The $\mathfrak{g}_1$ Invariant

Reminder

Make sure that you have Mathematica and that you play with these programs!

The main $\mathfrak{g}_k$ lemma

In $\mathfrak{g}^\epsilon = \langle h, e, l, f\rangle / ([e, l] = -e,\ [f, l] = f,\ [e, f] = h - 2\epsilon l,\ [h, *] = 0)$ and at $\epsilon^{k+1} = 0$, we have

1. $\mathbb{O}\left(e^{\gamma l+\beta e} \mid le\right) = \mathbb{O}\left(e^{\gamma l+e^{\gamma}\beta e} \mid e\, l\right)$,
2. $\mathbb{O}\left(e^{\gamma l+\beta f} \mid f\, l\right) = \mathbb{O}\left(e^{\gamma l+e^{\gamma}\beta f} \mid l\, f\right)$,
3. $\mathbb{O}\left(e^{\beta e+\alpha f+\delta e f} \mid f\, e\right) = \mathbb{O}\left(v e^{v(-\alpha\beta h+\beta e+\alpha f+\delta e f)} \Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta) \mid e\, f\right)$, with $v = (1 + h\,\delta)^{-1}$ and where for any fixed k, $\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)$ is a fixed polynomial of degree at most $4\,k$ in $e, \sqrt{l}, f, \alpha, \beta$, with scalar coefficients.

Comment. Even better, $\log(\Lambda_k)$ is of degree at most $2\,k + 2$ in said variables.

The Main $\mathfrak{g}_k$ Theorem

Raw Version. The $\mathfrak{g}_k$ invariant of any S-component tangle T can be written in the form $Z(T) = \mathbb{O}\left(\omega\, e^{L+Q+P} \mid \prod_{i\in S} e_i\, l_i\, f_i\right)$, where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i = e^{h_i}$), where $L = \sum a_{ij}\, h_i\, l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij}, where $Q = \sum b_{ij}\, e_i\, f_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij}, and where P is a polynomial in $\{\epsilon,\ e_i,\ l_i,\ f_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2\,d + 2$ in $\left\{e_i, \sqrt{l_i},\ f_i\right\}$. Furthermore, after setting $h_i = h$ and $t_i = t$ for all i, the invariant $Z(T)$ is poly-time computable.

Partial Proof. Indeed,

0. $R^\pm = ?$, $n^\pm = ?$.
1. $\mathbb{O}\left(\mathcal{P}(l, e)\, e^{\gamma l+\beta e} \mid l\, e\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l+e^{\gamma}\beta e} \mid e\, l\right)$,
2. $\mathbb{O}\left(\mathcal{P}(l, f)\, e^{\gamma l+\beta f} \mid f\, l\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l+e^{\gamma}\beta f} \mid l\, f\right)$,
3. $\mathbb{O}\left(\mathcal{P}(e, f)\, e^{\beta e+\alpha f+\delta e f} \mid f\, e\right) = \mathbb{O}\left(v\, \mathcal{P}(\partial_\beta, \partial_\alpha)\, e^{v(-\alpha\beta h+\beta e+\alpha f+\delta e f)} \Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta) \mid e\, f\right)$, with $v = (1 + \mathrm{h}\delta)^{-1}$, and $\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)$ as above.

Implementation at $k = 1$

$\mathbb{E}$1n[ω, L, Q, P] stands for $\omega\, e^{L+Q}(1+\epsilon\,\mathrm{P})$.

```
DP[x_ -> Dα_, y_ -> Dβ_][P_][f_] := Total[CoefficientRules[P, {x, y}] /. ({m_, n_} → c_) :→ c D[f, {α, m}, {β, n}]]
```

```
Λk_[h_, e_, l_, f_, α_, β_, δ_] := Λk[h, e, l, f, α, β, δ] = Module[{λ},
   λ = Normal@Series[e^((f α + e β)/(1 - α β ϵ)) (1 - α β ϵ)^(-2 l + h/ϵ), {ϵ, 0, k}] /. e- → 1;
   Collect[DP[α → Df, β → De][λ][e^((f α + e β + e f δ)/(1 + h δ))] /. e- → 1, ϵ, Simplify]];
```

```
ϵ /: ϵ^p_ /; p > 1 := 0;
```

```
CF[𝔼1n[ω_, L_, Q_, P_]] := 𝔼1n[Together[ω], Together[L], Together[Q], Together[P]];
𝔼1n /: 𝔼1n[ω1_, L1_, Q1_, P1_] 𝔼1n[ω2_, L2_, Q2_, P2_] := CF@𝔼1n[ω1 ω2, L1 + L2, Q1 + Q2, P1 + P2];
𝔼1n[ω1_, L1_, Q1_, P1_] ≡ 𝔼1n[ω2_, L2_, Q2_, P2_] := Simplify[ω1 == ω2 ∧ L1 == L2 ∧ Q1 == Q2 ∧ P1 == P2];
```

