Pensieve header: October 20: Some Hochschild Homology.

Today. Trees from triangulations, then some Hochschild homology, then whatever you may suggest, then EIWL 9-12, then, if time, Patterns.

Topics (in no particular order). Whatever you may suggest; whatever comes to my mind; ~~the Fibonacci numbers~~; the Catalan numbers; ~~the Jones polynomial~~; ~~a more efficient Jones algorithm~~; ~~a riddle on spheres~~; Khovanov homology; Γ-calculus; the Hopf fibration; Hilbert's 13th problem; non-commutative Gaussian elimination; free Lie algebras; the Baker-Campbell-Hausdorff formula; wacky numbers; an order 4 torus; the Schwarz Lantern; knot colourings; the Temperley-Lieb pairing; the dodecahedral link; sound experiments; barycentric subdivisions; a Peano curve; braid closures and Vogel's algorithm; the insolubility of the quintic; phase portraits; the Mandelbrot set; shadows of the Cantor aerogel; quilt plots; some image transformations; De Bruijn graphs; the Riemann series theorem; finite type invariants and the Willerton fish; ~~the Towers of Hanoi~~; Hochschild homology of (some) coalgebras; convolutions and image improvements.

An Image Manipulation Challenge

The image at http://drorbn.net/bbs/show?shot=17-1750-171016-111042.jpg is pathetic. Can you improve it? Whatever you do, should also work well with all other images at http://drorbn.net/bbs/show.php?prefix=17-1750.

img = 

```
img // FullForm
```

```
Image[RawArray["UnsignedInteger8",
  List[List[List[255, 255, 255], List[255, 255, 255], List[255, 255, 255], List[255, 255, 255],
    List[255, 255, 255], List[255, 255, 255], List[255, 255, 255], List[255, 255, 255], … 247 …,
    List[255, 255, 255], List[255, 255, 255], List[255, 255, 255], List[255, 255, 255],
    List[255, 255, 255], List[255, 255, 255], List[255, 255, 255]], … 190 …, … 1 …]], … 3 …]
```

large output | show less | show more | show all | set size limit...

```
imgd = img // ImageData
```

```
{{{1., 1., 1.}, {1., 1., 1.}, {1., 1., 1.}, {1., 1., 1.}, {1., 1., 1.}, … 252 …,
  {1., 1., 1.}, {1., 1., 1.}, {1., 1., 1.}, {1., 1., 1.}, {1., 1., 1.}}, … 190 …, {… 1 …}}
```

large output | show less | show more | show all | set size limit...

```
Dimensions[imgd]
```

```
{192, 262, 3}
```

```
imgd / 2
```

```
{{{0.5, 0.5, 0.5}, {0.5, 0.5, 0.5}, {0.5, 0.5, 0.5}, {0.5, 0.5, 0.5}, ⋯ 254 ⋯,
  {0.5, 0.5, 0.5}, {0.5, 0.5, 0.5}, {0.5, 0.5, 0.5}, {0.5, 0.5, 0.5}}, ⋯ 190 ⋯, {⋯ 1 ⋯}}
```

large output | show less | show more | show all | set size limit...

```
Image[imgd / 2]
```

```
? *Convolution*
```

▼ System`

ConvolutionLayer	LineIntegralConvolutionScale
LineIntegralConvolutionPlot	ListLineIntegralConvolutionPlot

```
? *Convolve*
```

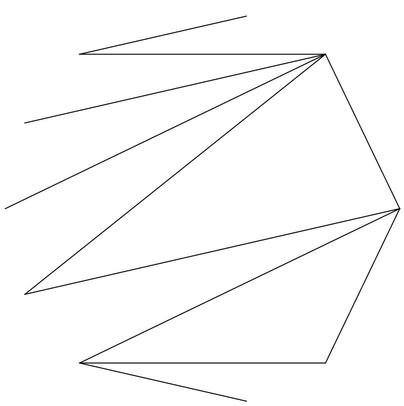
▼ System`

Convolve	DiscreteConvolve	ListConvolve
DirichletConvolve	ImageConvolve	MellinConvolve

