The $\mathfrak{g}_1$ Invariant

The main $\mathfrak{g}_k$ lemma

In $\mathfrak{g}^\epsilon = \langle h, e, l, f\rangle / ([e, l] = -e,\ [f, l] = f,\ [e, f] = h - 2\epsilon l,\ [h, *] = 0)$ and at $\epsilon^{k+1} = 0$, we have

1. $\mathbb{O}\left(e^{\gamma l + \beta e} \mid le\right) = \mathbb{O}\left(e^{\gamma l + e^\gamma \beta e} \mid e\, l\right)$,
2. $\mathbb{O}\left(e^{\gamma l + \beta f} \mid f\, l\right) = \mathbb{O}\left(e^{\gamma l + e^\gamma \beta f} \mid l\, f\right)$,
3. $\mathbb{O}\left(e^{\beta e + \alpha f + \delta e f} \mid f\, e\right) = \mathbb{O}\left(\nu e^{\nu(-\alpha\beta h + \beta e + \alpha f + \delta e f)} \Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta) \mid e\, l\, f\right)$, with $\nu = (1 + h\,\delta)^{-1}$ and where for any fixed k, $\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)$ is a fixed polynomial of degree at most $4k$ in e, $\sqrt{l}$, f, α, β, with scalar coefficients.

Comment. Even better, $\log(\Lambda_k)$ is of degree at most $2k + 2$ in said variables.

The Main $\mathfrak{g}_k$ Theorem

Raw Version. The $\mathfrak{g}_k$ invariant of any S-component tangle T can be written in the form $Z(T) = \mathbb{O}\left(\omega e^{L+Q+P} \mid \prod_{i\in S} e_i l_i f_i\right)$, where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i = e^{h_i}$), where $L = \sum a_{ij} h_i l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij}, where $Q = \sum b_{ij} e_i f_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij}, and where P is a polynomial in $\{\epsilon,\ e_i,\ l_i,\ f_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2d + 2$ in $\left\{e_i,\ \sqrt{l_i},\ f_i\right\}$. Furthermore, after setting $h_i = h$ and $t_i = t$ for all i, the invariant $Z(T)$ is poly-time computable.

Partial Proof. Indeed,

0. $R^\pm = ?$, $n^\pm = ?$.
1. $\mathbb{O}\left(\mathcal{P}(l, e)\, e^{\gamma l + \beta e} \mid l\, e\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^\gamma \beta e} \mid e\, l\right)$,
2. $\mathbb{O}\left(\mathcal{P}(l, f)\, e^{\gamma l + \beta f} \mid f\, l\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^\gamma \beta f} \mid l\, f\right)$,
3. $\mathbb{O}\left(\mathcal{P}(e, f)\, e^{\beta e + \alpha f + \delta e f} \mid f\, e\right) = \mathbb{O}\left(\nu\, \mathcal{P}(\partial_\beta, \partial_\alpha)\, e^{\nu(-\alpha\beta h + \beta e + \alpha f + \delta e f)} \Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta) \mid e\, f\right)$, with $\nu = (1 + \mathrm{h}\delta)^{-1}$, and $\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)$ as above.

Implementation at $k = 1$

$\mathbb{E}1\mathrm{n}[\omega, L, Q, P]$ stands for $\omega e^{L+Q}(1+\epsilon\,\mathrm{P})$.

```
DP[x_ → Dα_, y_ → Dβ_][P_][f_] := Total[CoefficientRules[P, {x, y}] /. ({m_, n_} → c_) ⧴ c D[f, {α, m}, {β, n}]]
```

```
Λk_[h_, e_, l_, f_, α_, β_, δ_] := Λk[h, e, l, f, α, β, δ] = Module[{λ},
   λ = Normal@Series[e^((f α + e β)/(1 - α β ϵ)) (1 - α β ϵ)^(-2 l + h/ϵ), {ϵ, 0, k}] /. e- → 1;
   Collect[DP[α → Df, β → De][λ][e^((f α + e β + e f δ)/(1 + h δ))] /. e- → 1, ϵ, Simplify]];
```

```
ϵ /: ϵ^p_ /; p > 1 := 0;
```

```
Table[ϵ^j, {j, 0, 7}]
```

```
{1, ϵ, 0, 0, 0, 0, 0, 0}
```

```
? Together
```

