

# The Polished $g_1$ Invariant

## Initialization

```
Once[<< KnotTheory`];
```

Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.

Read more at <http://katlas.org/wiki/KnotTheory>.

## NOE-It

```
DPx→Dα, y→Dβ[P-][f-] := (* means P[∂α, ∂β][f] *)
Total[CoefficientRules[P, {x, y}] /. ({m-, n-} → c-) ⇒ c D[f, {α, m}, {β, n}]]
```

```
CF[ $\mathbb{E}[\omega-, L-, Q-, P-]] :=
 $\mathbb{E}[\text{Expand@Together@}\omega, \text{Expand@Together@}L, \text{Expand@Together@}Q, \text{Expand@Together@}P];$$ 
```

```
 $\mathbb{E} /: \mathbb{E}[\omega 1-, L 1-, Q 1-, P 1-] \mathbb{E}[\omega 2-, L 2-, Q 2-, P 2-] := CF@ $\mathbb{E}[\omega 1 \omega 2, L 1 + L 2, \omega 2 Q 1 + \omega 1 Q 2, \omega 2^4 P 1 + \omega 1^4 P 2];$$ 
```

```
 $\Delta[k-] := ((t_k - 1) (2 (\alpha \beta + \delta \mu)^2 - \alpha^2 \beta^2) - 4 e_k l_k f_k \delta^2 \mu^2 -
\delta (1 + \mu) (f_k^2 \alpha^2 + e_k^2 \beta^2) - e_k^2 f_k^2 \delta^3 (1 + 3 \mu) -
2 (\alpha \beta + 2 \delta \mu + e_k f_k \delta^2 (1 + 2 \mu) + 2 l_k \delta \mu^2) (f_k \alpha + e_k \beta) - 4 (l_k \mu^2 + e_k f_k \delta (1 + \mu)) (\alpha \beta + \delta \mu)) (1 + t_k) / 4;$ 
```

```
 $N_{f_i e_j \rightarrow k-}[ $\mathbb{E}[\omega-, L-, Q-, P-]] := \text{With}[\{q = ((1 - t_k) \alpha \beta + \beta e_k + \delta e_k f_k + \alpha f_k) / \mu\}, CF[
\mathbb{E}[\mu \omega, L, \mu \omega q + \mu (Q /. f_i | e_j \rightarrow \theta), \mu^4 (DP_{f_i \rightarrow D_\alpha, e_j \rightarrow D_\beta}[P][e^q] /. e \rightarrow 1) + \omega^4 \Delta[k]] /. \mu \rightarrow 1 + (t_k - 1) \delta /.
\{\alpha \rightarrow \omega^{-1} (\partial_{f_i} Q /. e_j \rightarrow \theta), \beta \rightarrow \omega^{-1} (\partial_{e_j} Q /. f_i \rightarrow \theta), \delta \rightarrow \omega^{-1} \partial_{f_i, e_j} Q\}]]];$$ 
```

```
 $N_{l_j (x:e|f)_i \rightarrow k-}[ $\mathbb{E}[\omega-, L-, Q-, P-]] := \text{With}[\{q = e^\gamma \beta x_k + \gamma l_k\}, CF[
\mathbb{E}[\omega, \gamma l_k + (L /. l_j \rightarrow \theta), \omega e^\gamma \beta x_k + (Q /. x_i \rightarrow \theta), e^{-q} DP_{l_j \rightarrow D_\gamma, x_i \rightarrow D_\beta}[P][e^q]] /. \{\gamma \rightarrow \partial_{l_j} L, \beta \rightarrow \omega^{-1} \partial_{x_i} Q\}]]];$$ 
```

```
 $m_{i-, j \rightarrow k-}[Z_E] := \text{Module}[\{X, Z\},
CF[(Z // N_{f_i e_j \rightarrow x} // N_{l_i e_x \rightarrow x} // N_{f_x l_j \rightarrow x}) /. Z_{-i|j|x} \rightarrow Z_k]]$ 
```

```
 $R_{i-, j-}^+ := \mathbb{E}[1, \text{Log}[t_i] l_j, e_i f_j, e_i l_i f_j + l_i l_j + e_i^2 f_j^2 / 4];
R_{i-, j-}^- := \mathbb{E}[1, -\text{Log}[t_i] l_j, -t_i^{-1} e_i f_j, -l_i l_j + t_i^{-1} e_i l_i f_j - t_i^{-2} e_i^2 f_j^2 / 4];
(ur_{i-} := \mathbb{E}[t_i^{-1/2}, \theta, \theta, l_i t_i^2]; nr_{i-} := \mathbb{E}[t_i^{1/2}, \theta, \theta, -l_i t_i^2]);
ul- = nl- = rot[_ , \theta] = \mathbb{E}[1, \theta, \theta, \theta];
rot[i-, 1] := ur_{i-}; rot[i-, n_Integer] /; n > 1 := Module[{y}, rot[i, n - 1] rot[y, 1] // m_{i, y \rightarrow i}];
rot[i-, -1] := nr_{i-}; rot[i-, n_Integer] /; n < -1 := Module[{y}, rot[i, n + 1] rot[y, -1] // m_{i, y \rightarrow i}];$ 
```

