18-327 on Thursday October 25, hours 22-23: Compactness

September-11-10 12:29 PM

Today: (Above). 1. In $\mathbb{R}$? 2. Wherefore?

Read Along: M26-27.

HW5 on web by midnight.

"Compact": Every open cover has a finite sub-cover.

Thm. A continuous real function on a compact set is bounded.
PF1 Local to global.
PF2 Sneaky – $X=\bigcup_{n=1}^{\infty} f^{-1}(-n,n)$.

Example. A finite set is compact.

Thm. $[0,1]$ is compact.

Proof. Let $\mathcal{U}$ be an open cover of $[0,1]$.

Let $G=\{g\in[0,1] : \text{a finite subset of } \mathcal{U} \text{ covers } [0,g]\}$ Wish: $1\in G$.

G is non-empty and bounded, so $g_0=\sup(G)$ exists.

Step 1. $g_0>0$. Step 2. $g_0=1$. Step 3. $1\in G$.

Thm. A closed subset of a compact space is compact.

Thm. A compact subset of a T_2 space is closed.

Corollary. A compact T_2 space is T_3.

Corollary. A subset of $\mathbb{R}$ is compact iff it is closed and bounded.

Thm. The image of a compact set by a continuous function is compact. **Cor:** The max value Thm.

Thm. A finite product of compact spaces is compact (& vice-versa, if all spaces are non-empty)

Corollary. A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

done line

The Lebesgue Number Lemma. Given a cover $\mathcal{U}=\{U_\alpha\}$ of a compact metric space X, there exist $\delta_0>0$ s.t. $\forall x\in X\ \exists\alpha\ B(x,\delta_0)\subset U_\alpha$.

Proof. Set $\Delta(x) = \sup\{\delta \leq 1 : \exists \alpha \; B(x,\delta) \subset U_\alpha\}$

if $d(x,y) < \epsilon$, then $\Delta(y) \geq \Delta(x) - \epsilon$. $\quad$ [so $|\Delta(x) - \Delta(y)| < \epsilon$]

take $\delta_0 = \min \Delta$.

The Uniform Continuity Theorem. Def. A "uniformly cont. $f: X \to Y$, X, Y metric"

Thm X compact metric, Y metric, $f: X \to Y$ cont. $\Rightarrow$ f is uniformly cont.

The FIP. X is compact iff every collection of closed sets that has the fip has an non-trivial intersection.

Pf. (compact) $\Leftrightarrow$ (every collection of closed sets with empty intersection has a finite sub...) $\Leftrightarrow$ (The above.)