The $\mathfrak{g}_1$ Invariant

Reminder

Make sure that you have Mathematica and that you play with these programs!

Differential polynomials

```
DP_{x_→D_α_, y_→D_β_}[P_][f_] := Total[CoefficientRules[P, {x, y}] /. ({m_, n_} → c_) :→ c D[f, {α, m}, {β, n}]]
```

```
DP_{x→D_α, y→D_β}[(x + y)^3][e^(α u+β v)] // Simplify
```

$e^{u \alpha+v \beta} (u+v)^3$

The main $\mathfrak{g}_k$ lemma

In $\mathfrak{g}^\epsilon = \langle h, e, l, f\rangle / ([e, l] = -e,\ [f, l] = f,\ [e, f] = h - 2\epsilon l,\ [h, *] = 0)$ and at $\epsilon^{k+1} = 0$, we have

1. $\mathbb{O}\left(e^{\gamma l + \beta e} \mid le\right) = \mathbb{O}\left(e^{\gamma l + e^\gamma \beta e} \mid e\, l\right)$,
2. $\mathbb{O}\left(e^{\gamma l + \beta f} \mid f\, l\right) = \mathbb{O}\left(e^{\gamma l + e^\gamma \beta f} \mid l\, f\right)$,
3. $\mathbb{O}\left(e^{\beta e + \alpha f + \delta e f} \mid f\, e\right) = \mathbb{O}\left(\nu e^{\nu(-\alpha\beta h + \beta e + \alpha f + \delta e f)} \Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta) \mid e\, f\right)$, with $\nu = (1 + h\,\delta)^{-1}$ and where for any fixed k, $\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)$ is a fixed polynomial of degree at most $4\,k$ in e, $\sqrt{l}$, f, α, β, with scalar coefficients.

Comment. Even better, $\log(\Lambda_k)$ is of degree at most $2\,k + 2$ in said variables.

Finding the Logos

(Logos=Λὀγος, “a principle of order an knowledge”)

```
(* "D" for Detailed *)
DΛ_k_[h_, e_, l_, f_, α_, β_, δ_] := Module[
  {ρh, ρe, ρl, ρf, eqn, a, b, c, sol, λ, q, ν},
  ρh = ( -1  0 ; 0  -1 ); ρe = ( 0  0 ; -ϵ  0 ); ρl = ( -(1 + 1/ϵ)/2  0 ; 0  (1 - 1/ϵ)/2 ); ρf = ( 0  1 ; 0  0 );
  eqn = MatrixExp[α ρf].MatrixExp[β ρe] == MatrixExp[a ρe].MatrixExp[c (ρh - 2 ϵ ρl)].MatrixExp[b ρf];
  Echo[sol = Solve[Thread[Flatten /@ eqn], {a, b, c}][[1]] /. C[1] → 0];
  λ = Simplify[e^(-f α-e β+h α β) Normal@Series[e^(c h + a e - 2 ϵ c l + b f) /. sol, {ϵ, 0, k}]];
  q = e^(ν (f α+e β-h α β+e f δ));
  Collect[q^-1 DP_{α→D_f, β→D_e}[λ][q] /. ν → (1 + h δ)^-1, ϵ, Simplify]];
```

```
DΛ_1[h, e, l, f, α, β, δ]
```

» $\left\{a\$306 \to -\frac{\beta}{-1+\alpha\beta\epsilon},\ b\$306 \to -\frac{\alpha}{-1+\alpha\beta\epsilon},\ c\$306 \to \frac{\mathrm{Log}[1-\alpha\beta\epsilon]}{\epsilon}\right\}$

$$1 + \frac{1}{2\,(1+h\,\delta)^4}\Big(2\,e\,\alpha\,\beta^2 - h\,\alpha^2\,\beta^2 + 4\,e\,\beta\,\delta - 4\,h\,\alpha\,\beta\,\delta + 2\,e^2\,\beta^2\,\delta - 2\,h\,\delta^2 + 4\,e\,h\,\beta\,\delta^2 - 4\,h^2\,\alpha\,\beta\,\delta^2 + e^2\,h\,\beta^2\,\delta^2 - 4\,h^2\,\delta^3 - 2\,h^3\,\delta^4 + 4\,l\,(1+h\,\delta)^2\,\big(\alpha\,(\beta + f\,\delta) + \delta\,(1 + e\,\beta + e\,f\,\delta + h\,\delta)\big) + f^2\,\delta\,(\alpha + e\,\delta)\,\big(\alpha\,(2+h\,\delta) + e\,\delta\,(4 + 3\,h\,\delta)\big) + 2\,f\,\big(\alpha^2\,\beta + 2\,\alpha\,\delta\,(1 + h\,\delta + e\,\beta\,(2+h\,\delta)) + e\,\delta^2\,(4 + 6\,h\,\delta + 2\,h^2\,\delta^2 + e\,\beta\,(3 + 2\,h\,\delta))\big)\Big)\,\epsilon$$

```
Λk_[h_, e_, l_, f_, α_, β_, δ_] := Λk[h, e, l, f, α, β, δ] = Module[{λ},
    λ = Normal@Series[e^((f α + e β)/(1 - α β ε)) (1 - α β ε)^(-2 l + h/ε), {ε, 0, k}] /. e- → 1;
    Collect[DP_{α→Df, β→De}[λ][e^((f α + e β + e f δ)/(1 + h δ))] /. e- → 1, ε, Simplify]];
```

```
Simplify[DΛ2[h, e, l, f, α, β, δ] == Λ2[h, e, l, f, α, β, δ]]
```