## Rotational Virtual Knots and Z

```

RVK::usage =
  "RVK[xs, rots] represents a Rotational Virtual Knot with a list of n Xp/Xm crossings xs and
  a length 2n list of rotation numbers rots. Crossing sites are indexed 1 through
  2n, and rots[[k]] is the rotation between site k-1 and site k. RVK is also a casting
  operator converting to the RVK presentation from other knot presentations.";
RVK[pd_PD] := Module[{n, xs, x, rots, front, k},
  n = Length[pd];
  xs = List@@pd /. x_X => If[PositiveQ[x], Xp[x[[4]], x[[1]], Xm[x[[2]], x[[1]]];
  rots = Table[0, {2 n});
  front = {0};
  For[k = 0, k < 2 n, ++k,
    If[k == 0 ∨ FreeQ[front, -k],
      front = Flatten[front /. k → Catch[xs /. {
        Xp[k + 1, L_] | Xm[L_, k + 1] => Throw[{L, k + 1, 1 - L]},
        Xp[L_, k + 1] | Xm[k + 1, L_] => (++)rots[[L]; Throw[{1 - L, k + 1, L}]
      }]],
      If[MatchQ[front, {___, k, ___, -k, ___}], --rots[[k + 1]]
    ];
  RVK[xs, rots];
RVK[K_] := RVK[PD[K]];

```

```

t_ = t;
Z[K_] := Z[RVK@K];
Z[rvk_RVK] := Module[{todo, n, rots, ξ, done, st, x, ξ1, i, j, k, k1, k2, k3},
  {todo, rots} = List@@rvk;
  AppendTo[rots, 0];
  n = Length[todo];
  ξ = E[1, 0, 0, 0];
  done = {0};
  st = Range[0, 2 n + 1];
  While[todo != {},
    {x} = MaximalBy[todo, Length[done ∩ {#[[1]], #[[2]], #[[1]] - 1, #[[2]] - 1}] &, 1];
    Z$todo = todo; Z$x = x;
    {i, j} = List@@x;
    ξ1 = Switch[Head[x],
      Xp, mj,k→j[Ri,j+ (Rk3,k- nrk1 ulk2 // mk,k1→k // mk,k2→k // mk,k3→k)],
      Xm, mj,k→j[Ri,j- (Rk3,k3+ nrk1 ulk2 // mk,k1→k // mk,k2→k // mk,k3→k)
    ];
    ξ1 = rot[k, rots[[i]]] ξ1 // mk,i→i; rots[[i]] = 0;
    ξ1 = ξ1 rot[k, rots[[i + 1]]] // mi,k→i; rots[[i + 1]] = 0;
    ξ1 = rot[k, rots[[j]]] ξ1 // mk,j→j; rots[[j]] = 0;
    ξ1 = ξ1 rot[k, rots[[j + 1]]] // mj,k→j; rots[[j + 1]] = 0;
    ξ *= ξ1;
    If[MemberQ[done, i], ξ = ξ // mi,i+1→i; st = st /. st[[i + 2]] → st[[i + 1]];
    If[MemberQ[done, i - 1], ξ = ξ // mst[[i],i→st[[i]]; st = st /. st[[i + 1]] → st[[i]];
    If[MemberQ[done, j], ξ = ξ // mj,j+1→j; st = st /. st[[j + 2]] → st[[j + 1]];
    If[MemberQ[done, j - 1], ξ = ξ // mst[[j],j→st[[j]]; st = st /. st[[j + 1]] → st[[j]];
    done = done ∪ {i - 1, i, j - 1, j};
    todo = DeleteCases[todo, x];
  ];
  ξ /. {e0 → e, l0 → l, f0 → f};