Together[*expr*] puts terms in a sum over a common denominator, and cancels factors in the result. >>

```
CF[𝔼1n[ω_, L_, Q_, P_]] := 𝔼1n[Together[ω], Together[L], Together[Q], Together[P]];
𝔼1n /: 𝔼1n[ω1_, L1_, Q1_, P1_] 𝔼1n[ω2_, L2_, Q2_, P2_] := CF@𝔼1n[ω1 ω2, L1 + L2, Q1 + Q2, P1 + P2];
𝔼1n[ω1_, L1_, Q1_, P1_] ≡ 𝔼1n[ω2_, L2_, Q2_, P2_] := Simplify[ω1 == ω2 ∧ L1 == L2 ∧ Q1 == Q2 ∧ P1 == P2];
```

0. $R = \mathbb{O}\left(\exp\left(\text{hl} + \frac{e^h - 1}{h}\,\text{ef} + P\right) \,\middle|\, e\otimes\text{lf}\right)$:

```
𝔼1n[X⁺[i_, j_]] := 𝔼1n[1, h_i l_j, h_i^-1 (e^(h_i) - 1) e_i f_j, P⁺[i, j]];
𝔼1n[X⁻[i_, j_]] := 𝔼1n[1, -h_i l_j, h_i^-1 (e^(-h_i) - 1) e_i f_j, P⁻[i, j]];
𝔼1n[p_Times] := 𝔼1n /@ p;
```

```
𝔼1n[X⁺[4,1] X⁺[2,5] X⁺[6,3]]
```

$$\mathbb{E}1\text{n}\Big[1,\ h_4 l_1 + h_6 l_3 + h_2 l_5,\ \frac{1}{h_2 h_4 h_6}\left(-e_6 f_3 h_2 h_4 + e^{h_6} e_6 f_3 h_2 h_4 - e_4 f_1 h_2 h_6 + e^{h_4} e_4 f_1 h_2 h_6 - e_2 f_5 h_4 h_6 + e^{h_2} e_2 f_5 h_4 h_6\right),\ (P_{2,5})^+ + (P_{4,1})^+ + (P_{6,3})^+\Big]$$

1. $\mathbb{O}\left(\mathcal{P}(l, e)\, e^{\gamma l + \beta e} \,\middle|\, l e\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^\gamma \beta e} \,\middle|\, e\, l\right)$,
2. $\mathbb{O}\left(\mathcal{P}(l, f)\, e^{\gamma l + \beta f} \,\middle|\, f\, l\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^\gamma \beta f} \,\middle|\, l f\right)$.

```
NO_(x:f|e)i_ l j_[𝔼1n[ω_, L_, Q_, P_]] := With[{q = e^γ β x_i + γ l_j},
  CF[𝔼1n[ω, L,
     e^γ β x_i + (Q /. x_i → 0),
     e^-q DP_{l_j→D_γ, x_i→D_β}[P][e^q]
    ] /. {γ → ∂_{l_j} L, β → ∂_{x_i} Q}]];
```

3. $\mathbb{O}\left(e^{\beta\text{e} + \alpha\text{f} + \delta\text{ef}} \,\middle|\, \text{fe}\right) = \mathbb{O}\left(\nu e^{\nu(-\alpha\beta\text{h} + \beta\text{e} + \alpha\text{f} + \delta\text{ef})} \,\middle|\, \text{ef}\right)$, with $\nu = (1 + \text{h}\delta)^{-1}$:

```
NO_{f_i_ e_j_ → k_}[𝔼1n[ω_, L_, Q_, P_]] := With[{q = ν (-α β h_k + β e_k + α f_k + δ e_k f_k)},
  CF[𝔼1n[ν ω, L,
      q + (Q /. f_i | e_j → 0),
      e^-q DP_{f_i→D_α, e_j→D_β}[P][e^q] + (Λ1[h_k, e_k, l_k, f_k, α, β, δ] - 1 /. ϵ → 1)
    ] /. ν → (1 + h_k δ)^-1 /. {α → ∂_{f_i} Q /. e_j → 0, β → ∂_{e_j} Q /. f_i → 0, δ → ∂_{f_i, e_j} Q}]];
```

```
m_{i_, j_ → k_}[Z_] := Module[{x, z}, CF[(Z // NO_{f_i e_j → x} // NO_{l_i e_x} // NO_{f_x l_j}) /. z_{i|j|x} → z_k]]
```