Trees from Triangulations

```
triang =
  ds[d[9, 11], d[9, 12], d[0, 12], d[0, 9], d[2, 7], d[2, 6], d[3, 5], d[2, 5], d[2, 8], d[0, 8], d[0, 2]];
```

```
triang /. ds[ls___] → Graphics[{ls}] /. d[i_, j_] :> Line[{i, j}] /. j_Integer :> {Cos[2 π j / 14], Sin[2 π j / 14]}
```



```
d[0, 13] (Times @@ triang) ∏_{j=0}^{12} e[j, j + 1, ●]
```

```
d[0, 2] d[0, 8] d[0, 9] d[0, 12] d[0, 13] d[2, 5] d[2, 6] d[2, 7] d[2, 8] d[3, 5]
 d[9, 11] d[9, 12] e[0, 1, ●] e[1, 2, ●] e[2, 3, ●] e[3, 4, ●] e[4, 5, ●] e[5, 6, ●]
 e[6, 7, ●] e[7, 8, ●] e[8, 9, ●] e[9, 10, ●] e[10, 11, ●] e[11, 12, ●] e[12, 13, ●]
```

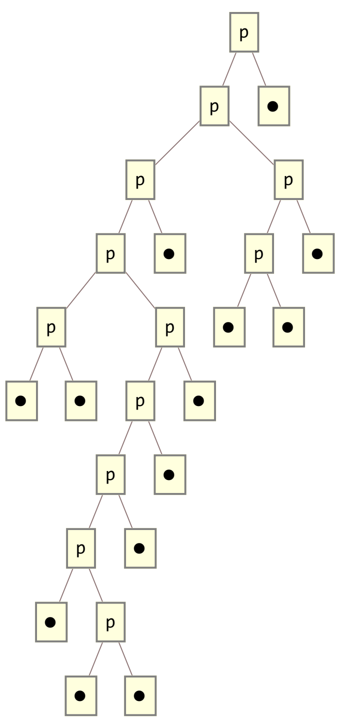
```
d[0, 13] (Times @@ triang) ∏_{j=0}^{12} e[j, j + 1, ●] /. e[i_, j_, t1_] e[j_, k_, t2_] d[i_, k_] :→ e[i, k, p[t1, t2]]
```

```
d[0, 8] d[0, 9] d[0, 12] d[0, 13] d[2, 5] d[2, 6] d[2, 7] d[2, 8] d[3, 5]
 d[9, 11] d[9, 12] e[0, 2, p[●, ●]] e[2, 3, ●] e[3, 4, ●] e[4, 5, ●] e[5, 6, ●]
 e[6, 7, ●] e[7, 8, ●] e[8, 9, ●] e[9, 10, ●] e[10, 11, ●] e[11, 12, ●] e[12, 13, ●]
```

```
d[0, 13] (Times @@ triang) ∏_{j=0}^{12} e[j, j + 1, ●] //. e[i_, j_, t1_] e[j_, k_, t2_] d[i_, k_] :→ e[i, k, p[t1, t2]]
```

```
e[0, 13, p[p[p[p[p[●, ●], p[p[p[p[●, p[●, ●]], ●], ●], ●]], ●], p[p[●, ●], ●]], ●]]
```

```
Last[d[0, 13] (Times @@ triang) ∏_{j=0}^{12} e[j, j + 1, ●] //.
   e[i_, j_, t1_] e[j_, k_, t2_] d[i_, k_] :→ e[i, k, p[t1, t2]]] // TreeForm
```



