Pensieve header: Sep 25: Further ways to compute Fibonacci.

Topics (in no particular order). Whatever you may suggest; whatever comes to my mind; the Fibonacci numbers; ~~the Jones polynomial~~; a more efficient Jones algorithm; a riddle on spheres; Khovanov homology; Γ-calculus; the Hopf fibration; Hilbert's 13th problem; non-commutative Gaussian elimination; free Lie algebras; the Baker-Campbell-Hausdorff formula; wacky numbers; an order 4 torus; the Schwarz Lantern; knot colourings; the Temperley-Lieb pairing; the dodecahedral link; sound experiments; barycentric subdivisions; a Peano curve; braid closures and Vogel's algorithm; the insolubility of the quintic; phase portraits; the Mandelbrot set; shadows of the Cantor Aerogel; quilt plots; some image transformations; De Bruijn graphs; the Riemann series theorem; finite type invariants and the Willerton fish.

Possible (Smallish) Projects

Get some neat things out of DBN: Classes: 2015-16: MAT 475 Problem Solving Seminar: Quiz 1:

Import["http://drorbn.net/17-1750/About.html"]

$\(\def\bbN{{\mathbb N}} \def\bbQ{{\mathbb Q}} \def\bbR{{\mathbb R}} \def\bbZ{{\mathbb Z}} \)$
© | Dror Bar-Natan : Classes : 2017-18 : Math 1750 - Shameless Mathematica : (1)   Next: Blackboards for Monday September 11
Previous: Class Home   About This Class
URL: http://drorbn.net/17-1750 and/or
  http://www.math.toronto.edu/~drorbn/classes/17-1750/ShamelessMathematica/ .
Agenda: Sophisticated mathematics with Mathematica!
Instructor: Dror Bar-Natan , drorbn@math.toronto.edu
  (for administrative matters only; math on email is slow and prone to
  misunderstandings, so I generally avoid it). Office: Bahen 6178, 416-946-5438.
Classes: Mondays 10-11 and Fridays 10-12 at Bahen 6183.
Intended for. Graduate students. While there are no formal prerequisites, a high
  level of mathematical sophistication will be assumed. Typically this would mean
  having taken a few core graduate classes here, or their parallels elsewhere.
Office hours. By appointment.
Software. By the time of the second class that you attend, you MUST arrange for
  yourself 24/7 access to Mathematica , either by installing the $150 student
  edition , or by whatever other means; otherwise please don't bother taking
  this class. There is a fair chance that the department will be able to help
  registered students with the purchase. Whatever you do, keep your receipts!
Textbook. An Elementary Introduction to the Wolfram Language , by Stephen Wolfram.
Course Description. Yes, we all dream of the day we will prove that
  powerful theorem, whose beauty and sophistication will leave our colleagues
  breathless. It will, of course, be a product or pure thought, affirming that
  our intellect rises far, oh so far, beyond everybody else's. Obviously, no
  computers will be used. We are artists and philosophers, not technicians.

As a temporary measure I have learned to work with computers, and I plan to
  share what I have learned with you. For me, computer-assisted mathematics
  is a powered exoskeleton (seen Avatar? Iron Man?) for the brain. It's still

my inner powers that everybody should admire, yet they reach much farther
now that I've learned how to integrate them so tightly with the machine.
Learn that too and reach far! I often use the platform "Mathematica" (though
not only), and hence that's what I'll teach (though perhaps not only).

The Monday classes will be a systemmatic introduction
to Mathematica following our textbook .

The Friday classes will be divided into chapters, each about some (mathematical) real
life problem that I have at some point encountered and solved with computers. The
typical chapter will start with a mathematical introduction (sometimes deep and
meaningful in itself). I will then pose a computational problem, and challenge you to
solve it better than the solution that I will present. Many (though not all) of the
problems will involve algebraic computations in knot theory, as this is what I know
best. There will also be graphics, and some interaction with the web and with $\TeX$.

