Do not turn this page until instructed.

MAT327 Introduction to Topology

Term Test

University of Toronto, October 16, 2018

Solve 4 of the 5 problems on the other side of this page.
Each problem is worth 25 points even though they are not equally difficult.
You have an hour and fifty minutes to write this test.

Notes

- No outside material other than stationary is allowed.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and consisting of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.
- Do not write on this examination form! Only what you write in the examination booklets counts towards your grade.
- Indicate clearly which problems you wish to have marked; otherwise an arbitrary subset of the problems you solved will be used.
- In green: at-exam questions and answers.
- In red: post-exam additions/notes. These are staff-internal and not binding.

Good Luck!

Solve 4 of the following 5 problems. Each problem is worth 25 points. You have an hour and fifty minutes. **Neatness counts! Language counts!**

Problem 1.

1. Define the "finite-complement topology" on a given set X. (Q. Do I need to prove that the finite-complement topology is a topology? A. No.)

2. Let X and Y be sets taken with their finite-complement topologies. Prove that a function $f\colon X \to Y$ is continuous if and only if it is either constant or "finite to one" (meaning that $\forall y \in Y$, $|f^{-1}(y)| < \infty$).

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Marked by Jamal. Approximate scheme: 5 marks for part (a), 10 marks for each direction of part (b).

Comment: Be careful when stating definitions! It is not enough to say that a set is open in the finite complement topology iff its complement is finite, since this excludes the empty set when the whole space is infinite.

Problem 2.

1. Define "a topological space X is Hausdorff (T_2)".

2. Prove that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x)\colon x \in X\}$ is a closed subset of $X \times X$.

Tip. "If and only if" always means that there are two things to prove.

Marked by Jamal. Approximate scheme: 5 marks for part (a), 10 marks for each direction of part (b).

Problem 3. Let B be the set of bounded sequences of real numbers. It is a subset of the set $X = \mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers.

1. Prove that if X is taken with the box topology, then B is both an open and a closed subset. (Q. Can I use the fact that if $\operatorname{Bd} A = \emptyset$ then B is clopen? A. Yes.)

2. Prove that if X is taken with the product (cylinders) topology, then B is neither open nor closed.

Marked by Dror. Approximate scheme: One point for understanding B and X, then 12 for each part, divided 3/5/4 for understanding the topology and for the first/second assertion of each part.

Problem 4. Given a set X equipped with a metric d, prove that there exists a unique topology on X for which the following two properties hold:

1. For every $x \in X$, the function $f_x \colon X \to \mathbb{R}$ defined by $f_x(y) = d(x, y)$ is continuous. Q. (Can I take as given that $d \colon X \times X \to \mathbb{R}$ is continuous? A. No.)

2. If Z is any other topological space, and $g \colon Z \to X$ is a function for which for every $x \in X$ the function $h_x \colon Z \to \mathbb{R}$ defined by $h_x(z) = d(x, g(z))$ is continuous, then g itself is continuous.

Tip. "There exists a unique" means two things: "there exists", and "if/once exists, it is unique". Both require a proof!

Marked by Jamal. Approximate scheme: 15 marks for proving the existence of a topology satisfying the two conditions listed (specifically, 5 marks for defining a topology and 10 for checking it works). 10 marks for proving such a topology is unique.

Comment: Remember, to check that two topologies on the same set are equal it is enough to check that the identity map is a homeomorphism between two such spaces. This fact is particularly useful for this problem.

Problem 5. Let (X_n, d_n) be a sequence of metric spaces whose diameters are at most 1: $\forall n \in \mathbb{N}$, $\forall x, y \in X_n$, $d_n(x, y) \leq 1$. Prove that the product $X = \prod_n X_n$ is metrizable. (Q. Do I need to prove that $\sup \frac{1}{n} d_n(x_n, y_n)$ is a metric? A. No.)

Tip. Once you have finished writing an exam, if you have time left, it is always a good idea to go back and re-read and improve everything you have written, and maybe even completely rewrite any parts that came out messy.

Marked by Jamal. Approximate scheme: 5 marks for defining an appropriate metric on the infinite product, 10 marks for checking that the corresponding metric topology refines the product topology, and 10 marks for the reverse direction.

Comment: For a topological space to be metrizable, it is not enough to simply define a metric on that space! You also need to check that the metric topology agrees with the underlying topology. Also, remember that we always equip a product of topological spaces with the product topology (unless otherwise stated).

Good Luck!