0. $R = \mathbb{O}\left(\exp\left(\mathrm{hl} + \frac{e^{h}-1}{h}\,\mathrm{ef} + P\right) \,\middle|\, e \otimes \mathrm{lf}\right)$:

```
𝔼1n[X⁺_{i_,j_}] := 𝔼1n[1, h_i l_j, h_i^-1 (ℯ^{h_i} - 1) e_i f_j, P⁺];
𝔼1n[X⁻_{i_,j_}] := 𝔼1n[1, -h_i l_j, h_i^-1 (ℯ^{-h_i} - 1) e_i f_j, P⁻];
𝔼1n[p_Times] := 𝔼1n /@ p;
```

```
𝔼1n[X⁺_{4,1} X⁺_{2,5} X⁺_{6,3}]
```

$\mathbb{E}1n\Big[1,\ h_4 l_1 + h_6 l_3 + h_2 l_5,\ \frac{1}{h_2 h_4 h_6}\left(-e_6 f_3 h_2 h_4 + e^{h_6} e_6 f_3 h_2 h_4 - e_4 f_1 h_2 h_6 + e^{h_4} e_4 f_1 h_2 h_6 - e_2 f_5 h_4 h_6 + e^{h_2} e_2 f_5 h_4 h_6\right),\ 3 P^{+}\Big]$

1. $\mathbb{O}\left(\mathcal{P}(l, e)\, e^{\gamma l + \beta e} \,\middle|\, l e\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^{\gamma} \beta e} \,\middle|\, e l\right)$,
2. $\mathbb{O}\left(\mathcal{P}(l, f)\, e^{\gamma l + \beta f} \,\middle|\, f l\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^{\gamma} \beta f} \,\middle|\, l f\right)$.

```
NO_{(x:f|e)_{i_} l_{j_}}[𝔼1n[ω_, L_, Q_, P_]] := With[{q = ℯ^γ β x_i + γ l_j},
  CF[𝔼1n[ω, L,
     ℯ^γ β x_i + (Q /. x_i → 0),
     ℯ^-q DP_{l_j→D_γ, x_i→D_β}[P][ℯ^q]
   ] /. {γ → ∂_{l_j} L, β → ∂_{x_i} Q}]];
```

3. $\mathbb{O}\left(e^{\beta \mathrm{e} + \alpha \mathrm{f} + \delta \mathrm{ef}} \,\middle|\, \mathrm{fe}\right) = \mathbb{O}\left(\nu e^{\nu(-\alpha\beta \mathrm{h} + \beta \mathrm{e} + \alpha \mathrm{f} + \delta \mathrm{ef})} \,\middle|\, \mathrm{ef}\right)$, with $\nu = (1 + \mathrm{h}\delta)^{-1}$:

```
NO_{f_{i_} e_{j_}→k_}[𝔼1n[ω_, L_, Q_, P_]] := With[{q = ν (-α β h_k + β e_k + α f_k + δ e_k f_k)},
  CF[𝔼1n[ν ω, L,
      q + (Q /. f_i | e_j → 0),
      ℯ^-q DP_{f_i→D_α, e_j→D_β}[P][ℯ^q] + (Λ1[h_k, e_k, l_k, f_k, α, β, δ] - 1 /. ϵ → 1)
    ] /. ν → (1 + h_k δ)^-1 /. {α → ∂_{f_i} Q /. e_j → 0, β → ∂_{e_j} Q /. f_i → 0, δ → ∂_{f_i,e_j} Q}]];
```

```
m_{i_,j_→k_}[Z_] := Module[{x, z}, CF[(Z // NO_{f_i e_j→x} // NO_{l_i e_x} // NO_{f_x l_j}) /. z_i|j|x → z_k]]
```

Meta-associativity

```
Total[CoefficientRules[Λ1[h, e, l, f, α, β, δ], {ϵ, e, l, f}] /. (p_ → c_) :→ Times @@ {ϵ, e, l, f}^p]
```

$1 + \epsilon + e\,\epsilon + e^2 \epsilon + f\,\epsilon + e f\,\epsilon + e^2 f\,\epsilon + f^2 \epsilon + e f^2 \epsilon + e^2 f^2 \epsilon + l\,\epsilon + e\, l\,\epsilon + f\, l\,\epsilon + e f l\,\epsilon$

```
ζ = 𝔼1n[ω, Σ_{i=1}^{4} Σ_{j=1}^{4} a_{i,j} h_i l_j, Σ_{i=1}^{4} Σ_{j=1}^{4} b_{i,j} e_i f_j, a + Σ_{i=1}^{4} (a_i l_i + b_i e_i + c_i f_i)]
```

$\mathbb{E}1n\big[\omega,\ h_1 l_1 a_{1,1} + h_1 l_2 a_{1,2} + h_1 l_3 a_{1,3} + h_1 l_4 a_{1,4} + h_2 l_1 a_{2,1} + h_2 l_2 a_{2,2} + h_2 l_3 a_{2,3} + h_2 l_4 a_{2,4} + h_3 l_1 a_{3,1} + h_3 l_2 a_{3,2} + h_3 l_3 a_{3,3} + h_3 l_4 a_{3,4} + h_4 l_1 a_{4,1} + h_4 l_2 a_{4,2} + h_4 l_3 a_{4,3} + h_4 l_4 a_{4,4},$
$e_1 f_1 b_{1,1} + e_1 f_2 b_{1,2} + e_1 f_3 b_{1,3} + e_1 f_4 b_{1,4} + e_2 f_1 b_{2,1} + e_2 f_2 b_{2,2} + e_2 f_3 b_{2,3} + e_2 f_4 b_{2,4} + e_3 f_1 b_{3,1} + e_3 f_2 b_{3,2} + e_3 f_3 b_{3,3} + e_3 f_4 b_{3,4} + e_4 f_1 b_{4,1} + e_4 f_2 b_{4,2} + e_4 f_3 b_{4,3} + e_4 f_4 b_{4,4},$
$a + b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + c_1 f_1 + c_2 f_2 + c_3 f_3 + c_4 f_4 + a_1 l_1 + a_2 l_2 + a_3 l_3 + a_4 l_4\big]$

```
Short[ℨ // m1,2→1, 5] // Timing
```