Possible Topics (in no particular order). Whatever you may suggest; whatever
comes to my mind; the Fibonacci numbers; the Jones polynomial; a more efficient
Jones algorithm; a riddle on spheres; Khovanov homology; Γ-calculus; the
Hopf fibration; Hilbert's 13th problem; non-commutative Gaussian elimination;
free Lie algebras; the Baker-Campbell-Hausdorff formula; wacky numbers;
an order 4 torus; the Schwarz Lantern; knot colourings; the Temperley-Lieb
pairing; the dodecahedral link; sound experiments; barycentric subdivisions;
a Peano curve; braid closures and Vogel's algorithm; the insolubility of
the quintic; phase portraits; the Mandelbrot set; shadows of the Cantor
Aerogel; quilt plots; some image transformations; De Bruijn graphs; the
Riemann series theorem; finite type invariants and the Willerton fish.

The Final Grade. Impress me! And if you impress me enough and convince me that you've
learned something, your grade will be high. Throughout the class I will suggest
ideas for projects, and you can come up with your own ideas too. One idea you can
always follow is to "one up" something that was done in class: make it better,
cleaner, meaner, prettier, more efficient, more general, something. I will expect
you to submit 4-5 projects during the semester. Elegance and cleanliness count!

Projects are to be submitted via email to my normal email address
drorbn@math.toronto.edu . Each project MUST be a single file (though it can be a
zip file that expands to a whole directory structure). Filenames MUST be of the
form "lastname_firstname_date_title.ext", where "lastname" is your last name,
"firstname" is your first name, "date" is the date of submission in the form yymmdd
(so the first day of classes, for example, is 170908), "title" is a very short
descriptive title, and "ext" is some filename extension such as "nb" or "zip".

By default all submissions will be made public at
http://drorbn.net/AcademicPensieve/Classes/17-1750-ShamelessMathematica/
StudentProjects/ , though you may request that
specific projects will be kept private.

Class Photo. To help us learn each other's names, I will take a class photo on
Friday of the third week of classes. I will post the picture on the class'
web site and you will be required to identify yourself in the picture. With
your individual consent, I will also post your names on the picture page.

Apologies.

There are places where exoskeleta simply don't work. Try swimming with one.
It's a huge pain to learn exoskeleta. You think "I'll just wear it and start
punching". Oh no, the balance is completely different; at first you'll simply
fall over. Exoskeleta are bulky, rough and unpleasant. There are sharp
ends, even, no, especially on the inside. Air conditioning? Don't make me

```
   laugh.  Try showing up to the conference dinner wearing your exoskeleton.
   Nobody will want to sit next to you. Deep inside, everybody will think that
   you no longer belong.  If ever you get too used to wearing your exoskeleton,
   you'll find that you are pink and vulnerable when you are not wearing it.
   If you want to remain tough, face the elements wearing only your sword.
```

 Solution lines of the system
$\dot{x}=a_1x^2+\sqrt{2}b_1xy+c_1y^2$,
$\dot{y}=a_2x^2+\sqrt{2}b_2xy+c_2y^2$,
with independent normal random a_i, b_i, c_i.

Import["**http://drorbn.net/AcademicPensieve/Classes/16-475-ProblemSolving/Quiz-01.pdf**"]

$\left\{ \right\}$

```
Rasterize[
 Import[
   "http://drorbn.net/AcademicPensieve/Classes/16-475-ProblemSolving/Quiz-01.pdf"][[1]],
 ImageResolution → 120
]
```

Name (Last, First): _____   Student ID: _____

Dror Bar-Natan: Classes: 2015-16: MAT 475 Problem Solving Seminar: http://drorbn.net/16-475

Quiz 1 on January 14, 2016:   Search for a Pattern . You have 25 minutes to solve one of the two problems below.
Please write on both sides of the page.   **Good Luck!**

Problem 1. (Larson's 1.1.8; my estimate: medium) Prove that a list can be made of all the subsets of a nite set in such
a way that   (i) the empty set is rst in the list,   (ii) each subset occurs exactly once, and   (iii) each subset in
the list is obtained from the preceding one either by adding one element or my removing one element.

