17-1350-AKT Fri Mar 3, Brute Hour 7: Proof of the Main Lemma

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The Main $\mathfrak{g}_0$ Theorem.

Raw Version. The $\mathfrak{g}_0$ invariant of any S-component tangle T can be written in the form $Z(T) = \mathbb{O}\left(\omega e^{L+Q} \mid \prod_{i\in S} e_i l_i f_i\right)$, where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i = e^{h_i}$), where $L = \sum a_{ij} h_i l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij} and where $Q = \sum b_{ij} e_i f_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij}. Furthermore, after setting $h_i = h$ and $t_i = t$ for all i, the invariant $Z(T)$ is poly-time computable.

Proof. Indeed,

0. $R^s = e^{s(h\otimes l+e\otimes f)} = \mathbb{O}\left(\exp\left(s\,\mathrm{hl} + \frac{e^{sh}-1}{h}\,\mathrm{ef}\right) \Big| \, e\otimes \mathrm{lf}\right)$,
1. $\mathbb{O}\left(e^{\gamma l+\beta e} \mid \mathrm{le}\right) = \mathbb{O}\left(e^{\gamma l+e^{\gamma}\beta e} \mid \mathrm{el}\right)$,
2. $\mathbb{O}\left(e^{\gamma l+\beta f} \mid \mathrm{fl}\right) = \mathbb{O}\left(e^{\gamma l+e^{\gamma}\beta f} \mid \mathrm{lf}\right)$,
3. $\mathbb{O}\left(e^{\beta e+\alpha f+\delta ef} \mid \mathrm{fe}\right) = \mathbb{O}\left(\nu e^{\nu(-\alpha\beta h+\beta e+\alpha f+\delta ef)} \mid \mathrm{ef}\right)$, with $\nu = (1+h\delta)^{-1}$,

and the rest is straight-forward.

summary version on board.

Review of the computational methodology and of lemmas 0-3 above.

Claim under $[e,f]=h$, h central, $\mathbb{O}\left(e^{tef+\alpha f+\beta e} \mid fe\right) = \mathbb{O}\left(\nu e^{\nu(tef-\alpha\beta h+\alpha f+\beta e)} \mid ef\right)$, where $\nu = (1+ht)^{-1}$.

Recurring trick: $\phi(x)e^{\alpha x} = \phi\left(\frac{\partial}{\partial\alpha}\right)e^{\alpha x}$

$\phi(x) = \phi\left(\frac{\partial}{\partial\alpha}\right)e^{\alpha x}\Big|_{\alpha=0}$

especially useful if you want to evaluate a lin. functional.

Proofs 4 & 5:

Think: Enough to reorder poly's; hence enough to re-order exps, and this we already know

$$\mathbb{O}\left(e^{tef+\alpha f+\beta e}/fe\right) \overset{\downarrow}{=} \mathbb{O}\left(e^{t\partial_\alpha\partial_\beta}e^{\alpha f+\beta e}/fe\right) = e^{t\partial_\alpha\partial_\beta}\,\mathbb{O}\left(e^{\alpha f+\beta e}/fe\right)$$

$$= e^{t\partial_\alpha\partial_\beta}\,\mathbb{O}\left(e^{-\alpha\beta h+\alpha f+\beta e}/ef\right) = \mathbb{O}\left(e^{t\partial_\alpha\partial_\beta}e^{-\alpha\beta h+\alpha f+\beta e}/ef\right) = \mathbb{O}\left(\Psi_t/ef\right)$$

So we are left with a first-year calculus question — Compute Ψ_t.

Sol'n 1 Ψ_t satisfies a. $\partial_t\Psi_t = \partial_\alpha\partial_\beta\Psi_t$ b. $\Psi_0 = e^{-\alpha\beta h+\alpha f+\beta e}$

and is determined by these conditions. So it is enough to guess & verify.

Verification in 170303-g0LemmaAndInvariant.nb.

Sol'n 2 use (170211b) Gaussian pairing: $\left\langle \exp\left(\frac{x\subset}{2}\right) \mid \exp\left(\frac{\supset y}{2} + \sum_i \multimap i\right)\right\rangle =$

$$\exp\left(\log\left(\frac{1}{1-xy}\right)\bigcirc + \sum_{i,j}\frac{x\,i\!\multimap\!\!\multimap j}{1-xy}\right).$$

Continue as in 170303-g0LemmaAndInvariant.nb.

Figs: