Things You Should Already Know

I will assume that you are familiar with all of the terms and symbols on this handout. Our first tutorials will go over everything here, just in case something is missing.

1 Basic Set Theory

In the following, A, B, X, Y are sets, I is an indexing set and $\{A_\alpha : \alpha \in I\}$ and $\{B_\alpha : \alpha \in I\}$ are families of sets indexed by I.

- Empty set: $\emptyset$, the set with no elements.
- Subset: $A \subseteq B$ means "$x \in A \implies x \in B$"
- Union: $A \cup B \coloneqq \{x : x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B \coloneqq \{x : x \in A \text{ and } x \in B\}$
- Complement: If $A \subseteq X$, then $X \setminus A \coloneqq \{x : x \in X \text{ and } x \notin A\}$
- Indexed union: $\bigcup_{\alpha \in I} A_\alpha \coloneqq \{x : \exists \alpha \in I,\ x \in A_\alpha\}$
- Indexed intersection: $\bigcap_{\alpha \in I} A_\alpha \coloneqq \{x : \forall \alpha \in I, x \in A_\alpha\}$
- Cartesian product of two sets: $X \times Y \coloneqq \{(x, y) : x \in X,\ y \in Y\}$
- Powers of sets: Y^X is the set of all function $f : X \to Y$.
- The power set of X: $\mathcal{P}(X) \coloneqq \{A : A \subseteq X\} \leftrightarrow \{0, 1\}^X$.

2 Functions

In the following, let X and Y be sets, and let $f : X \to Y$ be a function.

- X is the domain of f.
- Y is the target space or codomain of f.
- $f(X) = \{f(x) : x \in X\} \subseteq Y$ is the range or image of f.
- f is injective (or one-to-one, or an injection)
  $$\forall a, b \in X, \quad f(a) = f(b) \implies a = b.$$
- f is surjective (or onto, or a surjection) if its range is its entire codomain.
- f is bijective (or a bijection) if it is both injective and a surjective.
- The composition of two injective functions is again injective.
- The composition of two surjective functions is again surjective.
- The composition of two bijective functions is again bijective.
- Given a subset $B \subseteq Y$, the preimage of B is the set $f^{-1}(B) \coloneqq \{x \in X : f(x) \in B\}$.
- If f is an injection with range Y, then its inverse function $f^{-1} : Y \to X$ is (1) a function; and (2) bijective.

3 De Morgan's Laws and some Further Relations

The following two expressions are generalized versions of what are called De Morgan's Laws. They describe how unions and intersections interact with complementation.

- $X \setminus \left(\bigcup_{\alpha \in I} A_\alpha \right) = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$
- $X \setminus \left(\bigcap_{\alpha \in I} A_\alpha \right) = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$

The following are elementary facts about how functions interact with operations on subsets of their domains, codomains and ranges. Throughout the following, let X and Y be sets, let $f : X \to Y$ be a function, and let $A, B \subseteq X$ and $C, D \subseteq Y$.

- $A \subseteq B$ implies $f(A) \subseteq f(B)$
- $C \subseteq D$ implies $f^{-1}(C) \subseteq f^{-1}(D)$
- $f(A \cup B) = f(A) \cup f(B)$
- $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
- $f(A \cap B) \subseteq f(A) \cap f(B)$
- $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
- $f(A) \setminus f(B) \subseteq f(A \setminus B)$
- $f^{-1}(C \setminus D) = f^{-1}(C) \setminus f^{-1}(D)$
- $f(X \setminus f^{-1}(Y \setminus C)) \subseteq C$
- $A \subseteq f^{-1}(f(A))$, (with equality if f is injective)
- $f(f^{-1}(C)) \subseteq C$, (with equality if f is surjective)
- $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$

4 Countability and Uncountability

We will spend some time on this in class, but I do expect these words to be familiar to you.

Definition 1. *A set A is said to be countably infinite if there exists a bijection $f : \mathbb{N} \to A$. A set A is said to be countable if it is finite or countably infinite. If A is infinite but not countably infinite, A is said to be uncountable.*

The following theorem gives some equivalent conditions for being countable:

Theorem 2. *For an infinite set A, the following are equivalent:*

1. *A is countable.*
2. *There is an injection $f : A \to \mathbb{N}$.*

3. There is a surjection $g : \mathbb{N} \to A$.

Fact: The following sets are countable:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, the set of algebraic numbers.
- Any infinite subset of a countable set.
- The Cartesian product of two countable sets (and, inductively, the Cartesian product of a finite number of countable sets).
- The union of finitely many countable sets.
- The union of a countable collection of countable sets.
- The countable union of some countable sets and some finite sets.

Fact: The following sets are uncountable:

- $\mathbb{R}$, $\mathbb{R} \setminus \mathbb{Q}$ (the irrational numbers), the set of non-algebraic numbers (i.e. the set of transcendental numbers), $\mathbb{R}^n$.
- Any superset of an uncountable set.
- The power set of any infinite set (countable or otherwise), e.g. $\mathcal{P}(\mathbb{N})$.
- The set $\mathbb{N}^{\mathbb{N}}$ of functions from $\mathbb{N}$ to $\mathbb{N}$.

5 Selected Basic Facts About $\mathbb{R}$

First recall: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. (For us, $0 \notin \mathbb{N}$.)

Fact: Between any two distinct real numbers:

- There are infinitely many rational numbers.
- There are infinitely many irrational numbers.

Fact: Here are some useful facts from calculus:

- $\bigcup_{n\in\mathbb{N}} [\frac{1}{n}, 1] = (0, 1]$.
- $\bigcup_{n\in\mathbb{N}} [0, n] = [0, \infty)$.
- $\sum_{n\in\mathbb{N}} 2^{-n} = 1$.

6 Acknowledgement

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