```
(*S=e^(2A-B),T=e^(2B-A)*)
```

```
Ast = 1/3 Log[S^2 T]; Bst = 1/3 Log[S T^2];
R⁺i_,j_ := 𝔼[X_i x_j + Y_i y_j + Z_i z_j + Ast a_j + Bst b_j]
```

```
CF[ω_. 𝔼[Q_]] :=
  Simplify[ω, Assumptions → {S > 0, T > 0}] 𝔼[Simplify[Q, Assumptions → {T > 0, S > 0}]];
𝔼 /: 𝔼[Q1_] 𝔼[Q2_] := CF@𝔼[Q1 + Q2];
ω1_. 𝔼[Q1_] ≡ ω2_. 𝔼[Q2_] :=
  Simplify[ω1 == ω2 ∧ Q1 == Q2, TimeConstraint → Infinity, Assumptions → {S > 0, T > 0}];
```

```
bazyx[i_, e_, d_, c_, k_][ω_. 𝔼[Q_]] :=
  CF[ω 𝔼[Q /. {x_c -> ⅇ^(γ̄_1) x_k, y_d -> ⅇ^(γ̄_2) y_k, z_e -> ⅇ^(γ̄_3) z_k, b_i → b_k, a_i → a_k}] /.
    {γ̄_1 -> (-2 ∂_(a_i) Q + ∂_(b_i) Q), γ̄_2 -> (∂_(a_i) Q - 2 ∂_(b_i) Q), γ̄_3 -> (-∂_(a_i) Q - ∂_(b_i) Q)}];
baXYZ[i_, j_, c_, d_, e_, k_][ω_. 𝔼[Q_]] :=
  CF[ω 𝔼[Q /. {X_c -> ⅇ^(-γ̄_1) X_k, Y_d -> ⅇ^(-γ̄_2) Y_k, Z_e -> ⅇ^(-γ̄_3) Z_k, b_i → b_k, a_i → a_k, a_j → a_k, b_j → b_k}] /.
    {γ̄_1 -> (-2 ∂_(a_i) Q + ∂_(b_i) Q), γ̄_2 -> (∂_(a_i) Q - 2 ∂_(b_i) Q), γ̄_3 -> (-∂_(a_i) Q - ∂_(b_i) Q)}];
zyxba[e_, d_, c_, j_, k_][ω_. 𝔼[Q_]] :=
  CF[ω 𝔼[Q /. {x_c -> ⅇ^(-γ̄_1) x_k, y_d -> ⅇ^(-γ̄_2) y_k, z_e -> ⅇ^(-γ̄_3) z_k, b_j → b_k, a_j → a_k}] /.
    {γ̄_1 -> (-2 ∂_(a_j) Q + ∂_(b_j) Q), γ̄_2 -> (∂_(a_j) Q - 2 ∂_(b_j) Q), γ̄_3 -> (-∂_(a_j) Q - ∂_(b_j) Q)}];
xX[i_, j_, k_][ω_. 𝔼[Q_]] :=
  CF[ω ν 𝔼[α β ν (1 - S) + α ν x_k + β ν X_k + δ ν X_k x_k + Q /. {X_j | x_i -> 0}]] /.
   {α -> ∂_(x_i) Q /. X_j -> 0, β -> ∂_(X_j) Q /. x_i -> 0, δ -> ∂_(x_i,X_j) Q, ν -> (1 - (1 - S) ∂_(x_i,X_j) Q)^-1};
yY[i_, j_, k_][ω_. 𝔼[Q_]] :=
  CF[ω ν 𝔼[α β ν (1 - T) + α ν y_k + β ν Y_k + δ ν Y_k y_k + Q /. {Y_j | y_i -> 0}]] /.
   {α -> ∂_(y_i) Q /. Y_j -> 0, β -> ∂_(Y_j) Q /. y_i -> 0, δ -> ∂_(y_i,Y_j) Q, ν -> (1 - (1 - T) ∂_(y_i,Y_j) Q)^-1};
xZ[i_, j_, k_][ω_. 𝔼[Q_]] :=
  CF[ω ν 𝔼[-α β ν S Y_k + α ν x_k + β ν Z_k + δ ν Z_k x_k + Q /. {Z_j | x_i -> 0}]] /.
   {α -> ∂_(x_i) Q /. Z_j -> 0, β -> ∂_(Z_j) Q /. x_i -> 0, δ -> ∂_(x_i,Z_j) Q, ν -> (1 + S Y_k ∂_(x_i,Z_j) Q)^-1};
yZ[i_, j_, k_][ω_. 𝔼[Q_]] := CF[ω ν 𝔼[α β ν X_k + α ν y_k + β ν Z_k + δ ν Z_k y_k + Q /. {Z_j | y_i -> 0}]] /.
   {α -> ∂_(y_i) Q /. Z_j -> 0, β -> ∂_(Z_j) Q /. y_i -> 0, δ -> ∂_(y_i,Z_j) Q, ν -> (1 - X_k ∂_(y_i,Z_j) Q)^-1};
zZ[i_, j_, k_][ω_. 𝔼[Q_]] :=
  CF[ω ν 𝔼[α β ν (1 - S T) + α ν z_k + β ν Z_k + δ ν Z_k z_k + Q /. {Z_j | z_i -> 0}]] /.
   {α -> ∂_(z_i) Q /. Z_j -> 0, β -> ∂_(Z_j) Q /. z_i -> 0, δ -> ∂_(z_i,Z_j) Q, ν -> (1 - (1 - S T) ∂_(z_i,Z_j) Q)^-1};
xy[i_, j_, k_][ω_. 𝔼[Q_]] := CF[ω ν 𝔼[α β ν z_k + α ν x_k + β ν y_k + δ ν y_k x_k + Q /. {y_j | x_i -> 0}]] /.
   {α -> ∂_(x_i) Q /. y_j -> 0, β -> ∂_(y_j) Q /. x_i -> 0, δ -> ∂_(x_i,y_j) Q, ν -> (1 - z_k ∂_(x_i,y_j) Q)^-1};
m_(i_,j_->k_)[ω_. 𝔼[S_]] := Module[{h, l, m, n, o, p, q, r, f}, CF @@
    {((ω 𝔼[S] // bazyx[i, i, i, i, l] // baXYZ[l, j, j, j, j, m] // xX[l, m, n] // yY[l, m, n] //
          xZ[n, m, o] // yY[n, o, h] // yZ[h, o, p] // zZ[l, p, q] // zyxba[q, p, o, m, r] //
       xy[r, j, f]) /. {x_(h|i|j|l|m|n|o|p|q|r|f) → x_k, y_(h|i|j|l|m|n|o|p|q|r|f) → y_k,
      z_(h|i|j|l|m|n|o|p|q|r|f) → z_k, a_(h|i|j|l|m|n|o|p|q|r|f) → a_k, b_(h|i|j|l|m|n|o|p|q|r|f) → b_k,
      X_(h|i|j|l|m|n|o|p|q|r|f) → X_k, Y_(h|i|j|l|m|n|o|p|q|r|f) → Y_k, Z_(h|i|j|l|m|n|o|p|q|r|f) → Z_k}}]
```

```
Q0 = 𝔼[Sum[η_(i,j) X_i x_j + ζ_(i,j) Y_i y_j + θ_(i,j) Z_i z_j, {i, 3}, {j, 3}] + Sum[σ_i a_i + ξ_i b_i, {i, 3}]]
```

