18-327 on Tuesday September 18, hour 6: Order, products, subspaces

Remember blackboard shots!

September-11-10 12:29 PM

Today: order, products, subspaces.

Read Along: Munkres 14-16, Preread 17, 18.

HW1 due Thursday.

Def $\mathcal{B} \subset P(X)$ "basis for a topology"

1. $\forall x \in X \; \exists B \in \mathcal{B} \;\; x \in B$
2. $\forall B_1, B_2 \in \mathcal{B}, x \in B_1 \cap B_2 \; \exists B \in \mathcal{B} \;\; x \in B \subset B_1 \cap B_2$

Defines a topology

$\mathcal{T}_{\mathcal{B}} = \{U \subset X : \forall x \in U \; \exists B \in \mathcal{B} \;\; x \in B \subset U\}$

Does every topology have a basis? Is it unique?

Example: $\mathcal{B} = \{(a,b) : a,b \in \mathbb{R} \;\; a<b\}$

Leftover: Given a topological space X & a basis $\mathcal{B}$ for a topology on Y, $f : X \to (Y, \mathcal{T}_{\mathcal{B}})$ is cont. iff $\forall B \in \mathcal{B}$ $f^{-1}(B)$ is open in X.

The order topology: A "complete order" (or "simple order") (or "linear order") on X is a relation on X s.t.

1. For any $x,y \in X$ exactly one of $x=y$, $x<y$, $y<x$ holds.
2. If $x<y$ & $y<z$ then $x<z$.

Examples 1. $(\mathbb{R}, <)$; $(\mathbb{Q}, <)$

2. English ~~words~~ strings in dictionary order: ton<top<topology

3. $\{0,1\} \times \mathbb{N}$ in dict. order.

4. $\mathbb{R} \times \mathbb{R}$ in dict. order.

Def The "order topology $\mathcal{T}_<$" on an ordered set X

is defined by

$$\mathcal{B}_{<} = \{(a,b) : a<b\}$$
$$\cup\{[a_0,b) : a_0 \text{ is "minimal" in } X\}$$
$$\cup\{(a,b_0] : b_0 \text{ is "maximal" in } X\}$$

"The product topology"

Given X, Y topological spaces, we seek a topology on $X\times Y$ s.t.

1. $X\times Y \xrightarrow{\pi_X} X$, $X\times Y \xrightarrow{\pi_Y} Y$ are cont.

2. $f,g : Z \to X, Y$ cont. $\Rightarrow f\times g : Z \to X\times Y$ is cont.

switch to ~~"constructive"~~. no

Thm Such a topology exists and is unique.

done line sans proof

Claim $X \cong X\times\{y_0\}$ & $Y\cong\{x_0\}\times Y$.

The subspace Topology. Given a T.s. X and a subset $Y\subset X$, we seek a topology on Y s.t.

1. $i_Y : Y \hookrightarrow X$ is cont.

2. Given $f : Z \to Y \overset{i_Y}{\hookrightarrow} X$, if $i_Y\circ f$ is cont., then so is f.

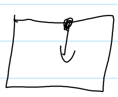
switch to ~~"constructive"~~. no

Thm Such a topology exists and is unique.

Compatibilities. Sub & Sub; Sub & product; Sub & order (in the convex case)

prove Leave as HW

Example The I^2_{dict} is different from $I^2_{ind}\subset\mathbb{R}^2_{dict}$.

 not open open