MAT1750: Khovanov Homology

Charlene Chu - December 6, 2017

Problem

Challenge: Implement Khovanov Homology as it is defined in the handout. Your implementation should fit in one page and should be good enough to show that Kh(5_1) ≠ Kh(10_{132}).

Solution

Please note that the core of this source code is from Dror’s lecture (http://www.math.toronto.edu/drorbn/classes/17-1750-ShamelessMathematica/NB-1204.html). The function Kh[*K*] returns the the vertices and edges for the Khovanov Homology of a knot *K*. We can use the edges to construct a graph, as seen in the examples. To show Kh(5_1) ≠ Kh(10_{132}), we will compare the number of vertices and edges of Kh[K51]. and Kh[K10132].

```
Kh[K_] := Module[{VerticesAndEdges, Vertices, Edges},
  SetAttributes[{B, P}, Orderless]; t = 0; ϵ /: ϵ^p_ /; p > 1 := 0;
  VerticesAndEdges = (List @@ (Expand[
            Times @@ (K /. X[i_, j_, k_, l_] :>
                (++t; P[i, j] P[k, l] + B[t] P[i, l] P[j, k] + ϵ S[t] X[i, j, k, l]))
           ] //. B[i__] B[j__] :> B[i, j] /. P[a_, b_] :> P[a, b][Min[a, b]] //.
          P[a_, b_][m1_] P[b_, c_][m2_] :> P[a, c][Min[m1, m2]] /.
         {P[i_, i_][m_] :> c[m], P[_, _][m_]^2 :> c[m]})) /. {
      (B[b__] S[s_] X[i_, j_, k_, l_] P[i_, l_][m1_] P[k_, j_][m2_] :>
        {B[b] c[Min[m1, m2]], B[b, s] c[m1] c[m2]}), (S[s_] X[i_, j_, k_, l_]
        P[l_, k_][m1_] P[j_, i_][m2_] :> {c[m2] c[m1], B[s] c[Min[m1, m2]]})
     } /. {ϵ fr_, ϵ to_} :> ϵ (fr → to);
  Vertices = Select[VerticesAndEdges, ! MemberQ[#, ϵ] &];
  Edges = Select[VerticesAndEdges, MemberQ[#, ϵ] & ] /. ϵ :> 1;
  {Vertices, Edges}]
```

Examples

We will look at the following knots: the Trefoil Knot 3_1, the Cinquefoil Knot 5_1, and the knot 10_{132}.

```
K31 = PD[X[3, 1, 4, 6], X[1, 5, 2, 4], X[5, 3, 6, 2]];
K51 = PD[X[1, 6, 2, 7], X[3, 8, 4, 9], X[5, 10, 6, 1], X[7, 2, 8, 3], X[9, 4, 10, 5]];
K10132 = PD[X[4, 2, 5, 1], X[8, 4, 9, 3], X[5, 12, 6, 13], X[15, 18, 16, 19], X[9, 16, 10, 17],
   X[17, 10, 18, 11], X[13, 20, 14, 1], X[19, 14, 20, 15], X[11, 6, 12, 7], X[2, 8, 3, 7]];
```

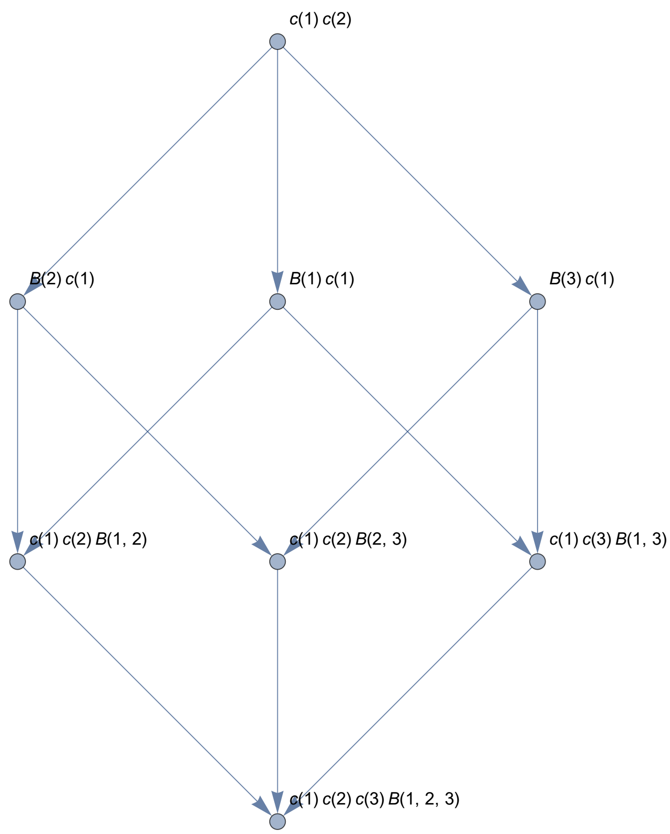
Here, we can see the Kh[*K*] returns a list of two list, the first being a list of vertices and the second being a list of edges.

```
KhTrefoil = Kh[K31]

{{B[1] c[1], B[2] c[1], B[3] c[1], c[1] c[2], B[1, 2] c[1] c[2],
  B[2, 3] c[1] c[2], B[1, 3] c[1] c[3], B[1, 2, 3] c[1] c[2] c[3]},
 {c[1] c[2] → B[2] c[1], B[1] c[1] → B[1, 2] c[1] c[2], B[3] c[1] → B[2, 3] c[1] c[2],
  B[1, 3] c[1] c[3] → B[1, 2, 3] c[1] c[2] c[3], c[1] c[2] → B[3] c[1],
  B[2] c[1] → B[2, 3] c[1] c[2], B[2] c[1] → B[1, 2] c[1] c[2], B[3] c[1] → B[1, 3] c[1] c[3],
  B[2, 3] c[1] c[2] → B[1, 2, 3] c[1] c[2] c[3], B[1] c[1] → B[1, 3] c[1] c[3],
  B[1, 2] c[1] c[2] → B[1, 2, 3] c[1] c[2] c[3], c[1] c[2] → B[1] c[1]}}
```

We can use this information to graph the vertices and edges.

```
Graph[KhTrefoil[[2]], VertexLabels → "Name"]
```



The graphs of Kh[K51] and Kh[K10132] are too large to display nicely in this document, so we will just compare the vertices and edges to show that $Kh(5_1) \neq Kh(10_{132})$.

```
Length[#] & /@ Kh[K51]

{32, 80}

Length[#] & /@ Kh[K10132]

{1024, 5120}
```

We can see that the Khovanov Homology of the Cinquefoil Knot 5_1 has a different number of vertices and edges from the knot 10_{132}. Therefore, we can conclude that $Kh(5_1) \neq Kh(10_{132})$.