A.

So $B = A \rtimes C$

$$1 \longrightarrow F_{n-1} \longrightarrow PB_n \longrightarrow PB_{n-1} \longrightarrow 1$$

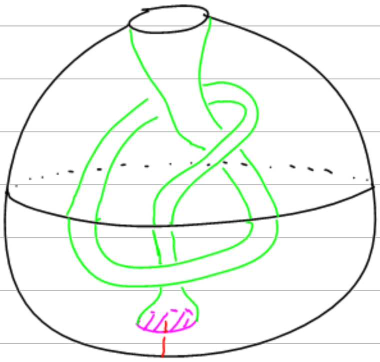
1. split?
2. Combing.

PvB_n, PwB_n, explain PwB_n, explain PvB_n

explain $PwB_n \longrightarrow PvB_n \longrightarrow PwB_n$ [not a complex?]

Theorem. Let $S = \{\tau\}$ be the symmetric group. Then vB is both
$$PvB \rtimes S \cong B * S \Big/ \Big(\gamma_i\tau = \tau\gamma_j \text{ when } \tau i = j,\ \tau(i+1) = (j+1)\Big)$$

Hour ??, ?day December ?: Decomposition into primes, correction.

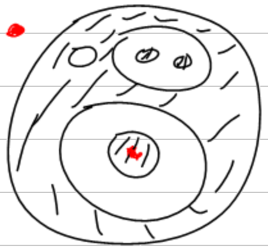


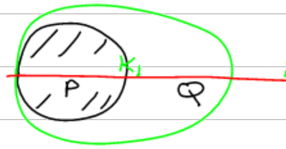
Thm If $P+Q=K_1+K_2$ w/ P prime then either $K_1=P+L$ & $Q=L+K_2$ or $K_2=P+L$ & $Q=L+K_1$

Proof γ: the curve.

B: a ball w/ P inside & Q outside. $S:=\partial B$

Σ: a sphere separating K_1 & K_2

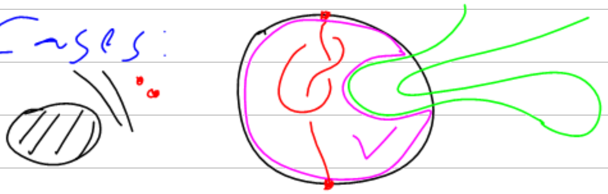


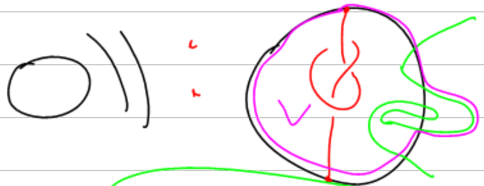


If $S\cap\Sigma=\emptyset$, we are done.

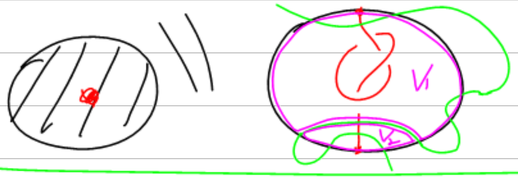
Otherwise, Σ wears a pajama:

Cases:



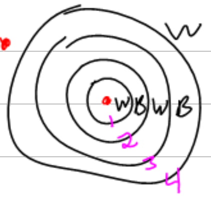


Let V be the new B!

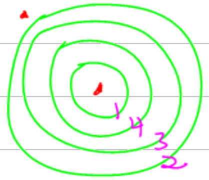


one of V_1 or V_2 is the new B

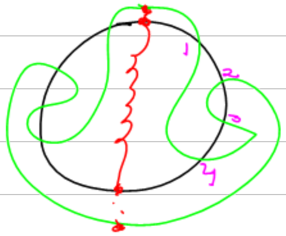
on Σ:

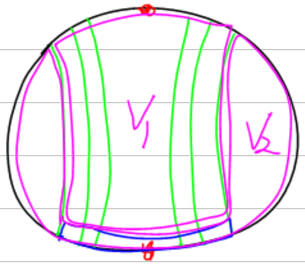


on S:



side view:





one of V_1 or V_2 is the new B

Hour 35, Monday December 7.
Goal for the remaining two classes: More on (uvw)B. Prove that PwB has a "Taylor Expansion".
But first an apology regarding unique factorization, following
http://drorbn.net/AcademicPensieve/Classes/20-1350-KnotTheory/LickorishOnUniqueFactorization.pdf