Problem 2 (Larson's 1.1.9; my estimate: easy). Determine the number of odd coefficients in the expansion of $(x + y)^{1000}$.

```
Rasterize[
  Import[
    "http://drorbn.net/AcademicPensieve/Classes/16-475-ProblemSolving/Quiz-01.pdf"][[1]],
  ImageResolution → 120
] // ImageCrop
```

Name (Last, First): _____   Student ID: _____

Dror Bar-Natan: Classes: 2015-16: MAT 475 Problem Solving Seminar: http://drorbn.net/16-475

Quiz 1 on January 14, 2016:   Search for a Pattern . You have 25 minutes to solve one of the two problems below.
Please write on both sides of the page.   **Good Luck!**

Problem 1. (Larson's 1.1.8; my estimate: medium) Prove that a list can be made of all the subsets of a finite set in such
a way that   (i) the empty set is first in the list,   (ii) each subset occurs exactly once, and   (iii) each subset in
the list is obtained from the preceding one either by adding one element or my removing one element.

Problem 2 (Larson's 1.1.9; my estimate: easy). Determine the number of odd coefficients in the expansion of $(x + y)^{1000}$.

Further ways to compute Fibonacci

The Naive Way

```
f0[0] = f0[1] = 1; f0[n_] := f0[n - 1] + f0[n - 2];
```

```
Table[Timing[n → f0[n]], {n, 20, 30}] // MatrixForm
```

$$
\begin{pmatrix}
0. & 20 \to 10\,946 \\
0.03125 & 21 \to 17\,711 \\
0.046875 & 22 \to 28\,657 \\
0.0625 & 23 \to 46\,368 \\
0.109375 & 24 \to 75\,025 \\
0.15625 & 25 \to 121\,393 \\
0.265625 & 26 \to 196\,418 \\
0.46875 & 27 \to 317\,811 \\
0.90625 & 28 \to 514\,229 \\
1.4375 & 29 \to 832\,040 \\
2.53125 & 30 \to 1\,346\,269
\end{pmatrix}
$$

The Naive Way, Corrected

```
f1[0] = f1[1] = 1; f1[n_] := (f1[n] = f1[n - 1] + f1[n - 2]);
```

```
Timing[f1[100]]
```

{0., 573 147 844 013 817 084 101}

"prev", "cur", and "While".

```
f2[n_] := (
  {k, prev, cur} = {1, 1, 1};
  While[k < n, {prev, cur} = {cur, prev + cur}; ++k];
  cur
)
```

```
f2[100]
```

573 147 844 013 817 084 101

```
f3[n_] := Module[{k, prev, cur},
  {k, prev, cur} = {1, 1, 1};
  While[k < n, {prev, cur} = {cur, prev + cur}; ++k];
  cur
 ]
```

```
f3[40]
```

165 580 141

"prev", "cur", and "For".

```
For[k = cur = prev = 1, k < 100, ++k, {prev, cur} = {cur, prev + cur}]; cur
```

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"prev", "cur", and "Do".

```
{prev, cur} = {1, 1};
Do[{prev, cur} = {cur, prev + cur}, 99];
cur
```

573 147 844 013 817 084 101

A "While" loop for $\{f_1, f_2, ...\}$ (using negative indices)

```
f6[n_] := Module[{fs = {1, 1}},
  While[Length[fs] ≤ n, fs = Append[fs, fs[[-1]] + fs[[-2]]]];
  Last[fs]
 ]
```

```
f6[100]
```

573 147 844 013 817 084 101

```
f7[n_] := Module[{fs = {1, 1}},
  While[Length[fs] ≤ n, fs = Append[fs, fs[[-1]] + fs[[-2]]]];
  fs
 ]
```

```
f7[20]
```

{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10 946}

```
f8[n_] := Module[{fs = {1, 1}},
  While[Length[fs] ≤ n, AppendTo[fs, fs[[-1]] + fs[[-2]]]];
  fs
 ]
```

```
f8[10]
```
{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89}

