18-327 on Tuesday October 30, hour 24: Uniform Continuity

September-11-10 12:29 PM

Today: (Above).

Read Along: M27, 37.

"Compact": Every open cover has a finite sub-cover.

cmpct $\subset \mathbb{R}^n \Leftrightarrow$ closed & bndd.

cont. on cmpct $\Rightarrow$ bndd. X, Y cmpct $\Rightarrow X\times Y$ cmpct. zoo

closed $\subset$ cmpct $\Rightarrow$ cmpct

cmpct $\subset T_2 \Rightarrow$ closed $[T_2 \Rightarrow T_3]$

Cont(cmpct) is cmpct [The MVT]

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order of proceedings.

* Unif. cont. as a variation of cont. [most variations are too dramatic]
* The Unif. Cont. Theorem
* The Lebesgue Number Lemma
* Compactness in terms of closed sets.
* If time: ∞-many prisonners on an island.

(Diverted by a student to do "Adrian's proof" see BBS.)

The Lebesgue Number Lemma. Given a cover $\mathcal{U}=\{U_\alpha\}$ of a compact metric space X, there exist $\delta_0>0$ st. $\forall x\in X\ \exists\alpha\ B(x,\delta_0)\subset U_\alpha$.

Proof. Set $\Delta(x)=\sup\{\delta\leq 1 : \exists\alpha\ B(x,\delta)\subset U_\alpha\}$

if $d(x,y)<\epsilon$, then $\Delta(y)\geq\Delta(x)-\epsilon$. [so $|\Delta(x)-\Delta(y)|<\epsilon$]

take $\delta_0=\min\Delta$.

(not done)

The Uniform Continuity Theorem. Def. A "uniformly cont. $f:X\to Y$, X,Y metric"

Thm X compact metric, Y metric, $f:X\to Y$ cont. $\Rightarrow$ f is uniformly cont.

(done not as planned)

f is uniformly cont.

not done.

The FIP. X is compact iff every collection of closed sets that has the fip has an non-trivial intersection.

Pf. (compact) $\Leftrightarrow$ (every collection of closed sets with empty intersection has a finite sub...) $\Leftrightarrow$ (The above.)

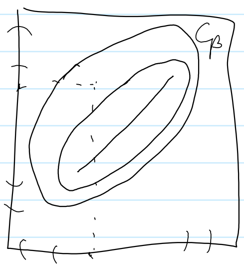
Thm. (Tychonoff) If X_α is compact for every α, then so is $\prod X_\alpha$.

Example $\{0,1\}^\omega$ is compact

EX. $\{0,1\}^\omega \simeq C$, the Cantor set.

Aside on Zorn's Lemma: In a partially-ordered set in which every chain has a bound, there is a maximal element.

Proof. Given $\mathcal{C}$, choose a maximal collection $\mathcal{A}$ having the FIP and containing $\mathcal{C}$, of not-necessarily-closed sets (using Zorn).



Claim $\mathcal{A}$ is closed under finite intersections.

Claim If $B \cap A \neq 0$ for all $A \in \mathcal{A}$, then $B \in \mathcal{A}$.

Now $\forall \alpha$, $\{\overline{\pi_\alpha(A)} : A \in \mathcal{A}\}$ has the FIP,

so choose $x_\alpha \in \bigcap_{A \in \mathcal{A}} \overline{\pi_\alpha(A)}$. Want: $(x_\alpha) \in \bigcap_{C \in \mathcal{C}} C$

Claim If U is a nbd of x_α in X_α, then $\pi^{-1}(U) \in \mathcal{A}$.

Pf $\forall A \in \mathcal{A}$, $U \cap \overline{\pi_\alpha(A)} \neq \emptyset$, so $U \cap \pi_\alpha(A) \neq \emptyset$, so $\pi^{-1}(U) \cap A \neq \emptyset$.

Claim Every basic nbd of (x_α) is in $\mathcal{A}$.

Claim $(x_\alpha) \in \bigcap_{C \in \mathcal{C}} C$ $\square$