"silly braids"

$PuB := \ker(uB \longrightarrow S)$

$PvB = \langle \overset{\text{"i over j, +"}}{\sigma_{ij}} : \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij},\ \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \rangle$

"pure virtual braids"

$PwB := PvB / \sigma_{ij}\sigma_{ik} = \sigma_{ik}\sigma_{ij}$ ← "Overcrossings commute (OC)".

"pure w-braids"

σ_{ij}

"The group of (horizontal) flying rings"

R3 & OC for PwB

philosophy for virtual knots

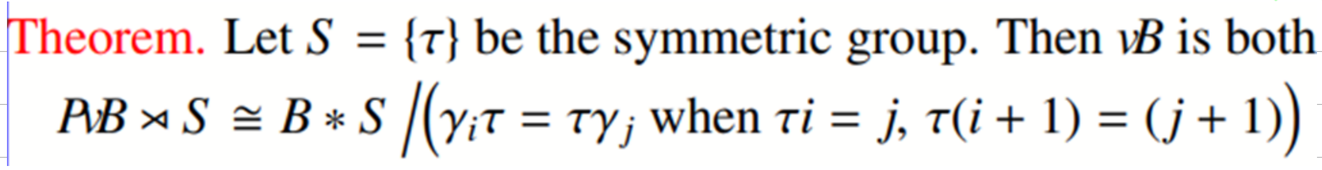
- * PD without the P.
- * virtual crossings.
- * many invts extend.
- * Not an Abelian monoid!
- * Two different mirrors!
- * Two π_1's!

done

The γ/τ presentation of vB / wB

loosely done

not done



Theorem. Let $S = \{\tau\}$ be the symmetric group. Then *vB* is both

$$PvB \rtimes S \cong B * S \Big/\Big(\gamma_i\tau = \tau\gamma_j \text{ when } \tau i = j,\ \tau(i+1) = (j+1)\Big)$$

In summary:

$$\begin{tikzcd}
& 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] \\
& PuB \arrow[r] \arrow[d] & PvB \arrow[r] \arrow[d] & PwB \arrow[d] \\
& uB \arrow[r] \arrow[d] \arrow[dl] & vB \arrow[r] \arrow[d] & wB \arrow[d] \\
\text{Knots \& links} & S_n \arrow[r] \arrow[d] & S_n \arrow[r] \arrow[d] & S_n \arrow[d] \\
& 1 & 1 & 1
\end{tikzcd}$$

G, $\mathbb{Q}G$, I, $\mathcal{A}=\mathcal{A}(G)$, expansions, homo, co-homo

- $C^\infty(\mathbb{R}^n)$
- PB_n & Finite type invariants.

Hour 36 and last, Wednesday December 9.
No time for the full story of "Taylor Expansion for PwB".
Goal: Explain what is a "Taylor Expansion" and mumble about why, for PwB, it is useful.

$$S = \langle \tau_i : \tau_i^2 = 1,\ \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1},\ \tau_i\tau_j = \tau_j\tau_i\ |i-j|>1 \rangle$$

