18-327 on Thursday November 15, hours 28-29: Separation Axioms

September-11-10 12:29 PM

Today: separation axioms at high TPH

Read Along: M31–33

HW5 returned, HW6 due, HW7 on web by midnight.

$T_0 \Leftarrow T_1 \Leftarrow T_2 \Leftarrow T_3 \Leftarrow T_4$

$T_4 \Leftrightarrow T_{4.5}$ a gem!

$T_3 \Leftarrow T_{3.5} \Leftarrow T_{4.5}$

T_3/Regular: T_1 & can sep pts & closed sets

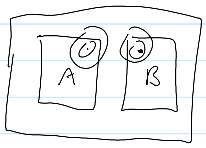
T_4/Normal: T_1 & can sep. pairs of closed sets.

F8: Theorem. X compact $T_2 \Rightarrow X$ normal.

Pf. The usual.

F9: Theorem. X metric $\Longrightarrow X$ normal.

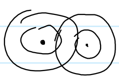
Pf.



For $a \in A$ find ϵ_a s.t. $B(a, \epsilon_a) \subset B^c$; set $U = \bigcup B(a, \frac{\epsilon_a}{2})$

For $b \in B$ find ϵ_b s.t. $B(b, \epsilon_b) \subset A^c$; $V = \bigcup B(b, \epsilon_b/2)$

U & V are disjoint: $B(a, \frac{\epsilon_a}{2}) \cap B(b, \frac{\epsilon_b}{2}) = \emptyset$

 ....

G10: Urysohn's Lemma. $T4 \Rightarrow T4.5$: If X is normal & A, B are disjoint & closed, then $\exists f: X \to I$ cont. s.t. $f(A)=0$, $f(B)=1$.

Lemma: X Normal $\Leftrightarrow$ $\forall$ disjoint closed A, B $\exists$ open U s.t. $A \subset U \subset \bar{U} \subset B^C$

Proof. For every rational $0 \le q \le 1$ we'll construct an open U_q, s.t. $q < r \Rightarrow A \subset U_q \subset \bar{U}_q \subset U_r \subset B^C$

1. For $q=1$, take $U_1 = B^C$

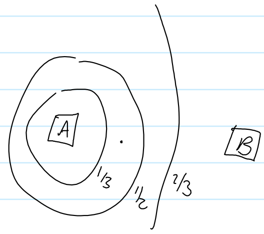
2. For $q=0$, take U_0 separating A & $U_1^C = B$.

3. Order the rationals $q_1 = 1$, $q_2 = 0$, q_3, q_4, ..., assume

$U_{q_1} \dots U_{q_{k-1}}$ were found. Let a be the biggest of $q_1 \dots q_{k-1}$ smaller the q_k, b the smallest of $q_1 \dots q_{k-1}$ larger than q_k. Need $\overline{U_a} \subset U_{q_k} \subset \overline{U_{q_k}} \subset U_b$; find it by separating $\overline{U_a}$ & U_b^c.

4 Set $U_{<0} = \emptyset$, $U_{>1} = X$ and

$$f(x) = \inf\{q : x \in U_q\}$$



$f(A)=0$, $f(B)=1$,

5. Note: $x \in U_q \Rightarrow f(x) \leq q$; $f(x) < q \Rightarrow x \in U_q$

$x \in U_q^c \Rightarrow f(x) \geq q$; $f(x) > q \Rightarrow x \in \overline{U_q}^c$

6. f is cont; indeed, given x & $\epsilon > 0$, find $a, b \in \mathbb{Q}$ with $f(x) - \epsilon < a < f(x) < b < f(x) + \epsilon$, and then $\overline{U_a}^c \cap U_b = V$ is a nbd of x for which $f(V) \subset (f(x) - \epsilon, f(x) + \epsilon)$.

All done!