$$\left\{2.21875, \mathbb{E}1n\left[\frac{\omega}{1+h_1 b_{2,1}}, h_1 l_1 a_{1,1} + h_1 l_1 a_{1,2} + h_1 l_3 a_{1,3} + h_1 l_4 a_{1,4} + h_1 l_1 a_{2,1} + h_1 l_1 a_{2,2} + h_1 l_3 a_{2,3} + h_1 l_4 a_{2,4} + h_3 l_1 a_{3,1} + h_3 l_1 a_{3,2} + h_3 l_3 a_{3,3} + h_3 l_4 a_{3,4} + h_4 l_1 a_{4,1} + h_4 l_1 a_{4,2} + h_4 l_3 a_{4,3} + h_4 l_4 a_{4,4}, \frac{\ll 1 \gg}{1 + h_1 b_{2 \ll 1 \gg} \ll 1 \gg}, \frac{2\,a + 2\,b_1\,e_1 + 2\,e^{h_1 a_{1,1} + h_1 a_{\ll 1 \gg} + \ll 1 \gg + h_4 a_{4,1}}\,b_1\,e_1 + \ll 404 \gg}{2\left(1+h_1 b_{2,1}\right)^4}\right]\right\}$$

```
Short[lhs = ℨ // m1,2→1 // m1,3→1, 5] // Timing
```

$$\left\{275.031, \mathbb{E}1n\left[\omega \Big/ \left(1 + h_1 b_{2,1} + e^{h_1 a_{1,2} + h_1 a_{2,2} + h_1 a_{3,2} + h_4 a_{4,2}}\,h_1 b_{3,1} - h_1^2 b_{2,2} b_{3,1} + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2}\right), h_1 l_1 a_{1,1} + h_1 l_1 a_{1,2} + h_1 l_1 a_{1,3} + h_1 l_4 a_{1,4} + h_1 l_1 a_{2,1} + h_1 l_1 a_{2,2} + h_1 l_1 a_{2,3} + \ll 3 \gg + h_1 l_1 a_{3,3} + h_1 l_4 a_{3,4} + h_4 l_1 a_{4,1} + h_4 l_1 a_{4,2} + h_4 l_1 a_{4,3} + h_4 l_4 a_{4,4}, \frac{\ll 1 \gg}{\ll 1 \gg}, \frac{2\,a + \ll 16247 \gg}{2\left(1 + h_1 b_{2,1} + \ll 3 \gg + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2}\right)^4}\right]\right\}$$

```
Short[rhs = ℨ // m2,3→2 // m1,2→1, 5] // Timing
```

$$\left\{208.391, \mathbb{E}1n\left[\omega \Big/ \left(1 + h_1 b_{2,1} + e^{h_1 a_{1,2} + h_1 a_{2,2} + h_1 a_{3,2} + h_4 a_{4,2}}\,h_1 b_{3,1} - h_1^2 b_{2,2} b_{3,1} + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2}\right), h_1 l_1 a_{1,1} + h_1 l_1 a_{1,2} + h_1 l_1 a_{1,3} + h_1 l_4 a_{1,4} + h_1 l_1 a_{2,1} + h_1 l_1 a_{2,2} + h_1 l_1 a_{2,3} + \ll 3 \gg + h_1 l_1 a_{3,3} + h_1 l_4 a_{3,4} + h_4 l_1 a_{4,1} + h_4 l_1 a_{4,2} + h_4 l_1 a_{4,3} + h_4 l_4 a_{4,4}, \frac{\ll 1 \gg}{\ll 1 \gg}, \frac{2\,a + \ll 16247 \gg}{2\left(1 + h_1 b_{2,1} + \ll 3 \gg + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2}\right)^4}\right]\right\}$$

```
lhs ≡ rhs
```

```
True
```

Note. Meta-associativity does not work for arbitrary Λ, yet is does work for any scalar multiple of “our” Λ, including for Λ = 0. I don’t know what other constraints one must impose on a meta-monoid for there to be “interesting” solutions for *R* within it. Even worse, I’m not sure what’s “interesting”.