```

Testing  $3_1$ ,  $8_{17}$ , and  $T_{9,5}$ **Timing**[Z[Knot[3, 1]]]

KnotTheory: Loading precomputed data in PD4Knots`.

$$\left\{ 1.01563, \mathbb{E} \left[ -1 + \frac{1}{t} + t, 0, 0, -16 + 2ef - \frac{2}{t^4} - \frac{2ef}{t^4} + \frac{2l}{t^4} + \frac{7}{t^3} + \frac{4ef}{t^3} - \frac{6l}{t^3} - \frac{14}{t^2} - \frac{6ef}{t^2} + \frac{10l}{t^2} + \frac{18}{t} + \frac{2ef}{t} - \frac{8l}{t} + 10t - 6eft + 8lt - 4t^2 + 4eft^2 - 10lt^2 + t^3 - 2eft^3 + 6lt^3 - 2lt^4 \right] \right\}$$

**Timing**[Z[Knot[8, 17]]]

$$\left\{ 9.84375, \mathbb{E} \left[ 11 - \frac{1}{t^3} + \frac{4}{t^2} - \frac{8}{t} - 8t + 4t^2 - t^3, 0, 0, \right. \right. \\ 35802ef - \frac{3}{t^{12}} - \frac{6ef}{t^{12}} + \frac{6l}{t^{12}} + \frac{44}{t^{11}} + \frac{82ef}{t^{11}} - \frac{88l}{t^{11}} - \frac{320}{t^{10}} - \frac{558ef}{t^{10}} + \frac{640l}{t^{10}} + \frac{1539}{t^9} + \frac{2520ef}{t^9} - \frac{3078l}{t^9} - \frac{5472}{t^8} - \\ \frac{8424ef}{t^8} + \frac{10944l}{t^8} + \frac{15204}{t^7} + \frac{21984ef}{t^7} - \frac{30408l}{t^7} - \frac{34023}{t^6} - \frac{46062ef}{t^6} + \frac{68046l}{t^6} + \frac{62240}{t^5} + \frac{78418ef}{t^5} - \frac{124480l}{t^5} - \\ \frac{93296}{t^4} - \frac{108174ef}{t^4} + \frac{186592l}{t^4} + \frac{113052}{t^3} + \frac{117930ef}{t^3} - \frac{226104l}{t^3} - \frac{105720}{t^2} - \frac{93510ef}{t^2} + \frac{211440l}{t^2} + \frac{64656}{t} + \\ \left. \frac{35802ef}{t} - \frac{129312l}{t} - 64656t - 93510eft + 129312lt + 105720t^2 + 117930eft^2 - 211440lt^2 - 113052t^3 - \right. \\ \left. 108174eft^3 + 226104lt^3 + 93296t^4 + 78418eft^4 - 186592lt^4 - 62240t^5 - 46062eft^5 + 124480lt^5 + \right. \\ \left. 34023t^6 + 21984eft^6 - 68046lt^6 - 15204t^7 - 8424eft^7 + 30408lt^7 + 5472t^8 + 2520eft^8 - 10944lt^8 - \right. \\ \left. 1539t^9 - 558eft^9 + 3078lt^9 + 320t^{10} + 82eft^{10} - 640lt^{10} - 44t^{11} - 6eft^{11} + 88lt^{11} + 3t^{12} - 6lt^{12} \right] \left. \right\}$$