Meta-associativity

```
Λ1[h, e, l, f, α, β, δ]
```

$$1 + \frac{1}{2\,(1 + \text{h}\,\delta)^4}\Big(4\,\text{l}\,(1 + \text{h}\,\delta)^2\left((\alpha + \text{e}\,\delta)(\beta + \text{f}\,\delta) + \delta\,(1 + \text{h}\,\delta)\right) + 2\,\text{f}\,(\alpha + \text{e}\,\delta)(1 + \text{h}\,\delta)\left((\alpha + \text{e}\,\delta)(\beta + \text{f}\,\delta) + 2\,\delta\,(1 + \text{h}\,\delta)\right) + 2\,\text{e}\,(\beta + \text{f}\,\delta)(1 + \text{h}\,\delta)\left((\alpha + \text{e}\,\delta)(\beta + \text{f}\,\delta) + 2\,\delta\,(1 + \text{h}\,\delta)\right) - \text{h}\left((\alpha + \text{e}\,\delta)^2(\beta + \text{f}\,\delta)^2 + 4\,\delta\,(\alpha + \text{e}\,\delta)(\beta + \text{f}\,\delta)(1 + \text{h}\,\delta) + 2\,\delta^2\,(1 + \text{h}\,\delta)^2\right)\Big)\,\epsilon$$

```
CoefficientRules[Λ1[h, e, l, f, α, β, δ], {ϵ, e, l, f}]
```

$$\left\{\{1,2,0,2\}\to\frac{2\delta^3}{(1+h\delta)^4}+\frac{3h\delta^4}{2(1+h\delta)^4},\ \{1,2,0,1\}\to\frac{3\beta\delta^2}{(1+h\delta)^4}+\frac{2h\beta\delta^3}{(1+h\delta)^4},\right.$$
$$\{1,2,0,0\}\to\frac{\beta^2\delta}{(1+h\delta)^4}+\frac{h\beta^2\delta^2}{2(1+h\delta)^4},\ \{1,1,1,1\}\to\frac{2\delta^2}{(1+h\delta)^4}+\frac{4h\delta^3}{(1+h\delta)^4}+\frac{2h^2\delta^4}{(1+h\delta)^4},$$
$$\{1,1,1,0\}\to\frac{2\beta\delta}{(1+h\delta)^4}+\frac{4h\beta\delta^2}{(1+h\delta)^4}+\frac{2h^2\beta\delta^3}{(1+h\delta)^4},\ \{1,1,0,2\}\to\frac{3\alpha\delta^2}{(1+h\delta)^4}+\frac{2h\alpha\delta^3}{(1+h\delta)^4},$$
$$\{1,1,0,1\}\to\frac{4\alpha\beta\delta}{(1+h\delta)^4}+\frac{4\delta^2}{(1+h\delta)^4}+\frac{2h\alpha\beta\delta^2}{(1+h\delta)^4}+\frac{6h\delta^3}{(1+h\delta)^4}+\frac{2h^2\delta^4}{(1+h\delta)^4},$$
$$\{1,1,0,0\}\to\frac{\alpha\beta^2}{(1+h\delta)^4}+\frac{2\beta\delta}{(1+h\delta)^4}+\frac{2h\beta\delta^2}{(1+h\delta)^4},\ \{1,0,1,1\}\to\frac{2\alpha\delta}{(1+h\delta)^4}+\frac{4h\alpha\delta^2}{(1+h\delta)^4}+\frac{2h^2\alpha\delta^3}{(1+h\delta)^4},$$
$$\{1,0,1,0\}\to\frac{2\alpha\beta}{(1+h\delta)^4}+\frac{2\delta}{(1+h\delta)^4}+\frac{4h\alpha\beta\delta}{(1+h\delta)^4}+\frac{6h\delta^2}{(1+h\delta)^4}+\frac{2h^2\alpha\beta\delta^2}{(1+h\delta)^4}+\frac{6h^2\delta^3}{(1+h\delta)^4}+\frac{2h^3\delta^4}{(1+h\delta)^4},$$
$$\{1,0,0,2\}\to\frac{\alpha^2\delta}{(1+h\delta)^4}+\frac{h\alpha^2\delta^2}{2(1+h\delta)^4},\ \{1,0,0,1\}\to\frac{\alpha^2\beta}{(1+h\delta)^4}+\frac{2\alpha\delta}{(1+h\delta)^4}+\frac{2h\alpha\delta^2}{(1+h\delta)^4},$$
$$\left.\{1,0,0,0\}\to-\frac{h\alpha^2\beta^2}{2(1+h\delta)^4}-\frac{2h\alpha\beta\delta}{(1+h\delta)^4}-\frac{h\delta^2}{(1+h\delta)^4}-\frac{2h^2\alpha\beta\delta^2}{(1+h\delta)^4}-\frac{2h^2\delta^3}{(1+h\delta)^4}-\frac{h^3\delta^4}{(1+h\delta)^4},\ \{0,0,0,0\}\to 1\right\}$$

```
Total[CoefficientRules[Λ1[h, e, l, f, α, β, δ], {ϵ, e, l, f}] /. (p_ → c_) :→ Times @@ {ϵ, e, l, f}^p]
```