Profiling

```
<< "C:\\drorbn\\AcademicPensieve\\Projects\\Profile\\Profile.m"
```

```
This is Profile.m of http://drorbn.net/AcademicPensieve/Projects/Profile/.
  This version: March 2017. Original version: July 1994.
```

```
CF[𝔼1n[ω_, L_, Q_, P_]] := PP_CF@𝔼1n[Together[ω], Together[L], Together[Q], Together[P]];
NO_(x:f|e)i_ 1j_[𝔼1n[ω_, L_, Q_, P_]] := PP_NOx1@With[{q = e^γ β x_i + γ 1_j},
    CF[𝔼1n[ω, L,
        e^γ β x_i + (Q /. x_i → 0),
        e^-q DP_{1_j→D_γ, x_i→D_β}[P][e^q]
       ] /. {γ → ∂_{1_j} L, β → ∂_{x_i} Q}]];
NO_{f_i_ e_j_→k_}[𝔼1n[ω_, L_, Q_, P_]] := PP_NOfe@With[{q = ν (-α β h_k + β e_k + α f_k + δ e_k f_k)},
    CF[𝔼1n[ν ω, L,
         q + (Q /. f_i | e_j → 0),
         e^-q DP_{f_i→D_α, e_j→D_β}[P][e^q] + (Λ1[h_k, e_k, l_k, f_k, α, β, δ] - 1 /. ϵ → 1)
        ] /. ν → (1 + h_k δ)^-1 /. {α → ∂_{f_i} Q /. e_j → 0, β → ∂_{e_j} Q /. f_i → 0, δ → ∂_{f_i,e_j} Q}]];
```

```
BeginProfile[];
Short[ℨ // m1,2→1 // m1,3→1]
EndProfile[];
PrintProfile[]
```

$\mathbb{E}1\text{n}\left[\frac{\omega}{1+\ll 5\gg+h_1^2\, b_{2,1}\, b_{3,2}},\ h_1\, l_1\, a_{1,1}+\ll 14\gg+h_4\, l_4\, a_{4,4},\ \frac{\ll 1\gg}{\ll 1\gg},\ \frac{2\,a+\ll 16\,247\gg}{2\,(\ll 1\gg)^4}\right]$

```
CF: called 8 times, time in 273.297/273.297
  Parents:
    ( 2)  108.880/ 108.880 under NOfe
    ( 4)  153.580/ 153.580 under NOxl
    ( 2)  10.844/ 10.844 under ProfileRoot
NOxl: called 4 times, time in 11.265/164.843
  Parents:
    ( 4)  11.265/ 164.840 under ProfileRoot
  Children:
    ( 4)  153.580/ 153.580 above CF
NOfe: called 2 times, time in 0.187/109.062
  Parents:
    ( 2)   0.187/ 109.060 under ProfileRoot
  Children:
    ( 2)  108.880/ 108.880 above CF
ProfileRoot: called 0 times, time in 0./0.
  Children:
    ( 2)  10.844/ 10.844 above CF
    ( 2)   0.187/ 109.060 above NOfe
    ( 4)  11.265/ 164.840 above NOxl
```

Pragmatic Simplifications

$\mathbb{E}1\text{n}[\omega, L, Q, P]$ stands for $\omega\, e^{L+Q}(1+\epsilon P)$; $\mathbb{E}1\text{p}[\omega, L, Q, P]$ stands for $\omega^{-1}\, e^{L+\omega^{-1}Q}\left(1+\epsilon\,\omega^{-4} P\right)$ /. $e_i \to \frac{t_i-1}{h_i} e_i$, all written in $t_i = e^{h_i}$.

```
𝔼1n[𝔼1p[ω_, L_, Q_, P_]] := CF[PowerExpand /@ CF[𝔼1n[ω^-1, L, ω^-1 Q, ω^-4 P] /. e_i_ :> (t_i - 1)/h_i e_i /. t_i_ :> e^(h_i)]];
CF[𝔼1p[ω_, L_, Q_, P_]] := 𝔼1p[Together[ω], Together[L], Together[Q], Together[P]];
𝔼1p[𝔼1n[ω_, L_, Q_, P_]] := CF[𝔼1p[ω^-1, L, ω^-1 Q, ω^-4 P] /. e_i_ :> h_i/(t_i - 1) e_i /. h_i_ :> Log[t_i]];
𝔼1p[ω1_, L1_, Q1_, P1_] ≡ 𝔼1p[ω2_, L2_, Q2_, P2_] := Simplify[ω1 == ω2 ∧ L1 == L2 ∧ Q1 == Q2 ∧ P1 == P2];
```

```
{X⁺1,2 // 𝔼1n , X⁺1,2 // 𝔼1n // 𝔼1p}
```

$\left\{\mathbb{E}1\text{n}\left[1,\ h_1\, l_2,\ \frac{(-1+e^{h_1})\, e_1\, f_2}{h_1},\ P^+\right],\ \mathbb{E}1\text{p}[1,\ \text{Log}[t_1]\, l_2,\ e_1\, f_2,\ P^+]\right\}$

```
ℨ = ReplacePart[ℨ, 4 → P]
```