A "While" loop for {f_1, f_2, ...} (using "Total" and "Most")

```
f9[n_] := Module[{fs = {1, 1}},
  While[Length[fs] ≤ n, AppendTo[fs, 1 + Total@Drop[fs, -1]]];
  fs
 ]
```

```
f9[10]
```
{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89}

```
f10[n_] := Module[{fs = {1, 1}},
  While[Length[fs] ≤ n, AppendTo[fs, 1 + Total@Most@fs]];
  fs
 ]
```

```
f10[10]
```
{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89}

"Series" and $\frac{1}{1-x-x^2}$

```
Series[ 1/(1 - x - x²), {x, 0, 10}]
```
$1 + x + 2 x^2 + 3 x^3 + 5 x^4 + 8 x^5 + 13 x^6 + 21 x^7 + 34 x^8 + 55 x^9 + 89 x^{10} + O[x]^{11}$

```
D[ 1/(1 - x - x²), {x, 100}] / 100! /. x → 0
```
573 147 844 013 817 084 101

```
Series[ 1/(1 - x - x²), {x, 0, 10}]〚10〛
```

⊞ Part: Part 10 of $1 + x + 2 x^2 + 3 x^3 + 5 x^4 + 8 x^5 + 13 x^6 + 21 x^7 + 34 x^8 + 55 x^9 + 89 x^{10} + O[x]^{11}$ does not exist.

$\left(1 + x + 2 x^2 + 3 x^3 + 5 x^4 + 8 x^5 + 13 x^6 + 21 x^7 + 34 x^8 + 55 x^9 + 89 x^{10} + O[x]^{11}\right)$〚10〛

```
Series[ 1/(1 - x - x²), {x, 0, 10}] // FullForm
```
SeriesData[x, 0, List[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89], 0, 11, 1]

```
Series[ 1/(1 - x - x²), {x, 0, 100}]〚3, 101〛
```
573 147 844 013 817 084 101

"SeriesCoefficient" and $\frac{1}{1-x-x^2}$

? SeriesCoefficient

> SeriesCoefficient[*series*, *n*] finds the coefficient of
> the nth-order term in a power series in the form generated by Series.
> SeriesCoefficient[*f*, {*x*, x_0, *n*}] finds the coefficient of $(x - x_0)^n$
> in the expansion of *f* about the point x = x_0.
> SeriesCoefficient[*f*, {*x*, x_0, n_x}, {*y*, y_0, n_y}, ...] finds a
> coefficient in a multivariate series. ≫

SeriesCoefficient$\left[1 \big/ \left(1 - x - x^2\right), \{x, 0, 100\}\right]$

573 147 844 013 817 084 101

SeriesCoefficient$\left[\dfrac{1}{1 - x - x^2}, \{x, 0, 100\}\right]$

573 147 844 013 817 084 101

$$\dfrac{1}{1 - x - x^2}$$
$$\dfrac{1}{1 - x - x^2}$$

A Sum of Binomial Coefficients

? Sum

> Sum[*f*, {*i*, i_{max}}] evaluates the sum $\sum_{i=1}^{i_{max}} f$.
>
> Sum[*f*, {*i*, i_{min}, i_{max}}] starts with $i = i_{min}$.
> Sum[*f*, {*i*, i_{min}, i_{max}, *di*}] uses steps *di*.
> Sum[*f*, {*i*, {i_1, i_2, ...}}] uses successive values i_1, i_2, ...
>
> Sum[*f*, {*i*, i_{min}, i_{max}}, {*j*, j_{min}, j_{max}}, ...] evaluates the multiple sum $\sum_{i=i_{min}}^{i_{max}} \sum_{j=j_{min}}^{j_{max}} ... f$.
>
> Sum[*f*, *i*] gives the indefinite sum $\sum_i f$. ≫

? Floor

> Floor[*x*] gives the greatest integer less than or equal to *x*.
> Floor[*x*, *a*] gives the greatest multiple of *a* less than or equal to *x*. ≫

? Binomial

> Binomial[*n*, *m*] gives the binomial coefficient $\binom{n}{m}$. ≫

f13[*n*_] := **Sum**[**Binomial**[*n* − k, k], {k, 0, **Floor**[*n* / 2]}];
f13[**100**]

573 147 844 013 817 084 101

$$\mathbf{f14}[\mathit{n}\_] := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathbf{Binomial}[\mathit{n} - k, k]$$

Solve for an "explicit" formula, then use it.