$$1+\epsilon+e\,\epsilon+e^2\,\epsilon+f\,\epsilon+e\,f\,\epsilon+e^2\,f\,\epsilon+f^2\,\epsilon+e\,f^2\,\epsilon+e^2\,f^2\,\epsilon+l\,\epsilon+e\,l\,\epsilon+f\,l\,\epsilon+e\,f\,l\,\epsilon$$

$$\mathcal{E}=\mathbb{E}1n\left[\omega,\ \sum_{i=1}^{4}\sum_{j=1}^{4}a_{i,j}\,h_i\,l_j,\ \sum_{i=1}^{4}\sum_{j=1}^{4}b_{i,j}\,e_i\,f_j,\ a+\sum_{i=1}^{4}\left(a_i\,l_i+b_i\,e_i+c_i\,f_i\right)\right]$$

$$\mathbb{E}1n[\omega,\ h_1 l_1 a_{1,1}+h_1 l_2 a_{1,2}+h_1 l_3 a_{1,3}+h_1 l_4 a_{1,4}+h_2 l_1 a_{2,1}+h_2 l_2 a_{2,2}+h_2 l_3 a_{2,3}+h_2 l_4 a_{2,4}+h_3 l_1 a_{3,1}+h_3 l_2 a_{3,2}+h_3 l_3 a_{3,3}+h_3 l_4 a_{3,4}+h_4 l_1 a_{4,1}+h_4 l_2 a_{4,2}+h_4 l_3 a_{4,3}+h_4 l_4 a_{4,4},$$
$$e_1 f_1 b_{1,1}+e_1 f_2 b_{1,2}+e_1 f_3 b_{1,3}+e_1 f_4 b_{1,4}+e_2 f_1 b_{2,1}+e_2 f_2 b_{2,2}+e_2 f_3 b_{2,3}+e_2 f_4 b_{2,4}+e_3 f_1 b_{3,1}+e_3 f_2 b_{3,2}+e_3 f_3 b_{3,3}+e_3 f_4 b_{3,4}+e_4 f_1 b_{4,1}+e_4 f_2 b_{4,2}+e_4 f_3 b_{4,3}+e_4 f_4 b_{4,4},$$
$$a+b_1 e_1+b_2 e_2+b_3 e_3+b_4 e_4+c_1 f_1+c_2 f_2+c_3 f_3+c_4 f_4+a_1 l_1+a_2 l_2+a_3 l_3+a_4 l_4]$$

```
Short[ℰ // m1,2→1, 5] // Timing
```