$$\mathbb{E}[b_1\,\xi_1 + b_2\,\xi_2 + b_3\,\xi_3 + a_1\,\sigma_1 + a_2\,\sigma_2 + a_3\,\sigma_3 + y_1\,Y_1\,\zeta_{1,1} + y_2\,Y_1\,\zeta_{1,2} + y_3\,Y_1\,\zeta_{1,3} + y_1\,Y_2\,\zeta_{2,1} + y_2\,Y_2\,\zeta_{2,2} + y_3\,Y_2\,\zeta_{2,3} + y_1\,Y_3\,\zeta_{3,1} + y_2\,Y_3\,\zeta_{3,2} + y_3\,Y_3\,\zeta_{3,3} + x_1\,X_1\,\eta_{1,1} + x_2\,X_1\,\eta_{1,2} + x_3\,X_1\,\eta_{1,3} + x_1\,X_2\,\eta_{2,1} + x_2\,X_2\,\eta_{2,2} + x_3\,X_2\,\eta_{2,3} + x_1\,X_3\,\eta_{3,1} + x_2\,X_3\,\eta_{3,2} + x_3\,X_3\,\eta_{3,3} + z_1\,Z_1\,\theta_{1,1} + z_2\,Z_1\,\theta_{1,2} + z_3\,Z_1\,\theta_{1,3} + z_1\,Z_2\,\theta_{2,1} + z_2\,Z_2\,\theta_{2,2} + z_3\,Z_2\,\theta_{2,3} + z_1\,Z_3\,\theta_{3,1} + z_2\,Z_3\,\theta_{3,2} + z_3\,Z_3\,\theta_{3,3}]$$