## Timing[Z[TorusKnot[9, 5]]]

$$\begin{aligned}
& \{860.469, \mathbb{E} \left[ -1 + \frac{1}{t^{16}} - \frac{1}{t^{15}} + \frac{1}{t^{11}} - \frac{1}{t^{10}} + \frac{1}{t^7} - \frac{1}{t^5} + \frac{1}{t^2} + t^2 - t^5 + t^7 - t^{10} + t^{11} - t^{15} + t^{16}, 0, 0, \right. \\
& 7580 + 280 e f - \frac{32 e f}{t^{64}} + \frac{32 l}{t^{64}} - \frac{1}{t^{63}} + \frac{94 e f}{t^{63}} - \frac{126 l}{t^{63}} + \frac{3}{t^{62}} - \frac{92 e f}{t^{62}} + \frac{186 l}{t^{62}} - \frac{3}{t^{61}} + \frac{30 e f}{t^{61}} - \frac{122 l}{t^{61}} + \frac{1}{t^{60}} + \frac{30 l}{t^{60}} + \\
& \frac{1}{t^{59}} - \frac{118 e f}{t^{59}} + \frac{118 l}{t^{59}} - \frac{8}{t^{58}} + \frac{346 e f}{t^{58}} - \frac{464 l}{t^{58}} + \frac{18}{t^{57}} - \frac{338 e f}{t^{57}} + \frac{684 l}{t^{57}} - \frac{16}{t^{56}} + \frac{110 e f}{t^{56}} - \frac{448 l}{t^{56}} + \frac{6}{t^{55}} - \frac{110 e f}{t^{55}} + \\
& \frac{220 l}{t^{55}} - \frac{3}{t^{54}} + \frac{52 e f}{t^{54}} - \frac{162 l}{t^{54}} - \frac{12}{t^{53}} + \frac{476 e f}{t^{53}} - \frac{424 l}{t^{53}} + \frac{44}{t^{52}} - \frac{668 e f}{t^{52}} + \frac{1144 l}{t^{52}} - \frac{45}{t^{51}} + \frac{250 e f}{t^{51}} - \frac{918 l}{t^{51}} + \frac{25}{t^{50}} - \\
& \frac{400 e f}{t^{50}} + \frac{650 l}{t^{50}} - \frac{41}{t^{49}} + \frac{678 e f}{t^{49}} - \frac{1078 l}{t^{49}} + \frac{28}{t^{48}} + \frac{294 e f}{t^{48}} - \frac{384 l}{t^{48}} + \frac{62}{t^{47}} - \frac{1022 e f}{t^{47}} + \frac{1316 l}{t^{47}} - \frac{89}{t^{46}} + \frac{220 e f}{t^{46}} - \\
& \frac{1242 l}{t^{46}} + \frac{45}{t^{45}} - \frac{230 e f}{t^{45}} + \frac{450 l}{t^{45}} - \frac{105}{t^{44}} + \frac{1200 e f}{t^{44}} - \frac{1430 l}{t^{44}} + \frac{135}{t^{43}} - \frac{90 e f}{t^{43}} + \frac{1290 l}{t^{43}} - \frac{50}{t^{42}} - \frac{1350 e f}{t^{42}} + \frac{1260 l}{t^{42}} - \\
& \frac{125}{t^{41}} - \frac{120 e f}{t^{41}} - \frac{1230 l}{t^{41}} - \frac{20}{t^{40}} + \frac{780 e f}{t^{40}} - \frac{900 l}{t^{40}} - \frac{90}{t^{39}} + \frac{1248 e f}{t^{39}} - \frac{468 l}{t^{39}} + \frac{284}{t^{38}} - \frac{576 e f}{t^{38}} + \frac{1824 l}{t^{38}} + \frac{3}{t^{37}} - \\
& \frac{1982 e f}{t^{37}} + \frac{1406 l}{t^{37}} - \frac{188}{t^{36}} + \frac{250 e f}{t^{36}} - \frac{2232 l}{t^{36}} - \frac{179}{t^{35}} + \frac{1720 e f}{t^{35}} - \frac{1470 l}{t^{35}} + \frac{86}{t^{34}} + \frac{972 e f}{t^{34}} - \frac{748 l}{t^{34}} + \frac{437}{t^{33}} - \frac{1074 e f}{t^{33}} + \\
