18-327 on Thursday September 13, hours 4-5: Topology, Continuity, Bases

September-11-10 12:29 PM

Remember blackboard shots!

Today: "Bases" & some further examples of topologies.

Read Along: Munkres 12-14, Preread 15-16.

HW1 on web by midnight.

Def "Topology": $\mathcal{T} \subset P(X)$ s.t. $\emptyset, X \in \mathcal{T}$ & closed under arbit unions & finite intersections

Def $f: X \to Y$ cont. $\Leftrightarrow$ U open $\Rightarrow f^{-1}(U)$ is open

Examples 0. $\mathcal{T}_{std}$ on $\mathbb{R}$ $[\mathbb{R}_{std}]$

1. $\mathcal{T}_{disc}$ on X $[X_{disc}]$

2. $\mathcal{T}_{triv.}$ on X $[X_{triv}]$

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3. The finite-complement topology $\mathcal{T}_3$.

Def Homeomorphism, homeomorphic

Examples. 1. $(-\frac{\pi}{2}, \frac{\pi}{2})$ is homeomorphic to $\mathbb{R}$.

2. $(-1,1)$ is homeomorphic to $(-\frac{\pi}{2}, \frac{\pi}{2})$

3. $\mathbb{R}$ is homeomorphic to $(-1,1)$.

Definition $\mathcal{T}_1 \supset \mathcal{T}_2$ is "$\mathcal{T}_1$ is finer / bigger / stronger than $\mathcal{T}_2$" while "$\mathcal{T}_2$ is coarser / smaller / weaker" than $\mathcal{T}_1$.

The identity is continuous iff it goes from the finer topology to the weaker one.

Claim $Id: (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is a homeo iff $\mathcal{T}_1 = \mathcal{T}_2$.

Def A "basis for a topology on a set X" is a collection $\mathcal{B} \subset P(X)$ s.t.

not same as linalg!

1. $\forall x \in X \; \exists B \in \mathcal{B}$ s.t. $x \in B$

2. $\forall B_{1,2}\in\mathcal{B}\ \forall x\in B_1\cap B_2\ \exists B_3\in\mathcal{B}$ s.t. $x\in B_3\subset B_1\cap B_2$

Examples 1. $\{(a,b)\}$ 2. $\{[a,b)\}$ "the lower limit topology".

Thm Given a basis for a topology on X,

1. There exists a unique minimal topology $\mathcal{T}_{\mathcal{B}}$ containing $\mathcal{B}$. "characterization of $\mathcal{T}_{\mathcal{B}}$"

2. $\mathcal{T}_{\mathcal{B}}$ is the collection of all unions of elements of $\mathcal{B}$. "construction"

3. $U\in\mathcal{T}_{\mathcal{B}} \iff \forall x\in U\ \exists B\in\mathcal{B}$ s.t. $x\in B\subset U$ "description" of open sets

That was a mistake - too early for this level of abstraction. Should have done 3, then 2, then mention 1.

HW (informal): For any $\mathcal{T}_1, \mathcal{T}_2$ considered today on $\mathbb{R}$, find what it means to be "a continuous function $f:(\mathbb{R},\mathcal{T}_1)\longrightarrow(\mathbb{R},\mathcal{T}_2)$".

done line

The order topology: A "complete order" (or "simple order") on X is a relation on X s.t.

1. For any $x,y\in X$ exactly one of $x=y$, $x<y$, $y<x$ holds.
2. If $x<y$ & $y<z$ then $x<z$.

Examples 1. $(\mathbb{R},<)$; $(\mathbb{Q},<)$

2. English ~~words~~ strings in dictionary order: ton < top < topology

3. $\{0,1\}\times\mathbb{N}$ in dict. order.

4. $\mathbb{R}\times\mathbb{R}$ in dict. order.

Def The "order topology $\mathcal{T}_<$" on an ordered set X is defined by

$$\mathcal{B}_< = \{(a,b) : a<b\}$$

$\cup\{[a_0,b) : a_0$ is "minimal" in $X\}$

$\cup\{(a,b_0] : b_0$ is "maximal" in $X\}$

"The product topology"

Given X,Y topological spaces, we seek a topology on $X\times Y$ s.t.

1. $X\times Y \xrightarrow{\pi_X} X$, $X\times Y \xrightarrow{\pi_Y} Y$ are cont.

2. $f,g: Z\to X,Y$ cont. $\Rightarrow f\times g: Z\to X\times Y$ is cont.

switch to "constructive".

Thm Such a topology exists and is unique.

Claim $X\cong X\times\{y_0\}$ & $Y\cong\{x_0\}\times Y$.

The Subspace Topology. Given a T.S. X and a subset $Y\subset X$, we seek a topology on Y s.t.

switch to "constructive".

1. $i_Y: Y\hookrightarrow X$ is cont.

2. Given $f: Z\to Y \xhookrightarrow{i_Y} X$, if $i_Y\circ f$ is cont., Then so is f.

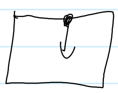
Thm such a topology exists and is unique.

Compatibilities. Sub & Sub; sub & product; sub & order (in the convex case)

Prove

Leave as HW

Example The I^2_{dict} is different from $I^2_{ind}\subset\mathbb{R}^2_{dict}$.



not open

open