Test Polynomial

```
Q1 = 𝔼[Sum[RandomInteger[{-4, 4}] X_i x_j +
      RandomInteger[{-4, 4}] Y_i y_j + RandomInteger[{-4, 4}] Z_i z_j, {i, 3}, {j, 3}] +
    Sum[RandomInteger[{-4, 4}] a_i + RandomInteger[{-4, 4}] b_i, {i, 3}]]
```

$$\mathbb{E}[3\,a_1 + 3\,a_3 - 4\,b_1 - 2\,b_2 + 4\,x_1\,X_1 + x_2\,X_1 + 3\,x_3\,X_1 - x_1\,X_2 + x_2\,X_2 + x_3\,X_2 + 3\,x_1\,X_3 - 3\,x_2\,X_3 - x_3\,X_3 + 2\,y_1\,Y_1 - 2\,y_2\,Y_1 + 4\,y_3\,Y_1 - 2\,y_1\,Y_2 + 4\,y_2\,Y_2 - y_3\,Y_2 + 3\,y_1\,Y_3 + y_2\,Y_3 + 3\,y_3\,Y_3 - 3\,z_2\,Z_1 - 2\,z_3\,Z_1 + 3\,z_1\,Z_2 - z_2\,Z_2 - z_3\,Z_2 + 2\,z_1\,Z_3 - 2\,z_3\,Z_3]$$

Associativity

```
Qtest = 𝔼[3 a₁ + 3 a₃ - 4 b₁ - 2 b₂ + 4 x₁ X₁ + x₂ X₁ + 3 x₃ X₁ - x₁ X₂ + x₂ X₂ +
    x₃ X₂ + 3 x₁ X₃ - 3 x₂ X₃ - x₃ X₃ + 2 y₁ Y₁ - 2 y₂ Y₁ + 4 y₃ Y₁ - 2 y₁ Y₂ + 4 y₂ Y₂ - y₃ Y₂ +
    3 y₁ Y₃ + y₂ Y₃ + 3 y₃ Y₃ - 3 z₂ Z₁ - 2 z₃ Z₁ + 3 z₁ Z₂ - z₂ Z₂ - z₃ Z₂ + 2 z₁ Z₃ - 2 z₃ Z₃];
Timing[(Qtest // m₁,₂→₁ // m₁,₃→₁) ≡ (Qtest // m₂,₃→₂ // m₁,₂→₁)]
```

{116.281250, True}

QYBE

```
Timing[
 (R⁺₁,₂ R⁺₃,₄ R⁺₅,₆ // m₃,₅→x // m₁,₆→y // m₂,₄→z) ≡ (R⁺₁,₂ R⁺₃,₄ R⁺₅,₆ // m₁,₃→x // m₂,₅→y // m₄,₆→z)]
```

{4.515625, True}

Trefoil

```
R⁺₁,₂ R⁺₃,₄ R⁺₅,₆ // m₁,₄→₁ // m₁,₅→₁ // m₁,₂→₁ // m₁,₃→₁ // m₁,₆→₁
```

$$\mathbb{E}\left[\mathrm{Log}\left[S^2\,T\right]\,a_1+\mathrm{Log}\left[S\,T^2\right]\,b_1+x_1\,X_1+S\,(1+S)\,x_1\,X_1+y_1\,Y_1+T\,y_1\,Y_1+T^2\,y_1\,Y_1+S\,(1+S)\,X_1\,Y_1\,z_1+S^2\,T\,X_1\,Y_1\,z_1+z_1\,Z_1+S\,T\,z_1\,Z_1+S^2\,T^2\,z_1\,Z_1\right]\Big/\left(\left(1-S+S^2\right)\left(1-T+T^2\right)\left(1-S\,T+S^2\,T^2\right)\right)$$