$\mathbb{E}1\text{n}[\omega,\ h_1 l_1 a_{1,1}+h_1 l_2 a_{1,2}+h_1 l_3 a_{1,3}+h_1 l_4 a_{1,4}+h_2 l_1 a_{2,1}+h_2 l_2 a_{2,2}+h_2 l_3 a_{2,3}+h_2 l_4 a_{2,4}+h_3 l_1 a_{3,1}+h_3 l_2 a_{3,2}+h_3 l_3 a_{3,3}+h_3 l_4 a_{3,4}+h_4 l_1 a_{4,1}+h_4 l_2 a_{4,2}+h_4 l_3 a_{4,3}+h_4 l_4 a_{4,4},\ e_1 f_1 b_{1,1}+e_1 f_2 b_{1,2}+e_1 f_3 b_{1,3}+e_1 f_4 b_{1,4}+e_2 f_1 b_{2,1}+e_2 f_2 b_{2,2}+e_2 f_3 b_{2,3}+e_2 f_4 b_{2,4}+e_3 f_1 b_{3,1}+e_3 f_2 b_{3,2}+e_3 f_3 b_{3,3}+e_3 f_4 b_{3,4}+e_4 f_1 b_{4,1}+e_4 f_2 b_{4,2}+e_4 f_3 b_{4,3}+e_4 f_4 b_{4,4},\ P]$

Simplify /@ $\mathbb{E}$1p[ζ]

$$\mathbb{E}1p\Big[\frac{1}{\omega},\ \text{Log}[t_1]\,l_3\,a_{1,3}+\text{Log}[t_1]\,l_4\,a_{1,4}+\text{Log}[t_2]\,l_3\,a_{2,3}+\text{Log}[t_2]\,l_4\,a_{2,4}+\text{Log}[t_3]\,l_3\,a_{3,3}+\text{Log}[t_3]\,l_4\,a_{3,4}+l_1\,(\text{Log}[t_1]\,a_{1,1}+\text{Log}[t_2]\,a_{2,1}+\text{Log}[t_3]\,a_{3,1}+\text{Log}[t_4]\,a_{4,1})+l_2\,(\text{Log}[t_1]\,a_{1,2}+\text{Log}[t_2]\,a_{2,2}+\text{Log}[t_3]\,a_{3,2}+\text{Log}[t_4]\,a_{4,2})+\text{Log}[t_4]\,l_3\,a_{4,3}+\text{Log}[t_4]\,l_4\,a_{4,4},$$
$$\frac{1}{\omega}\left(\frac{\text{Log}[t_1]\,e_1\,(f_1 b_{1,1}+f_2 b_{1,2}+f_3 b_{1,3}+f_4 b_{1,4})}{-1+t_1}+\frac{\text{Log}[t_2]\,e_2\,(f_1 b_{2,1}+f_2 b_{2,2}+f_3 b_{2,3}+f_4 b_{2,4})}{-1+t_2}+\frac{\text{Log}[t_3]\,e_3\,(f_1 b_{3,1}+f_2 b_{3,2}+f_3 b_{3,3}+f_4 b_{3,4})}{-1+t_3}+\frac{\text{Log}[t_4]\,e_4\,(f_1 b_{4,1}+f_2 b_{4,2}+f_3 b_{4,3}+f_4 b_{4,4})}{-1+t_4}\right),\ \frac{P}{\omega^4}\Big]$$

ζ ≡ (ζ // $\mathbb{E}$1p // $\mathbb{E}$1n)

True

Simplify /@ $\mathbb{E}$1n[$\mathbb{E}$1p @@ ζ]

$$\mathbb{E}1n\Big[\frac{1}{\omega},\ h_1\,(l_1 a_{1,1}+l_2 a_{1,2}+l_3 a_{1,3}+l_4 a_{1,4})+h_2\,(l_1 a_{2,1}+l_2 a_{2,2}+l_3 a_{2,3}+l_4 a_{2,4})+h_3 l_1 a_{3,1}+h_3 l_2 a_{3,2}+h_3 l_3 a_{3,3}+h_3 l_4 a_{3,4}+h_4 l_1 a_{4,1}+h_4 l_2 a_{4,2}+h_4 l_3 a_{4,3}+h_4 l_4 a_{4,4},$$
$$\frac{1}{\omega}\left(\frac{(-1+e^{h_1})\,e_1\,(f_1 b_{1,1}+f_2 b_{1,2}+f_3 b_{1,3}+f_4 b_{1,4})}{h_1}+\frac{(-1+e^{h_2})\,e_2\,(f_1 b_{2,1}+f_2 b_{2,2}+f_3 b_{2,3}+f_4 b_{2,4})}{h_2}+\frac{(-1+e^{h_3})\,e_3\,(f_1 b_{3,1}+f_2 b_{3,2}+f_3 b_{3,3}+f_4 b_{3,4})}{h_3}+\frac{(-1+e^{h_4})\,e_4\,(f_1 b_{4,1}+f_2 b_{4,2}+f_3 b_{4,3}+f_4 b_{4,4})}{h_4}\right),\ \frac{P}{\omega^4}\Big]$$

($\mathbb{E}$1p @@ ζ /. $h_{i\_}$:→ Log[t_i]) ≡ ($\mathbb{E}$1n[$\mathbb{E}$1p @@ ζ] // $\mathbb{E}$1p)

True

$\mathbb{E}$1p[ω, L, $e_1 f_2 + f_1 e_2 + \delta\, e_1 f_1$, 0] // $\mathbb{E}$1n

$$\mathbb{E}1n\Big[\frac{1}{\omega},\ L,\ \frac{1}{\omega\, h_1 h_2}\left(-e_2 f_1 h_1+e^{h_2} e_2 f_1 h_1-\delta\, e_1 f_1 h_2+e^{h_1}\delta\, e_1 f_1 h_2-e_1 f_2 h_2+e^{h_1} e_1 f_2 h_2\right),\ 0\Big]$$