& \frac{2046 l}{t^{33}} - \frac{32}{t^{32}} - \frac{3058 e f}{t^{32}} + \frac{1984 l}{t^{32}} - \frac{491}{t^{31}} + \frac{1840 e f}{t^{31}} - \frac{4898 l}{t^{31}} - \frac{214}{t^{30}} + \frac{2020 e f}{t^{30}} - \frac{180 l}{t^{30}} + \frac{402}{t^{29}} + \frac{512 e f}{t^{29}} + \frac{1508 l}{t^{29}} + \\
& \frac{593}{t^{28}} - \frac{1994 e f}{t^{28}} + \frac{2506 l}{t^{28}} - \frac{131}{t^{27}} - \frac{3128 e f}{t^{27}} + \frac{1134 l}{t^{27}} - \frac{1110}{t^{26}} + \frac{3320 e f}{t^{26}} - \frac{6448 l}{t^{26}} + \frac{75}{t^{25}} + \frac{1870 e f}{t^{25}} + \frac{1450 l}{t^{25}} + \\
& \frac{804}{t^{24}} - \frac{2 e f}{t^{24}} + \frac{1872 l}{t^{24}} + \frac{858}{t^{23}} - \frac{3406 e f}{t^{23}} + \frac{3404 l}{t^{23}} - \frac{738}{t^{22}} - \frac{1602 e f}{t^{22}} - \frac{1804 l}{t^{22}} - \frac{1669}{t^{21}} + \frac{3900 e f}{t^{21}} - \frac{5502 l}{t^{21}} + \frac{695}{t^{20}} + \\
& \frac{1450 e f}{t^{20}} + \frac{2450 l}{t^{20}} + \frac{1341}{t^{19}} - \frac{944 e f}{t^{19}} + \frac{2394 l}{t^{19}} + \frac{989}{t^{18}} - \frac{3662 e f}{t^{18}} + \frac{2718 l}{t^{18}} - \frac{1903}{t^{17}} - \frac{24 e f}{t^{17}} - \frac{3638 l}{t^{17}} - \frac{1900}{t^{16}} + \\
& \frac{3720 e f}{t^{16}} - \frac{3744 l}{t^{16}} + \frac{1613}{t^{15}} + \frac{930 e f}{t^{15}} + \frac{2790 l}{t^{15}} + \frac{2246}{t^{14}} - \frac{2178 e f}{t^{14}} + \frac{3108 l}{t^{14}} + \frac{122}{t^{13}} - \frac{2334 e f}{t^{13}} + \frac{156 l}{t^{13}} - \frac{2952}{t^{12}} + \\
& \frac{762 e f}{t^{12}} - \frac{3096 l}{t^{12}} - \frac{1811}{t^{11}} + \frac{2940 e f}{t^{11}} - \frac{2178 l}{t^{11}} + \frac{3013}{t^{10}} + \frac{130 e f}{t^{10}} + \frac{2810 l}{t^{10}} + \frac{2956}{t^9} - \frac{2174 e f}{t^9} + \frac{2304 l}{t^9} - \frac{1656}{t^8} - \\
& \frac{962 e f}{t^8} - \frac{1212 l}{t^8} - \frac{3603}{t^7} + \frac{816 e f}{t^7} - \frac{1778 l}{t^7} - \frac{1390}{t^6} + \frac{1740 e f}{t^6} - \frac{924 l}{t^6} + \frac{5340}{t^5} - \frac{460 e f}{t^5} + \frac{2200 l}{t^5} + \frac{2004}{t^4} - \\
& \frac{932 e f}{t^4} + \frac{472 l}{t^4} - \frac{3247}{t^3} - \frac{266 e f}{t^3} - \frac{666 l}{t^3} - \frac{3938}{t^2} + \frac{218 e f}{t^2} - \frac{484 l}{t^2} - \frac{219}{t} + \frac{280 e f}{t} - \frac{62 l}{t} - 281 t + 218 e f t + \\
& 62 l t - 4422 t^2 - 266 e f t^2 + 484 l t^2 - 3913 t^3 - 932 e f t^3 + 666 l t^3 + 2476 t^4 - 460 e f t^4 - 472 l t^4 + 7540 t^5 + \\
& 1740 e f t^5 - 2200 l t^5 - 2314 t^6 + 816 e f t^6 + 924 l t^6 - 5381 t^7 - 962 e f t^7 + 1778 l t^7 - 2868 t^8 - 2174 e f t^8 + \\
& 1212 l t^8 + 5260 t^9 + 130 e f t^9 - 2304 l t^9 + 5823 t^{10} + 2940 e f t^{10} - 2810 l t^{10} - 3989 t^{11} + 762 e f t^{11} + 2178 l t^{11} - \\