$$\left\{3.04688,\ \mathbb{E}1n\left[\frac{\omega}{1+h_1 b_{2,1}},\ h_1 l_1 a_{1,1}+h_1 l_1 a_{1,2}+h_1 l_3 a_{1,3}+h_1 l_4 a_{1,4}+h_1 l_1 a_{2,1}+h_1 l_1 a_{2,2}+h_1 l_3 a_{2,3}+\right.\right.$$
$$h_1 \ll1\gg a_{2\ll1\gg\ll1\gg}+\ll1\gg+h_3 l_1 a_{3,2}+h_3 l_3 a_{3,3}+h_3 l_4 a_{3,4}+h_4 l_1 a_{4,1}+h_4 l_1 a_{4,2}+h_4 l_3 a_{4,3}+h_4 l_4 a_{4,4},$$
$$\left.\left.\ll1\gg,\ \frac{2a+2b_1 e_1+2e^{h_1 a_{\ll1\gg}+\ll3\gg}b_1 e_1+\ll404\gg}{2\left(1+h_1 b_{2,1}\right)^4}\right]\right\}$$

```
Short[lhs = ℰ // m1,2→1 // m1,3→1, 5] // Timing
```

$$\left\{275.031,\ \mathbb{E}1n\left[\omega/\left(1+h_1 b_{2,1}+e^{h_1 a_{1,2}+h_1 a_{2,2}+h_1 a_{3,2}+h_4 a_{4,2}}h_1 b_{3,1}-h_1^2 b_{2,2} b_{3,1}+h_1 b_{3,2}+h_1^2 b_{2,1} b_{3,2}\right),\right.\right.$$
$$h_1 l_1 a_{1,1}+h_1 l_1 a_{1,2}+h_1 l_1 a_{1,3}+h_1 l_4 a_{1,4}+h_1 l_1 a_{2,1}+h_1 l_1 a_{2,2}+h_1 l_1 a_{2,3}+\ll3\gg+h_1 l_1 a_{3,3}+h_1 l_4 a_{3,4}+$$
$$\left.\left.h_4 l_1 a_{4,1}+h_4 l_1 a_{4,2}+h_4 l_1 a_{4,3}+h_4 l_4 a_{4,4},\ \frac{\ll1\gg}{\ll1\gg},\ \frac{2a+\ll16247\gg}{2\left(1+h_1 b_{2,1}+\ll3\gg+h_1 b_{3,2}+h_1^2 b_{2,1} b_{3,2}\right)^4}\right]\right\}$$

```
Short[rhs = ℰ // m2,3→2 // m1,2→1, 5] // Timing
```

$Aborted

```
lhs ≡ rhs
```

True

Note. Meta-associativity does not work for arbitrary Λ, yet is does work for any scalar multiple of “our” Λ, including for

$\Lambda = 0$. I don't know what other constraints one must impose on a meta-monoid for there to be “interesting” solutions for R within it. Even worse, I'm not sure what's “interesting”.

Profiling

```
<< "C:\\drorbn\\AcademicPensieve\\Projects\\Profile\\Profile.m"
```

```
This is Profile.m of http://drorbn.net/AcademicPensieve/Projects/Profile/.
  This version: March 2017. Original version: July 1994.
```

```
CF[𝔼1n[ω_, L_, Q_, P_]] := PP_CF@𝔼1n[Together[ω], Together[L], Together[Q], Together[P]];
NO_(x:f|e)_i_ 1_j_[𝔼1n[ω_, L_, Q_, P_]] := PP_NOxl@With[{q = e^γ β x_i + γ 1_j},
    CF[𝔼1n[ω, L,
       e^γ β x_i + (Q /. x_i → 0),
       e^-q DP_{1_j→D_γ, x_i→D_β}[P][e^q]
      ] /. {γ → ∂_{1_j} L, β → ∂_{x_i} Q}]];
NO_{f_i_ e_j_ → k_}[𝔼1n[ω_, L_, Q_, P_]] := PP_NOfe@With[{q = ν (-α β h_k + β e_k + α f_k + δ e_k f_k)},
    CF[𝔼1n[ν ω, L,
        q + (Q /. f_i | e_j → 0),
        e^-q DP_{f_i→D_α, e_j→D_β}[P][e^q] + (Λ_1[h_k, e_k, 1_k, f_k, α, β, δ] - 1 /. ϵ → 1)
       ] /. ν → (1 + h_k δ)^-1 /. {α → ∂_{f_i} Q /. e_j → 0, β → ∂_{e_j} Q /. f_i → 0, δ → ∂_{f_i,e_j} Q}]];
```

```
BeginProfile[];
Short[ℰ // m_{1,2→1} // m_{1,3→1}]
EndProfile[];
PrintProfile[]
```