Simplify /@ (($\mathbb{E}$1p[ω, L, $\omega\, e_1 \beta + \omega\, f_1 \alpha + \omega\, \delta\, e_1 f_1$, 0] // $\mathbb{E}$1n // $\text{NO}_{f_1 e_1 \to 0}$ // $\mathbb{E}$1p) /. {$t_0 \to t_1$, $h_0 \to h_1$, $l_0 \to l_1$})

$$\mathbb{E}1p\Big[\omega\,(1-\delta+\delta t_1),\ L,\ \omega\,(e_0\,(\beta+\delta f_0)+\alpha\,(\beta+f_0-\beta t_1)),$$
$$\frac{1}{2\,\text{Log}[t_1]}\,\omega^4\,(-1+t_1)\,\Big(\alpha^2\beta^2+4\alpha\beta\delta+2\delta^2-4\alpha\beta\delta^2-4\delta^3+2\delta^4+4\alpha\beta l_1+4\delta l_1-8\alpha\beta\delta l_1-12\delta^2 l_1+4\alpha\beta\delta^2 l_1+12\delta^3 l_1-4\delta^4 l_1-\alpha^2\beta^2 t_1-4\alpha\beta\delta t_1-2\delta^2 t_1+8\alpha\beta\delta^2 t_1+8\delta^3 t_1-6\delta^4 t_1+8\alpha\beta\delta l_1 t_1+12\delta^2 l_1 t_1-8\alpha\beta\delta^2 l_1 t_1-24\delta^3 l_1 t_1+12\delta^4 l_1 t_1-4\alpha\beta\delta^2 t_1^2-4\delta^3 t_1^2+6\delta^4 t_1^2+4\alpha\beta\delta^2 l_1 t_1^2+12\delta^3 l_1 t_1^2-12\delta^4 l_1 t_1^2-2\delta^4 t_1^3+4\delta^4 l_1 t_1^3+\alpha^2\delta f_0^2\,(2-\delta+\delta t_1)+2\alpha f_0\left(\alpha\beta+2\delta-2\delta^2+2\delta^2 t_1+2\delta l_1\,(1-\delta+\delta t_1)^2\right)+\delta e_0^2\,(\beta+\delta f_0)\,(\beta\,(2-\delta+\delta t_1)+\delta f_0\,(4-3\delta+3\delta t_1))+2 e_0\left(\alpha\delta^2 f_0^2\,(3-2\delta+2\delta t_1)+\beta\left(\alpha\beta+2\delta-2\delta^2+2\delta^2 t_1+2\delta l_1\,(1-\delta+\delta t_1)^2\right)+2\delta f_0\left(\delta l_1\,(1-\delta+\delta t_1)^2+(2-\delta+\delta t_1)\,(\alpha\beta+\delta-\delta^2+\delta^2 t_1)\right)\right)\Big)\Big]$$

Simplify /@ (($\mathbb{E}$1p[ω, L, $\omega\, \beta\, e_1 + \omega\, \alpha\, f_1 + \omega\, \delta\, e_1 f_1$, 0] // $\mathbb{E}$1n // $\text{NO}_{f_1 e_1 \to k}$ // $\mathbb{E}$1p) /. $t_1 \to t_k$)

$$\mathbb{E}1p\Big[\omega\,(1-\delta+\delta t_k),\ L,\ \omega\,(e_k\,(\beta+\delta f_k)+\alpha\,(\beta+f_k-\beta t_k)),$$
$$\frac{1}{2\,\text{Log}[t_k]}\,\omega^4\,(-1+t_k)\,\Big(\alpha^2\beta^2+4\alpha\beta\delta+2\delta^2-4\alpha\beta\delta^2-4\delta^3+2\delta^4+4\alpha\beta l_k+4\delta l_k-8\alpha\beta\delta l_k-12\delta^2 l_k+4\alpha\beta\delta^2 l_k+12\delta^3 l_k-4\delta^4 l_k-\alpha^2\beta^2 t_k-4\alpha\beta\delta t_k-2\delta^2 t_k+8\alpha\beta\delta^2 t_k+8\delta^3 t_k-6\delta^4 t_k+8\alpha\beta\delta l_k t_k+12\delta^2 l_k t_k-8\alpha\beta\delta^2 l_k t_k-24\delta^3 l_k t_k+12\delta^4 l_k t_k-4\alpha\beta\delta^2 t_k^2-4\delta^3 t_k^2+6\delta^4 t_k^2+4\alpha\beta\delta^2 l_k t_k^2+12\delta^3 l_k t_k^2-12\delta^4 l_k t_k^2-2\delta^4 t_k^3+4\delta^4 l_k t_k^3+\alpha^2\delta f_k^2\,(2-\delta+\delta t_k)+2\alpha f_k\left(\alpha\beta+2\delta-2\delta^2+2\delta^2 t_k+2\delta l_k\,(1-\delta+\delta t_k)^2\right)+\delta e_k^2\,(\beta+\delta f_k)\,(\beta\,(2-\delta+\delta t_k)+\delta f_k\,(4-3\delta+3\delta t_k))+2 e_k\left(\alpha\delta^2 f_k^2\,(3-2\delta+2\delta t_k)+\beta\left(\alpha\beta+2\delta-2\delta^2+2\delta^2 t_k+2\delta l_k\,(1-\delta+\delta t_k)^2\right)+2\delta f_k\left(\delta l_k\,(1-\delta+\delta t_k)^2+(2-\delta+\delta t_k)\,(\alpha\beta+\delta-\delta^2+\delta^2 t_k)\right)\right)\Big)\Big]$$