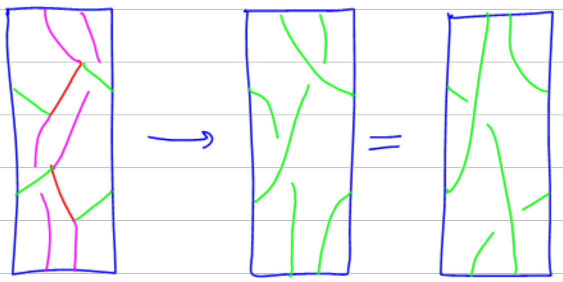
$$\begin{tikzcd}
uB = \langle \gamma_i : \gamma_i\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1},\ \gamma_i\gamma_j = \gamma_j\gamma_i\ |i-j|>1 \rangle \arrow[d] & P_uB = \ker\left(B \xrightarrow{\gamma_i \to \tau_i} S\right) \arrow[l, hook'] \arrow[d, leftrightarrow, "?"] \\
vB = B*S \Big/ \left\langle \begin{array}{l} \tau_i\gamma_{i+1}\tau_i = \tau_{i+1}\gamma_i\tau_{i+1} \\ \tau_i\gamma_j = \gamma_j\tau_i\ \ |i-j|>1 \end{array} \right\rangle \arrow[d] & P_vB = \left\langle \sigma_{ij} : \begin{array}{l} \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \\ \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \end{array} \right\rangle \arrow[l, hook', "?"'] \arrow[d] \\
wB = vB / \gamma_i\gamma_{i+1}\tau_i = \tau_{i+1}\gamma_i\gamma_{i+1} \rangle & P_wB = P_vB / \sigma_{ij}\sigma_{ik} = \sigma_{ik}\sigma_{ij} \arrow[l, hook', "?"']
\end{tikzcd}$$

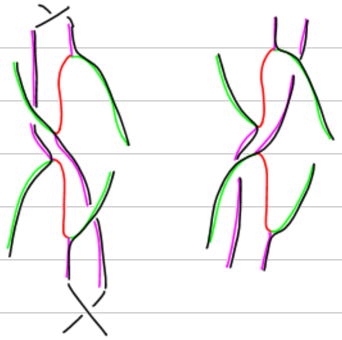
In fact, $vB = P_vB \rtimes S$ and $wB = P_wB \rtimes S$ [so a "good" invariant of PwB may lead to an invt of {links}]

G, QG, I, $A = A(G)$, expansions, homo, co-homo

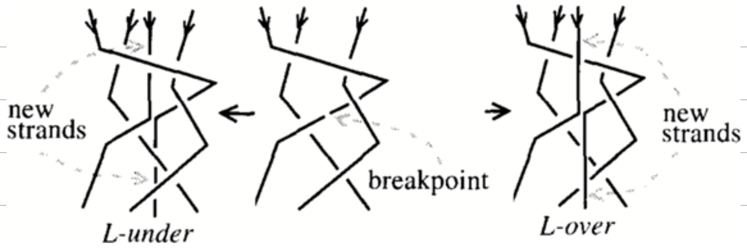
* $C^\infty(\mathbb{R}^n)$

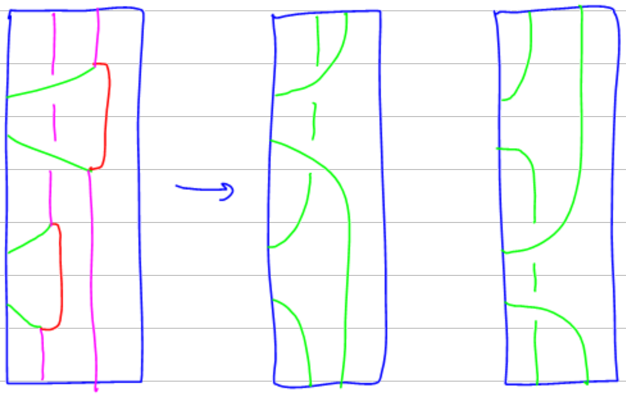
* PB_n & Finite type invariants.