$$\mathbb{E}1\mathrm{n}\left[\frac{\omega}{1 + \ll 5 \gg + h_1^2\, b_2 \ll 1 \gg \ll 1 \gg b_{3,2}}, \ll 1 \gg, \frac{\ll 1 \gg}{\ll 1 \gg}, \frac{2\,a + \ll 16247 \gg}{2 \ll 1 \gg^4}\right]$$

```
CF: called 8 times, time in 405.999/405.999
  Parents:
    ( 2)  155.920/ 155.920 under NOfe
    ( 4)  233.660/ 233.660 under NOxl
    ( 2)   16.422/  16.422 under ProfileRoot
NOxl: called 4 times, time in 18.22/251.876
  Parents:
    ( 4)   18.220/ 251.880 under ProfileRoot
  Children:
    ( 4)  233.660/ 233.660 above CF
NOfe: called 2 times, time in 0.22/156.141
  Parents:
    ( 2)    0.220/ 156.140 under ProfileRoot
  Children:
    ( 2)  155.920/ 155.920 above CF
ProfileRoot: called 0 times, time in 0./0.
  Children:
    ( 2)   16.422/  16.422 above CF
    ( 2)    0.220/ 156.140 above NOfe
    ( 4)   18.220/ 251.880 above NOxl
```