$$\mathbb{E}1p\ /:\ \mathbb{E}1p[\omega 1\_,\ L1\_,\ Q1\_,\ P1\_]\ \mathbb{E}1p[\omega 2\_,\ L2\_,\ Q2\_,\ P2\_] := \text{CF}@\mathbb{E}1p\left[\omega 1\, \omega 2,\ L1 + L2,\ \omega 2\, Q1 + \omega 1\, Q2,\ \omega 2^4\, P1 + \omega 1^4\, P2\right];$$

$$\text{NO}_{f_{i\_}\, e_{j\_} \to k\_}[\mathbb{E}1p[\omega\_,\ L\_,\ Q\_,\ P\_]] := \text{Module}\Big[\{\alpha, \beta, \delta, \mu, q, \Lambda\},$$
$$q = \left(\left(1 - t_k\right) \alpha \beta + \beta e_k + \delta e_k f_k + \alpha f_k\right) / \mu;$$
$$\Lambda = \frac{1}{2 \operatorname{Log}[t_k]} \omega^4 \left(-1 + t_k\right)$$
$$\Big(\alpha^2 \beta^2 + 4 \alpha \beta \delta + 2 \delta^2 - 4 \alpha \beta \delta^2 - 4 \delta^3 + 2 \delta^4 + 4 \alpha \beta l_k + 4 \delta l_k - 8 \alpha \beta \delta l_k - 12 \delta^2 l_k + 4 \alpha \beta \delta^2 l_k + 12 \delta^3 l_k -$$
$$4 \delta^4 l_k - \alpha^2 \beta^2 t_k - 4 \alpha \beta \delta t_k - 2 \delta^2 t_k + 8 \alpha \beta \delta^2 t_k + 8 \delta^3 t_k - 6 \delta^4 t_k + 8 \alpha \beta \delta l_k t_k + 12 \delta^2 l_k t_k -$$
$$8 \alpha \beta \delta^2 l_k t_k - 24 \delta^3 l_k t_k + 12 \delta^4 l_k t_k - 4 \alpha \beta \delta^2 t_k^2 - 4 \delta^3 t_k^2 + 6 \delta^4 t_k^2 + 4 \alpha \beta \delta^2 l_k t_k^2 + 12 \delta^3 l_k t_k^2 - 12 \delta^4 l_k t_k^2 -$$
$$2 \delta^4 t_k^3 + 4 \delta^4 l_k t_k^3 + \alpha^2 \delta f_k^2 \left(2 - \delta + \delta t_k\right) + 2 \alpha f_k \left(\alpha \beta + 2 \delta - 2 \delta^2 + 2 \delta^2 t_k + 2 \delta l_k \left(1 - \delta + \delta t_k\right)^2\right) +$$
$$\delta e_k^2 \left(\beta + \delta f_k\right) \left(\beta \left(2 - \delta + \delta t_k\right) + \delta f_k \left(4 - 3 \delta + 3 \delta t_k\right)\right) +$$
$$2 e_k \Big(\alpha \delta^2 f_k^2 \left(3 - 2 \delta + 2 \delta t_k\right) + \beta \left(\alpha \beta + 2 \delta - 2 \delta^2 + 2 \delta^2 t_k + 2 \delta l_k \left(1 - \delta + \delta t_k\right)^2\right) +$$
$$2 \delta f_k \left(\delta l_k \left(1 - \delta + \delta t_k\right)^2 + \left(2 - \delta + \delta t_k\right) \left(\alpha \beta + \delta - \delta^2 + \delta^2 t_k\right)\right)\Big)\Big);$$
$$\text{CF}\Big[$$
$$\mathbb{E}1p\Big[\mu \omega,\ L,\ \mu \omega q + \mu \left(Q\ /.\ f_i \mid e_j \to 0\right),\ \mu^4 \left(\text{DP}_{f_i \to D_\alpha, e_j \to D_\beta}[P]\left[e^q\right]\ /.\ e^{-} \to 1\right) + \omega^4 \Lambda[k]\Big]\ /.\ \mu \to 1 + \left(t_k - 1\right) \delta\ /.$$
$$\left\{\alpha \to \omega^{-1} \left(\partial_{f_i} Q\ /.\ e_j \to 0\right),\ \beta \to \omega^{-1} \left(\partial_{e_j} Q\ /.\ f_i \to 0\right),\ \delta \to \omega^{-1} \partial_{f_i, e_j} Q\right\}$$
$$\Big]\Big];$$

$$\text{NO}_{l_{j\_}\, (x:e|f)_{i\_} \to k\_}[\mathbb{E}1p[\omega\_,\ L\_,\ Q\_,\ P\_]] := \text{With}\Big[\left\{q = e^{\gamma} \beta x_k + \gamma l_k\right\},\ \text{CF}\Big[$$
$$\mathbb{E}1p\left[\omega,\ \gamma l_k + \left(L\ /.\ l_j \to 0\right),\ \omega e^{\gamma} \beta x_k + \left(Q\ /.\ x_i \to 0\right),\ e^{-q}\, \text{DP}_{l_j \to D_\gamma, x_i \to D_\beta}[P]\left[e^q\right]\right]\ /.\ \left\{\gamma \to \partial_{l_j} L,\ \beta \to \omega^{-1} \partial_{x_i} Q\right\}\Big]\Big];$$

