18-327 on Thursday November 1, hours 25-26: Tychonoff and the Axiom of Choice

September-11-10 12:29 PM

Today: (Above).

Read Along: M37, for fun: M9-11, incl "suppl. exercises"

HW5 due, HW6 on web by midnight!

The FIP. X is compact iff every collection of closed sets that has the fip has an non-trivial intersection.

Pf. (compact) $\Leftrightarrow$ (every collection of closed sets with empty intersection has a finite sub) $\Leftrightarrow$ (The above.)

The AC. $\forall\alpha\; X_\alpha \neq \emptyset \Rightarrow \prod X_\alpha$ is non-empty;

alternatively, if $\pi: X \to Y$ is surjective, then $\exists \sigma: Y \to X$ s.t. $\pi \circ \sigma = I_Y$.

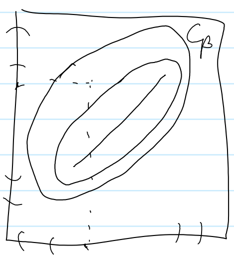
Zorn's Lemma: In a partially-ordered set in which every chain has a bound, there is a maximal element.

Example. Every v.s. has a basis.

Thm. (Tychonoff) If X_α is compact for every α, then so is $\prod X_\alpha$.

Example $\{0,1\}^\omega$ is compact EX. $\{0,1\}^\omega \simeq C$, the Cantor set.

Proof. Given $\mathcal{C}$, choose a maximal collection $\mathcal{A}$ having the FIP and containing $\mathcal{C}$, of not-necessarily-closed sets (using Zorn).



claim If $B \cap A \neq \emptyset$ for all $A \in \mathcal{A}$, then $B \in \mathcal{A}$. !

claim $\mathcal{A}$ is closed under finite intersections.

Now $\forall\alpha$, $\{\overline{\pi_\alpha(A)} : A \in \mathcal{A}\}$ has the FIP, so choose $x_\alpha \in \bigcap_{A\in\mathcal{A}} \overline{\pi_\alpha(A)}$.

Want: $(x_\alpha) \in \bigcap_{C\in\mathcal{C}} C$

Claim If U is a nbd of x_α in X_α, then $\pi^{-1}(U)\in\mathcal{A}$.

PF $\forall A\in\mathcal{A}$, $U\cap\overline{\pi_\alpha(A)}\neq\emptyset$, so $U\cap\pi_\alpha(A)\neq\emptyset$, so $\pi^{-1}(U)\cap A\neq\emptyset$.

Claim Every basic nbd of (x_α) is in $\mathcal{A}$.

done fine.

Claim $(x_\alpha)\in\bigcap_{C\in\mathcal{C}} C$

$\square$

The Evil Neptune

[©](http://drorbn.net) | [Dror Bar-Natan](http://drorbn.net): [Classes](http://drorbn.net): [2018-19](http://drorbn.net): [MAT327F - Introduction to Topology](http://drorbn.net): (34) [Next: Class Home](http://drorbn.net) [Previous: Blackboards for Tuesday October 30](http://drorbn.net)

The Evil Neptune

Neptune, the god of the seas, holds an infinite set H of human prisonners on an island in an ocean far away. One day, he assmebles the humans and tells them:

> Tomorrow I will place a hat on each of you; black or white it will be, and you will not know which it is. You will be able to see all the hats of all other humans, but if any of you will speak or communicate anything with anybody else, I will know, and you will all be doomed. I will then ask you to all call out at the same time the colour of the hat on your heads, and if all but finitely many of you will get it right, you will be set free. Otherwise [evil laughter]. You have until tomorrow to devise a strategy to counter my threat; I so much enjoy seeing humans squirm [more evil laughter]...

Any suggestions to our unlucky species-mates?

* What if all but finitely many hats had the same colour?
* What if $H=\mathbb{Z}$ & Neptune promises to be periodic?
* What if $H=\mathbb{Z}$ & Neptune promises to be periodic w/ finitely many exceptions?
* What if $H=\mathbb{N}$ so $\{0,1\}^{\mathbb{N}}=\mathbb{R}$, & Neptune promises to be rational?

* $X=\{0,1\}^H$ $\quad a\sim b \iff |\{h: a(h)\neq b(h)\}|<\infty$ } setup

$Y:=X/\sim \quad \pi: X\to Y$

$\sigma: Y\to X$ s.t. $\sigma /\!/ \pi = \pi\circ\sigma = I_Y$ } strategy

$v\in X$ } Evil.

$v_h(h')=\begin{cases} v(h') & h'\neq h \\ 0 & h=h'\end{cases}$ } survival.

$$\nu_h(h') = \begin{cases} \nu(h') & h' \neq h \\ 0 & h = h' \end{cases}$$

$$\mu(h) = \sigma(\pi(\nu_h))(h)$$

survival.

Indeed $\pi(\nu_h) = \pi(\nu)$ so $\mu = \sigma(\pi(\nu_h)) = \sigma(\pi(\nu))$ so

$\nu \sim \sigma(\pi(\nu)) = \mu$!

A discussion of the A.C. and infinity.