Pragmatic Simplifications

$\mathbb{E}1\mathrm{n}[\omega, L, Q, P]$ stands for $\omega\, e^{L+Q}(1+\epsilon P)$; $\mathbb{E}1\mathrm{p}[\omega, L, Q, P]$ stands for $\omega^{-1}\, e^{L+\omega^{-1} Q}\left(1+\epsilon\,\omega^{-4} P\right)$ /. $e_i \to \frac{t_i-1}{h_i} e_i$, all written in $t_i = e^{h_i}$.

```
𝔼1n[𝔼1p[ω_, L_, Q_, P_]] := CF[PowerExpand /@ CF[𝔼1n[ω^-1, L, ω^-1 Q, ω^-4 P] /. e_i_ :> (t_i - 1)/h_i e_i /. t_i_ :> ⅇ^(h_i)]];
CF[𝔼1p[ω_, L_, Q_, P_]] := 𝔼1p[Together[ω], Together[L], Together[Q], Together[P]];
𝔼1p[𝔼1n[ω_, L_, Q_, P_]] := CF[𝔼1p[ω^-1, L, ω^-1 Q, ω^-4 P] /. e_i_ :> h_i/(t_i - 1) e_i /. h_i_ :> Log[t_i]];
𝔼1p[ω1_, L1_, Q1_, P1_] ≡ 𝔼1p[ω2_, L2_, Q2_, P2_] := Simplify[ω1 == ω2 ∧ L1 == L2 ∧ Q1 == Q2 ∧ P1 == P2];
```

```
{X⁺_{1,2} // 𝔼1n , X⁺_{1,2} // 𝔼1n // 𝔼1p}

ζ = 𝔼1n[ω, Σ_{i=1}^{4} Σ_{j=1}^{4} a_{i,j} h_i l_j, Σ_{i=1}^{4} Σ_{j=1}^{4} b_{i,j} e_i f_j, a + Σ_{i=1}^{4} (a_i l_i + b_i e_i + c_i f_i)]

Simplify /@ 𝔼1p[ζ]

ζ ≡ (ζ // 𝔼1p // 𝔼1n)

Simplify /@ 𝔼1n[𝔼1p @@ ζ]

(𝔼1p @@ ζ /. h_i_ :> Log[t_i]) ≡ (𝔼1n[𝔼1p @@ ζ] // 𝔼1p)

𝔼1p[ω, L, e_1 f_2 + f_1 e_2 + δ e_1 f_1, 0] // 𝔼1n

Simplify /@ ((𝔼1p[ω, L, ω e_1 β + ω f_1 α + ω δ e_1 f_1, 0] // 𝔼1n // NO_{f_1 e_1→0} // 𝔼1p) /. {t_0 → t_1, h_0 → h_1, l_0 → l_1})

Simplify /@ ((𝔼1p[ω, L, ω β e_1 + ω α f_1 + ω δ e_1 f_1, 0] // 𝔼1n // NO_{f_1 e_1→k} // 𝔼1p) /. t_1 → t_k)
```

```
𝔼1p /: 𝔼1p[ω1_, L1_, Q1_, P1_] 𝔼1p[ω2_, L2_, Q2_, P2_] :=
  CF@𝔼1p[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2, ω2^4 P1 + ω1^4 P2];
```

```
NO_{f_i_ e_j_→k_}[𝔼1p[ω_, L_, Q_, P_]] := Module[{α, β, δ, μ, q, Λ},
   q = ((1 - t_k) α β + β e_k + δ e_k f_k + α f_k) / μ;
   Λ = 1/(2 Log[t_k]) ω^4 (-1 + t_k)
     (α^2 β^2 + 4 α β δ + 2 δ^2 - 4 α β δ^2 - 4 δ^3 + 2 δ^4 + 4 α β l_k + 4 δ l_k - 8 α β δ l_k - 12 δ^2 l_k + 4 α β δ^2 l_k + 12 δ^3 l_k -
       4 δ^4 l_k - α^2 β^2 t_k - 4 α β δ t_k - 2 δ^2 t_k + 8 α β δ^2 t_k + 8 δ^3 t_k - 6 δ^4 t_k + 8 α β δ l_k t_k + 12 δ^2 l_k t_k -
       8 α β δ^2 l_k t_k - 24 δ^3 l_k t_k + 12 δ^4 l_k t_k - 4 α β δ^2 t_k^2 - 4 δ^3 t_k^2 + 6 δ^4 t_k^2 + 4 α β δ^2 l_k t_k^2 + 12 δ^3 l_k t_k^2 - 12 δ^4 l_k t_k^2 -
       2 δ^4 t_k^3 + 4 δ^4 l_k t_k^3 + α^2 δ f_k^2 (2 - δ + δ t_k) + 2 α f_k (α β + 2 δ - 2 δ^2 + 2 δ^2 t_k + 2 δ l_k (1 - δ + δ t_k)^2) +
       δ e_k^2 (β + δ f_k) (β (2 - δ + δ t_k) + δ f_k (4 - 3 δ + 3 δ t_k)) +
       2 e_k (α δ^2 f_k^2 (3 - 2 δ + 2 δ t_k) + β (α β + 2 δ - 2 δ^2 + 2 δ^2 t_k + 2 δ l_k (1 - δ + δ t_k)^2) +
          2 δ f_k (δ l_k (1 - δ + δ t_k)^2 + (2 - δ + δ t_k) (α β + δ - δ^2 + δ^2 t_k))));
   CF[
    𝔼1p[μ ω, L, μ ω q + μ (Q /. f_i | e_j → 0), μ^4 (DP_{f_i→D_α, e_j→D_β}[P][ⅇ^q] /. ⅇ- → 1) + ω^4 Λ[k]] /. μ → 1 + (t_k - 1) δ /.
     {α → ω^-1 (∂_{f_i} Q /. e_j → 0), β → ω^-1 (∂_{e_j} Q /. f_i → 0), δ → ω^-1 ∂_{f_i,e_j} Q}
   ]];
```

```
NO_{l_j_ (x:e|f)_i_→k_}[𝔼1p[ω_, L_, Q_, P_]] := With[{q = ⅇ^γ β x_k + γ l_k}, CF[
    𝔼1p[ω, γ l_k + (L /. l_j → 0), ω ⅇ^γ β x_k + (Q /. x_i → 0), ⅇ^-q DP_{l_j→D_γ, x_i→D_β}[P][ⅇ^q]] /. {γ → ∂_{l_j} L, β → ω^-1 ∂_{x_i} Q}]];
```

```
lhs = ζ // NO_{f_1 e_1→5} // 𝔼1p

rhs = (ζ // 𝔼1p // NO_{f_1 e_1→5})

Simplify[lhs[[1]] == rhs[[1]]]
```