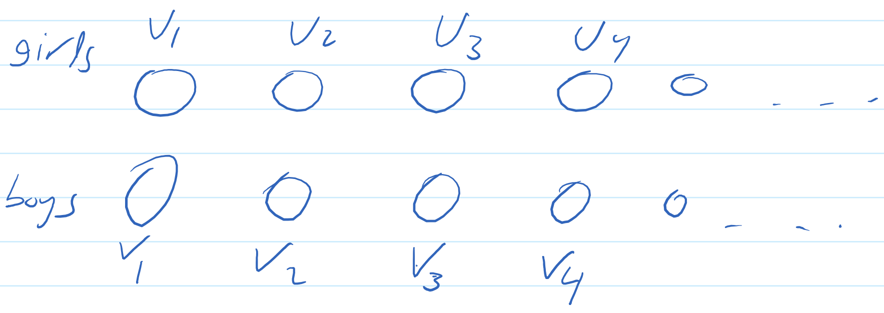
18-327 on Thursday November 29, housr 34-35: Compactness in Metric Spaces

September-11-10 12:29 PM

Today: Above Read: M28,43,45.

HW7 returned, HW8 due, HW9 on web by midnight,



Every girls unfriends all younger boys

Every boy unfriends all younger girls

Compactness in Metric Spaces [Munkres 28,43,45]

Theorem. The Following are Equivalent for a Metric X:

1. X is compact.
2. X is "limit-point-compact".
3. X is "sequentialy compact".
4. X is "totally bounded" & "satisfies Lebesgue's Lemma".
5. X is totally bounded & "complete".

Quoted phrases need to be defined!

$\Updownarrow$ bounded, in $\mathbb{R}^n_{std}$, $\Updownarrow$ closed

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Def X is "limit-point compact" if every infinite subset of X has a limit point.

Thm 1⇒2, compact ⇒ l.p. compact. [True even if X isn't metric]

pf If $A'=\emptyset$ then A is closed; for each $a\in A$ pick an open U_a s.t. $U_a\cap A=\{a\}$. Now $\{X\setminus A\}\cup\{U_a\}$ is an open cover of X, so finitely many of the U_a's cover A, so A is finite.

Def X is "sequencially compact" if every sequence in X has a convergent subsequence.

Thm 2⇒3, l.p. compact ⇒ seg. compact. [only in metrizable!]

Pf Find a l.p. x of $\{x_n\}$ and choose $x_{n_k} \in B_{1/k}(x)$

small mistake! What if $\{x_k\}$ is finite?

Def X is totally bndd if for every ϵ, X can be covered with finitely many ϵ-balls.

Prop: For $X \subset \mathbb{R}^n$, bndd ⟺ totally bndd.

Thm: Seq cmpct ⇒ totally bndd.

Pf Suppose by contradiction $\epsilon > 0$ is such that no finitely many ϵ-balls cover X. ....

Lebesgue's Lemma: If X is metric & compact & $\mathcal{U}$ is an open cover of X, then $\exists \delta > 0$ s.t. $\forall x \in X\ \exists U \in \mathcal{U}$ s.t. $B_\delta(x) \subset U$. (δ is called "the Lebesgue # of $\mathcal{U}$").

Q: How is this useful for uniform continuity?

Proof define $\delta(x) = \sup\{\delta \leq 1 : \exists U \in \mathcal{U}\ B_\delta(x) \subset U\}$.

Clearly $\delta(x)$ exists and is positive.

claim $\delta(x)$ is a conti. fnctn. Pf ...

So it attains its min ...

Thm metric & seq. cmpct ⇒ "satisfies Lebesgue's Lemma"

Lemma Seq cmpct ⇒ cont. functions are bndd & attain their bounds.

Thm Tot. bndd & satisfies Lebesgue's Lemma ⇒ Compact.

done line

Thm Tot. bnd & satisfies Lebesgue's Lemma $\Longrightarrow$ Compact.

done line

Def "Complete": Every Cauchy seq. converges.

Prop $X \subset \mathbb{R}^n$ is complete $\Longleftrightarrow$ X is closed.

Thm $3 \Rightarrow 5$ (Easy)

Thm $5 \Rightarrow 3$

Aside: Every metric space has a "completion": A complete metric space in which it embeds isometrically and densely.