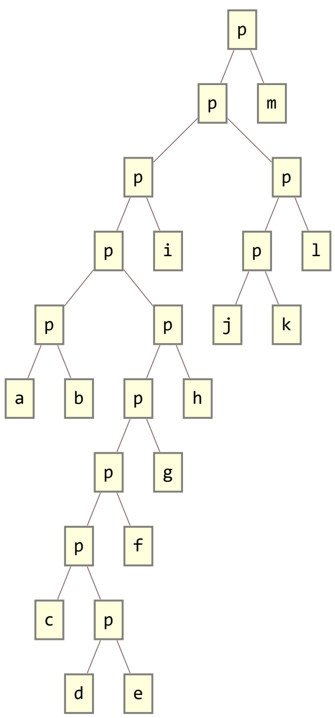
Last[d[0, 13] (Times @@ triang) $\prod_{j=0}^{12}$ e[j, j + 1, ] //.
e[i_, j_, t1_] e[j_, k_, t2_] d[i_, k_] :→ e[i, k, p[t1, t2]]]

p[p[p[p[p[, ],
p[p[p[p[, p[, ]], ], ], ]], ],
p[p[, ], ]], ]

FromLetterNumber[13]

m

Last[d[0, 13] (Times @@ triang) $\prod_{j=0}^{12}$ e[j, j + 1, FromLetterNumber[j + 1]] //.
e[i_, j_, t1_] e[j_, k_, t2_] d[i_, k_] :→ e[i, k, p[t1, t2]]] // TreeForm



Last[d[0, 13] (Times @@ triang) $\prod_{j=0}^{12}$ e[j, j + 1, FromLetterNumber[j + 1]] //.
e[i_, j_, t1_] e[j_, k_, t2_] d[i_, k_] :→ e[i, k, StringJoin["(", t1, t2, ")"]]]

((((ab)((((c(de))f)g)h))i)((jk)l))m)

Some Hochschild Homology

```
g[x_, y_] := f[y] - f[x + y] + f[x];
1 (g[y, z] - g[x + y, z] + g[x, y + z] - g[x, y])
```

0

```
Clear[g]
```

```
Clear[d]
```

```
d_{n_,k_}[ℰ_] := ℰ /. {x_{i_} /; i < k :> x_i, x_{i_} /; i == k :> x_k + x_{k+1}, x_{i_} /; i > k :> x_{i+1}}
```

```
d_{2,1}[g[x_1, x_2]]
```

$g[x_1 + x_2, x_3]$

```
d_{2,2}[g[x_1, x_2]]
```

$g[x_1, x_2 + x_3]$

```
d_{2,0}[g[x_1, x_2]]
```

$g[x_2, x_3]$

```
d_{2,3}[g[x_1, x_2]]
```

$g[x_1, x_2]$

```
d_{n_}[ℰ_] := Expand@Sum[(-1)^k d_{n,k}[ℰ], {k, 0, n + 1}]
```

```
d_2[g[x_1, x_2]]
```

$-g[x_1, x_2] + g[x_1, x_2 + x_3] + g[x_2, x_3] - g[x_1 + x_2, x_3]$

```
g[x_1, x_2] // d_2 // d_3
```

0

```
g[x_1, x_2, x_3] // d_3 // d_4
```

0

```
List @@ Expand[(x_1 + x_2 + x_3)^2]
```

$\{x_1^2, 2 x_1 x_2, x_2^2, 2 x_1 x_3, 2 x_2 x_3, x_3^2\}$

```
With[{n = 3, d = 4}, List @@ Expand[(Sum[x_i, {i, 1, n}])^d]]
```

$\{x_1^4, 4 x_1^3 x_2, 6 x_1^2 x_2^2, 4 x_1 x_2^3, x_2^4, 4 x_1^3 x_3, 12 x_1^2 x_2 x_3, 12 x_1 x_2^2 x_3, 4 x_2^3 x_3, 6 x_1^2 x_3^2, 12 x_1 x_2 x_3^2, 6 x_2^2 x_3^2, 4 x_1 x_3^3, 4 x_2 x_3^3, x_3^4\}$

```
C_{0,d_} := If[d == 0, {1}, {}];
C_{n_,d_} := Union @@ Table[x_n^k C_{n-1,d-k}, {k, 0, d}];
```

```
C_{1,2}
```