ζ

$$\mathbb{E}1n[\omega,\ h_1 l_1 a_{1,1} + h_1 l_2 a_{1,2} + h_1 l_3 a_{1,3} + h_1 l_4 a_{1,4} + h_2 l_1 a_{2,1} + h_2 l_2 a_{2,2} + h_2 l_3 a_{2,3} +$$
$$h_2 l_4 a_{2,4} + h_3 l_1 a_{3,1} + h_3 l_2 a_{3,2} + h_3 l_3 a_{3,3} + h_3 l_4 a_{3,4} + h_4 l_1 a_{4,1} + h_4 l_2 a_{4,2} + h_4 l_3 a_{4,3} + h_4 l_4 a_{4,4},$$
$$e_1 f_1 b_{1,1} + e_1 f_2 b_{1,2} + e_1 f_3 b_{1,3} + e_1 f_4 b_{1,4} + e_2 f_1 b_{2,1} + e_2 f_2 b_{2,2} + e_2 f_3 b_{2,3} + e_2 f_4 b_{2,4} +$$
$$e_3 f_1 b_{3,1} + e_3 f_2 b_{3,2} + e_3 f_3 b_{3,3} + e_3 f_4 b_{3,4} + e_4 f_1 b_{4,1} + e_4 f_2 b_{4,2} + e_4 f_3 b_{4,3} + e_4 f_4 b_{4,4},\ P]$$

$\text{lhs} = \zeta\ //\ \text{NO}_{f_1\, e_1 \to 5}\ //\ \mathbb{E}1p$

$$\mathbb{E}1p\Big[\frac{1 + \operatorname{Log}[t_5]\, b_{1,1}}{\omega},\ \operatorname{Log}[t_1]\, l_1 a_{1,1} + \operatorname{Log}[t_1]\, l_2 a_{1,2} + \operatorname{Log}[t_1]\, l_3 a_{1,3} + \langle\langle 11 \rangle\rangle + \operatorname{Log}[t_4]\, l_3 a_{4,3} + \operatorname{Log}[t_4]\, l_4 a_{4,4},$$
$$\frac{\langle\langle 1 \rangle\rangle}{\langle\langle 1 \rangle\rangle},\ \left(2 P - 4 P t_2 + \langle\langle 14198 \rangle\rangle + 2 \operatorname{Log}[t_4]^2 \langle\langle 7 \rangle\rangle\, b_{4,1}^2 - \operatorname{Log}[t_4]^2 \operatorname{Log}[t_5]\, e_4^2 f_4^2 t_2^2 t_3^2 t_5^2 b_{1,4}^2 b_{4,1}^2\right) \Big/$$
$$\left(2 \omega^4 \left(-1 + t_2\right)^2 \left(-1 + t_3\right)^2 \left(-1 + t_4\right)^2 \left(-1 + t_5\right)^2\right)\Big]$$

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$\text{rhs} = \left(\zeta\ //\ \mathbb{E}1p\ //\ \text{NO}_{f_1\, e_1 \to 5}\right)$

$$\mathbb{E}1p\Big[\frac{-1 + t_1 - \operatorname{Log}[t_1]\, b_{1,1} + \operatorname{Log}[t_1]\, t_5 b_{1,1}}{\omega \left(-1 + t_1\right)},$$
$$\operatorname{Log}[t_1]\, l_1 a_{1,1} + \operatorname{Log}[t_1]\, l_2 a_{1,2} + \operatorname{Log}[t_1]\, l_3 a_{1,3} + \langle\langle 11 \rangle\rangle + \operatorname{Log}[t_4]\, l_3 a_{4,3} + \operatorname{Log}[t_4]\, l_4 a_{4,4},$$
$$\frac{\langle\langle 1 \rangle\rangle}{\langle\langle 1 \rangle\rangle},\ \frac{P - 4 P t_1 + \langle\langle 51 \rangle\rangle + \langle\langle 1 \rangle\rangle - 4 t_1^3 \frac{\langle\langle 1 \rangle\rangle}{\langle\langle 1 \rangle\rangle}[5] + t_1^4 \frac{\left(-1 + t_5\right) \left(\langle\langle 1 \rangle\rangle\right)}{2 \omega^4 \operatorname{Log}[t_5]}[5]}{\omega^4 \left(-1 + t_1\right)^4}\Big]$$

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$\text{Simplify}[\text{lhs}[[1]] == \text{rhs}[[1]]]$

$$\frac{\left(\operatorname{Log}[t_1] - \operatorname{Log}[t_5] + \operatorname{Log}[t_5]\, t_1 - \operatorname{Log}[t_1]\, t_5\right) b_{1,1}}{\omega \left(-1 + t_1\right)} == 0$$