& 6048 t^{12} - 2334 e f t^{12} + 3096 l t^{12} + 278 t^{13} - 2178 e f t^{13} - 156 l t^{13} + 5354 t^{14} + 930 e f t^{14} - 3108 l t^{14} + 4403 t^{15} + \\
& 3720 e f t^{15} - 2790 l t^{15} - 5644 t^{16} - 24 e f t^{16} + 3744 l t^{16} - 5541 t^{17} - 3662 e f t^{17} + 3638 l t^{17} + 3707 t^{18} - 944 e f t^{18} - \\
& 2718 l t^{18} + 3735 t^{19} + 1450 e f t^{19} - 2394 l t^{19} + 3145 t^{20} + 3900 e f t^{20} - 2450 l t^{20} - 7171 t^{21} - 1602 e f t^{21} + \\
& 5502 l t^{21} - 2542 t^{22} - 3406 e f t^{22} + 1804 l t^{22} + 4262 t^{23} - 2 e f t^{23} - 3404 l t^{23} + 2676 t^{24} + 1870 e f t^{24} - 1872 l t^{24} + \\
& 1525 t^{25} + 3320 e f t^{25} - 1450 l t^{25} - 7558 t^{26} - 3128 e f t^{26} + 6448 l t^{26} + 1003 t^{27} - 1994 e f t^{27} - 1134 l t^{27} + \\
& 3099 t^{28} + 512 e f t^{28} - 2506 l t^{28} + 1910 t^{29} + 2020 e f t^{29} - 1508 l t^{29} - 394 t^{30} + 1840 e f t^{30} + 180 l t^{30} - 5389 t^{31} - \\
& 3058 e f t^{31} + 4898 l t^{31} + 1952 t^{32} - 1074 e f t^{32} - 1984 l t^{32} + 2483 t^{33} + 972 e f t^{33} - 2046 l t^{33} + 834 t^{34} + \\
& 1720 e f t^{34} - 748 l t^{34} - 1649 t^{35} + 250 e f t^{35} + 1470 l t^{35} - 2420 t^{36} - 1982 e f t^{36} + 2232 l t^{36} + 1409 t^{37} - \\
& 576 e f t^{37} - 1406 l t^{37} + 2108 t^{38} + 1248 e f t^{38} - 1824 l t^{38} - 558 t^{39} + 780 e f t^{39} + 468 l t^{39} - 920 t^{40} - 120 e f t^{40} + \\
& 900 l t^{40} - 1355 t^{41} - 1350 e f t^{41} + 1230 l t^{41} + 1310 t^{42} - 90 e f t^{42} - 1260 l t^{42} + 1425 t^{43} + 1200 e f t^{43} - \\
& 1290 l t^{43} - 1535 t^{44} - 230 e f t^{44} + 1430 l t^{44} + 495 t^{45} + 220 e f t^{45} - 450 l t^{45} - 1331 t^{46} - 1022 e f t^{46} + 1242 l t^{46} + \\
& 1378 t^{47} + 294 e f t^{47} - 1316 l t^{47} + 412 t^{48} + 678 e f t^{48} - 384 l t^{48} - 1119 t^{49} - 400 e f t^{49} + 1078 l t^{49} + 675 t^{50} + \\
& 250 e f t^{50} - 650 l t^{50} - 963 t^{51} - 668 e f t^{51} + 918 l t^{51} + 1188 t^{52} + 476 e f t^{52} - 1144 l t^{52} - 436 t^{53} + 52 e f t^{53} + \\
& 424 l t^{53} - 165 t^{54} - 110 e f t^{54} + 162 l t^{54} + 226 t^{55} + 110 e f t^{55} - 220 l t^{55} - 464 t^{56} - 338 e f t^{56} + 448 l t^{56} + \\
& 702 t^{57} + 346 e f t^{57} - 684 l t^{57} - 472 t^{58} - 118 e f t^{58} + 464 l t^{58} + 119 t^{59} - 118 l t^{59} + 31 t^{60} + 30 e f t^{60} - 30 l t^{60} - \\
& 125 t^{61} - 92 e f t^{61} + 122 l t^{61} + 189 t^{62} + 94 e f t^{62} - 186 l t^{62} - 127 t^{63} - 32 e f t^{63} + 126 l t^{63} + 32 t^{64} - 32 l t^{64} \}
\end{aligned}$$