18-327 on Thursday November 22, hours 31-32: Tietze's Theorem, T3.5 spaces

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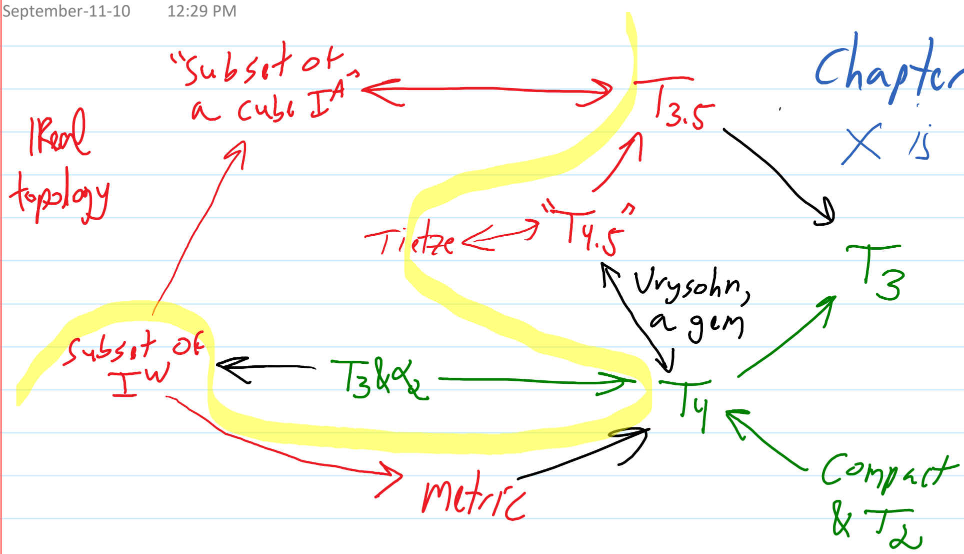


Chapter 4. X is T_1

Riddle: Find an express route metric $\to$ subset of I^A.

Today: Finish Tietze and embed in cubes

Save future generations! Do the course evals!

Read Along: M30-35.

HW6 returned, HW7 due, HW8 on web by midnight Friday.

Today @ GSS: "Nobody solves the quintic" w/ DBN.

Tietze's Theorem. T_4 X, $A \subset X$ closed, $f: A \to \mathbb{R}$ cont.

$\Rightarrow \exists \tilde{F}: X \to \mathbb{R}$ cont. w/ $\tilde{F}|_A = f$.

<u>F bndd:</u> Found $f_0, f_1, f_2 \ldots : X \to \mathbb{R}$ cont. s.t.

$|f - f_n| < \left(\frac{2}{3}\right)^n M$ on A $\quad |f_n - f_{n+1}| < \frac{1}{3}\left(\frac{2}{3}\right)^n M$ on X

Easy: $\tilde{F}(x) := \lim_{n\to\infty} f_n(x)$ exists, $\tilde{F}|_A = f$, $\|F - f_n\|_\infty \to 0$

<u>Lemma</u> A uniformly convergent sequence of cont. functions converges to a cont. limit.

$\mathbb{R}^X_{unif} \supset C = \left\{\begin{matrix}\text{cont.}\\ \text{Fncts}\end{matrix}\right\}$ is seq. closed.

<u>Claim</u> C is simply closed.

Proof .... Now do the unbounded case.....

Proposition. 1. X is $T_{3.5}$ iff $\{[f>0] : f: X \to I \text{ cont.}\}$ is a basis for the topology of X.

2. If X is $T_{3.5}$ & $Y \subset X$, then Y is $T_{3.5}$

3. If X_α is $T_{3.5}$ $\forall \alpha$, then so is $\prod X_\alpha$

Given X let $C_X = C(X, I) = \left\{\substack{\text{cont.}\\ f: X \to I}\right\}$ and let

$\phi: X \to I^{C_X}$ be $\prod_f f$; $\phi_f = f$; $\phi(x)_f = f(x)$.

Theorem. ϕ is an embedding iff X is $T_{3.5}$.

(def: embedding: homeomorphism into its image).

Proof. $\Rightarrow$ A subspace of $T_{3.5}$ is $T_{3.5}$.

$\Leftarrow$ ϕ is clearly 1-1. If $U \subset X$ is $\overbrace{\text{open}}^{\text{basic}}$, we need to show that $\phi(U)$ is open in $f(X)$. Indeed,

$$\phi([f>0]) = \phi(X) \cap [\pi_f > 0]$$

Def $\{f_\alpha\}_{\alpha \in A} \subset C_X$ is "adequate" if $[f_\alpha > 0]$ is a basis for the topology of X.

We have shown that $\phi: X \to I^A$ by $\phi_\alpha = f_\alpha$ is an embedding. Goal: Find smaller adequate sets.

Def X is called "second countable", or α_2, if it has a countable basis for its topology.

Examples: $\mathbb{R}^n$, $\mathbb{R}^{\mathbb{N}}$, but not $\mathbb{R}^{\mathbb{N}}_{box}$

done line

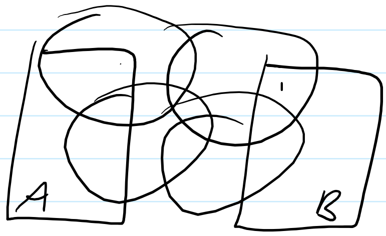
Thm $T_3 + \alpha_2 \Rightarrow T_4$ Pf Given A, B, Thm $T_3 + \alpha_2 \Rightarrow T_4$ Pf Given A, B,

Find a countable basic cover U_k of A, w/ $\overline{U_i} \cap B = \emptyset$

Find a countable basic cover V_k of B w/ $\overline{V_i} \cap A = \emptyset$

Set $U_k' = U_k \setminus \bigcup_{j=1}^{k} \overline{V_k}$ $V_k' = V_k \setminus \bigcup_{j=1}^{k} \overline{U_k}$

$U = \bigcup U_k'$ $V = \bigcup V_k'$

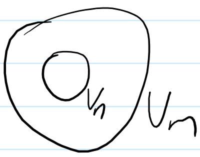


Thm $T_3 + \alpha_2 \Rightarrow \exists$ a countable adequate set.

pf Let $\{U_n\}$ be a basis. If $\overline{U_n} \subset U_m$, Find $F_{n,m}$ w/

$F_{n,m}|_{\overline{U_n}} = 1$, $F_{n,m}|_{U_m^c} = 0$. Then $[F_{n,m} > 0]$ is

a basis.



Corollary: The Urysohn metrization thm:

$T_3 + \alpha_2 \Rightarrow$ metrizable.