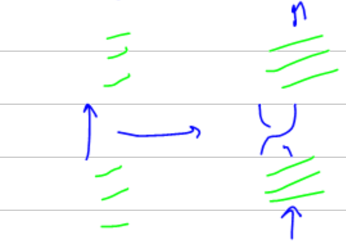


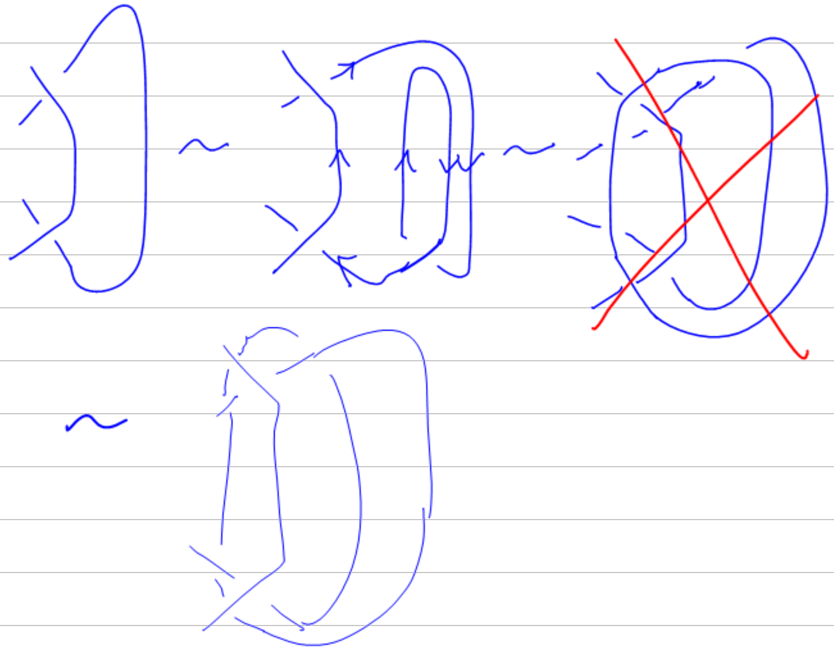
From Lambropoulou-Rourke:

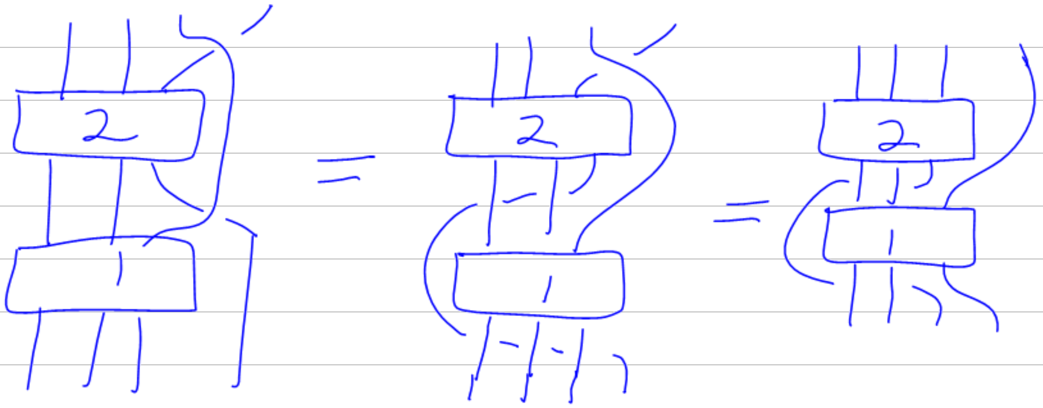


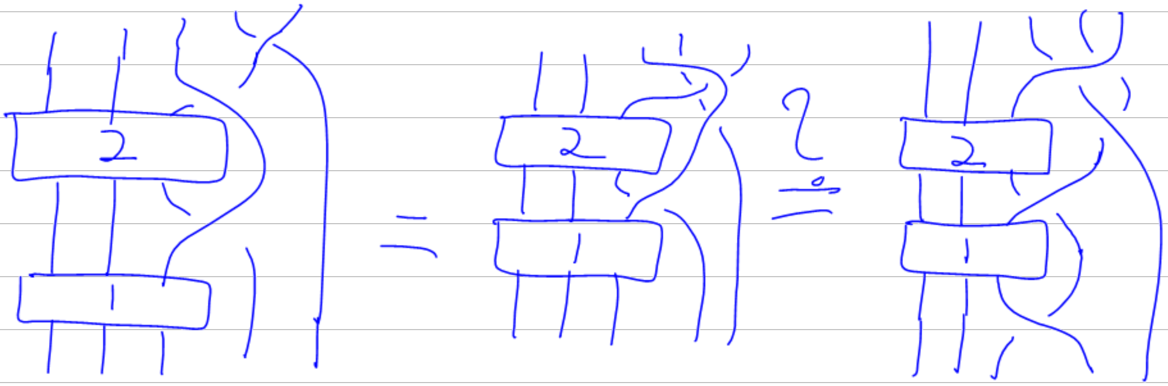


A variant:









From Birman-Brendle, page 27: