LAST (Family) NAME: ________________________________

FIRST (Given) NAME: ________________________________

STUDENT NUMBER: ________________________________

UNIVERSITY OF TORONTO
Faculty of Arts & Science

DECEMBER 2018 EXAMINATIONS

MAT327H1F

Duration: 3 hours
Aids Allowed: None

Exam Reminders:

- Fill out your name and student number on the top of this page.
- Do not begin writing the actual exam until the announcements have ended and the Exam Facilitator has started the exam.
- As a student, you help create a fair and inclusive writing environment. If you possess an unauthorized aid during an exam, you may be charged with an academic offence.
- Turn off and place all cell phones, smart watches, electronic devices, and unauthorized study materials in your bag under your desk. If it is left in your pocket, it may be an academic offence.
- When you are done your exam, raise your hand for someone to come and collect your exam. Do not collect your bag and jacket before your exam is handed in.
- If you are feeling ill and unable to finish your exam, please bring it to the attention of an Exam Facilitator so it can be recorded before leaving the exam hall.
- In the event of a fire alarm, do not check your cell phone when escorted outside.

Special Instructions:

Good Luck!

Exam Format and Grading Scheme: **Solve 5 of the 6 questions on the next page.** If you solve more than 5 questions indicate very clearly on the examination booklets (not on this form) which are the ones that you want marked, or else an arbitrary one may be excluded. The questions are of equal value of 20 points each, even though they might not be of equal difficulty.

Students must hand in all examination materials at the end

Red - post factum comments.

General. Point reductions, though mild, for irrelevant nonsense.

Problem 1. For the purpose of this problem, we say that a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is "Classically Continuous" (CC) if $\forall x \in \mathbb{R}^n\, \forall \epsilon > 0\, \exists \delta > 0\, \forall y \in \mathbb{R}^n\, |x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$, and we say that it is "Modern-Continuous" if whenever $U \subset \mathbb{R}^m$ is open, $f^{-1}(U)$ is an open subset of $\mathbb{R}^n$.

1. Define "$U \subset \mathbb{R}^m$ is open" (relative to the standard topology). (4 points)
2. Prove that CC⇒MC. (8 points)
3. Prove that MC⇒CC. (8 points)

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Tip. Neatness, cleanliness, and organization count, here and everywhere else!

Problem 2. Let (X, d) be a metric space. Define $\phi\colon X \to \mathbb{R}^X$ by $\phi(x)_y := d(x, y)$, whenever $x, y \in X$. Prove that the map ϕ is an embedding (namely, that it is a homeomorphism of X and $\phi(X)$).

Tip. 1-1? Continuous? Continuous inverse?

Tip. In a "fresh" exercise you are welcome to use anything proven in class or in any homework assignment or term test.

Q. Do I need to prove that d itself is a continuous function? **A.** No.

Marking: (4) Being oriented (4) Continuous (6) 1-1 (6) Continuous inverse.

(-1) In proving that ϕ is continuous, used the irrelevant fact that π_y is continuous.

(-4) To show that ϕ is open, showed that for al y, ϕ_y is open.

Problem 3. Let U be a subset of $\mathbb{R}^2$.

1. Define "U is connected". (4 points)
2. Define "U is path-connected". (4 points)
3. Show that if U is open and connected then it is path-connected. (12 points)

Hint. Fix $x_0 \in U$ and show that the set A of points in U that can be reached from x_0 by a path within U is clopen.

(-2) in 1, omitted $A \cup B = U$ and $A \neq \emptyset$ and $B \neq \emptyset$.

(-1) in 3, omitted $A \neq \emptyset$.

(-2) in 3, correct argument but no "$B_\epsilon(x) \subset U$" references.

(-4) Proved that A is closed but not open.

Problem 4. Prove that if Y is a compact topological space and X is an arbitrary topological space then the projection $\pi_1 \colon X \times Y \to X$, defined by $(x, y) \mapsto x$, is a closed map (namely, it sends closed sets to closed sets).

(2/20) Did only the case of $F = A \times B$.

(2/20) "π_1 is open therefore it is closed".

Problem 5. Let X be a T_1 topological space.

1. Define what it means for X to be T_3, T_4, or α_2. ["or" is slightly ambiguous] (8 points)
2. Prove that if X is T_3 and α_2, then it is also T_4. (12 points)

(-2) Correctly defined $T_{3.5}$, instead of T_3.
(-4) $U'_n := U_n \setminus \bigcup_{k \le n} V_k$, instead of $U'_n := U_n \setminus \bigcup_{k \le n} \overline{V_k}$.
(-11) Used Urysohn metrization.
(-11) A countable intersection is open.
(-8) "Half" of the unfriending construction.

Problem 6. Let (X, d) be a metric space.

1. Define "X is complete". (4 points)
2. Suppose that for some $\epsilon > 0$, every ϵ-ball in X has a compact closure. Show that X is complete. (8 points)
3. Suppose that for each $x \in X$ there is an $\epsilon > 0$ such that the ball $B_\epsilon(x)$ has compact closure. Show by means of an example that X need not be complete. (8 points)

Q. Can I use that compact plus metric implies complete? **A.** Yes.

Good Luck!