$\{x_1^2\}$

```
C_{3,4}
```

$\{x_1^4, x_1^3 x_2, x_1^2 x_2^2, x_1 x_2^3, x_2^4, x_1^3 x_3, x_1^2 x_2 x_3, x_1 x_2^2 x_3, x_2^3 x_3, x_1^2 x_3^2, x_1 x_2 x_3^2, x_2^2 x_3^2, x_1 x_3^3, x_2 x_3^3, x_3^4\}$

```
d_3 /@ C_{3,4}
```

$\{-4 x_1^3 x_2 - 6 x_1^2 x_2^2 - 4 x_1 x_2^3, x_1^3 x_2 - 3 x_1^2 x_2 x_3 - 3 x_1 x_2^2 x_3, x_1^2 x_2^2 + 2 x_1^2 x_2 x_3 - 2 x_1 x_2 x_3^2, x_1 x_2^3 + 3 x_1 x_2^2 x_3 + 3 x_1 x_2 x_3^2,$
$x_2^4 + 4 x_2^3 x_3 + 6 x_2^2 x_3^2 + 4 x_2 x_3^3 + x_3^4, -x_1^3 x_4 - 3 x_1^2 x_2 x_4 - 3 x_1 x_2^2 x_4, -2 x_1 x_2 x_3 x_4, 2 x_1 x_2 x_3 x_4,$
$3 x_2^2 x_3 x_4 + 3 x_2 x_3^2 x_4 + x_3^3 x_4, -2 x_1^2 x_3 x_4 - x_1^2 x_4^2 - 2 x_1 x_2 x_4^2, -2 x_1 x_2 x_3 x_4, -2 x_2^2 x_3 x_4 + 2 x_2 x_3 x_4^2 + x_3^2 x_4^2,$
$-3 x_1 x_3^2 x_4 - 3 x_1 x_3 x_4^2 - x_1 x_4^3, -3 x_2 x_3^2 x_4 - 3 x_2 x_3 x_4^2 + x_3 x_4^3, -4 x_3^3 x_4 - 6 x_3^2 x_4^2 - 4 x_3 x_4^3\}$

```
d4 /@ d3 /@ C3,4
```

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
? Coefficient
```

Coefficient[*expr*, *form*] gives the coefficient of *form* in the polynomial *expr*.
Coefficient[*expr*, *form*, *n*] gives the coefficient of *form*^*n* in *expr*. >>

```
Coefficient[(x + y)^3, x y^2]
```

3

```
M_{n_,p_} := Table[Coefficient[d_n[a], b], {b, C_{n+1,p}}, {a, C_{n,p}}]
```

$C_{2,3}$

$\{x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3\}$

d_2 /@ $C_{2,3}$

$\{-x_1^3 - 3 x_1^2 x_2 - 3 x_1 x_2^2, -2 x_1 x_2 x_3, 2 x_1 x_2 x_3, 3 x_2^2 x_3 + 3 x_2 x_3^2 + x_3^3\}$

$C_{3,3}$

$\{x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3, x_1^2 x_3, x_1 x_2 x_3, x_2^2 x_3, x_1 x_3^2, x_2 x_3^2, x_3^3\}$

Coefficient$[d_2[x_1 x_2^2], x_1 x_2 x_3]$

2

$d_2[x_1 x_2^2]$

$2 x_1 x_2 x_3$

$M_{2,3}$

{{-1, 0, 0, 0}, {-3, 0, 0, 0}, {-3, 0, 0, 0}, {0, 0, 0, 0},
{0, 0, 0, 0}, {0, -2, 2, 0}, {0, 0, 0, 3}, {0, 0, 0, 0}, {0, 0, 0, 3}, {0, 0, 0, 1}}

$M_{3,3}.M_{2,3}$

{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0},
{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0},
{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

```
β_{n_,p_} := Length[NullSpace[M_{n,p}]] - MatrixRank[M_{n-1,p}]
```

$\beta_{3,3}$

0

```
Table[β_{n,p}, {n, 1, 5}, {p, 1, 5}] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$