? Solve

Solve[*expr*, *vars*] attempts to solve the
     system *expr* of equations or inequalities for the variables *vars*.
Solve[*expr*, *vars*, *dom*] solves over the domain *dom*. Common choices
     of *dom* are Reals, Integers, and Complexes. ≫

Solve$\left[\lambda^2 == \lambda + 1, \ \lambda\right]$

$\left\{\left\{\lambda \to \frac{1}{2}\left(1 - \sqrt{5}\right)\right\}, \left\{\lambda \to \frac{1}{2}\left(1 + \sqrt{5}\right)\right\}\right\}$

λ /. **Solve**$\left[\lambda^2 == \lambda + 1, \ \lambda\right]$

$\left\{\frac{1}{2}\left(1 - \sqrt{5}\right), \frac{1}{2}\left(1 + \sqrt{5}\right)\right\}$

$\{\lambda_1, \lambda_2\} = \lambda$ /. **Solve**$\left[\lambda^2 == \lambda + 1, \ \lambda\right]$

$\left\{\frac{1}{2}\left(1 - \sqrt{5}\right), \frac{1}{2}\left(1 + \sqrt{5}\right)\right\}$

Solve$[\{\alpha + \beta == 1, \ \lambda_1 \alpha + \lambda_2 \beta == 1\}, \ \{\alpha, \beta\}]$

$\left\{\left\{\alpha \to \frac{1}{10}\left(5 - \sqrt{5}\right), \beta \to \frac{1}{10}\left(5 + \sqrt{5}\right)\right\}\right\}$

$\alpha \lambda_1^n + \beta \lambda_2^n$ /. **Solve**$[\{\alpha + \beta == 1, \ \lambda_1 \alpha + \lambda_2 \beta == 1\}, \ \{\alpha, \beta\}]$

$\left\{\frac{1}{5} \times 2^{-1-n}\left(1 - \sqrt{5}\right)^n\left(5 - \sqrt{5}\right) + \frac{1}{5} \times 2^{-1-n}\left(1 + \sqrt{5}\right)^n\left(5 + \sqrt{5}\right)\right\}$

f15[*n*_] := **Expand**$\left[\frac{1}{5} \times 2^{-1-n}\left(1 - \sqrt{5}\right)^n\left(5 - \sqrt{5}\right) + \frac{1}{5} \times 2^{-1-n}\left(1 + \sqrt{5}\right)^n\left(5 + \sqrt{5}\right)\right]$

f15[**100**]

573 147 844 013 817 084 101

Using "MatrixPower"

```
MatrixPower[( 1  1
              1  0 ), 100]
```

{{573 147 844 013 817 084 101, 354 224 848 179 261 915 075},
 {354 224 848 179 261 915 075, 218 922 995 834 555 169 026}}

```
f16[n_] := MatrixPower[( 1  1
                         1  0 ), 100][[1, 1]]
```

```
f16[100]
```

573 147 844 013 817 084 101

Using $f_{2n} = f_n^2 + f_{n-1}^2$ and $f_{2n+1} = f_n(f_{n+1} + f_{n-1})$

A "categorified" version (using lists)

A "categorified" version (using strings)

"ReplaceRepeated" on $\begin{pmatrix} n \\ f_{n-1} \\ f_n \end{pmatrix}$.

"NestWhile" on $\begin{pmatrix} n \\ f_{n-1} \\ f_n \end{pmatrix}$.

"Nest" on $\begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix}$.

Other Items

Continue looking at Charlene's project? (Probably not; enough Fibonacci!)

A look at Etienne's project? (Maybe)