» $\left\{a\$360 \to -\frac{\beta}{-1+\alpha\beta\epsilon},\ b\$360 \to -\frac{\alpha}{-1+\alpha\beta\epsilon},\ c\$360 \to \frac{\text{Log}[1-\alpha\beta\epsilon]}{\epsilon}\right\}$

True

Testing the Logos

```
NotebookOpen["C:\\drorbn\\AcademicPensieve\\Classes\\17-1350-AKT\\170317-TestingTheLogos.nb"]
```

NotebookObject[170317–TestingTheLogos.nb]

The Main $\mathfrak{g}_k$ Theorem

Raw Version. The $\mathfrak{g}_k$ invariant of any S-component tangle T can be written in the form $Z(T) = \mathbb{O}(\omega\, e^{L+Q+P} \mid \prod_{i\in S} e_i\, l_i\, f_i)$, where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i = e^{h_i}$), where $L = \sum a_{ij}\, h_i\, l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij}, where $Q = \sum b_{ij}\, e_i\, f_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij}, and where P is a polynomial in $\{\epsilon,\ e_i,\ l_i,\ f_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2d + 2$ in $\left\{e_i,\ \sqrt{l_i}\ ,\ f_i\right\}$. Furthermore, after setting $h_i = h$ and $t_i = t$ for all i, the invariant $Z(T)$ is poly-time computable.

Partial Proof. Indeed,

0. $R^\pm = ?$, $n^\pm = ?$.

1. $\mathbb{O}(\mathcal{P}(l, e)\, e^{\gamma l + \beta e} \mid l\, e) = \mathbb{O}(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^\gamma \beta e} \mid e\, l)$,

2. $\mathbb{O}(\mathcal{P}(l, f)\, e^{\gamma l + \beta f} \mid f\, l) = \mathbb{O}(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^\gamma \beta f} \mid l\, f)$,

3. $\mathbb{O}(\mathcal{P}(e, f)\, e^{\beta e + \alpha f + \delta e f} \mid f\, e) = \mathbb{O}(\nu\, \mathcal{P}(\partial_\beta, \partial_\alpha)\, e^{\nu(-\alpha\beta h + \beta e + \alpha f + \delta e f)}\, \Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta) \mid e\, f)$, with $\nu = (1 + h\delta)^{-1}$, and $\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)$ as above.

Implementation at $k = 1$

$\mathbb{E}$1n[ω, L, Q, P] stands for $\omega\, e^{L+Q}(1+\epsilon\, P)$.

```
ε /: ε^p_ /; p > 1 := 0;
```

```
CF[E1n[ω_, L_, Q_, P_]] := E1n[Together[ω], Together[L], Together[Q], Together[P]];
E1n /: E1n[ω1_, L1_, Q1_, P1_] E1n[ω2_, L2_, Q2_, P2_] := CF@E1n[ω1 ω2, L1 + L2, Q1 + Q2, P1 + P2];
E1n[ω1_, L1_, Q1_, P1_] ≡ E1n[ω2_, L2_, Q2_, P2_] := Simplify[ω1 == ω2 ∧ L1 == L2 ∧ Q1 == Q2 ∧ P1 == P2];
```

0. $R = \mathbb{O}\left(\exp\left(hl + \frac{e^h - 1}{h}\, ef + P\right) \mid e \otimes lf\right)$:

```
E1n[X⁺_{i_,j_}] := E1n[1, h_i l_j, h_i^-1 (e^(h_i) - 1) e_i f_j, P⁺];
E1n[X⁻_{i_,j_}] := E1n[1, -h_i l_j, h_i^-1 (e^(-h_i) - 1) e_i f_j, P⁻];
E1n[p_Times] := E1n /@ p;
```

```
E1n[X⁺_{4,1} X⁺_{2,5} X⁺_{6,3}]
```

$\mathbb{E}1n\left[1,\ h_4 l_1 + h_6 l_3 + h_2 l_5,\ \frac{1}{h_2 h_4 h_6}\left(-e_6 f_3 h_2 h_4 + e^{h_6} e_6 f_3 h_2 h_4 - e_4 f_1 h_2 h_6 + e^{h_4} e_4 f_1 h_2 h_6 - e_2 f_5 h_4 h_6 + e^{h_2} e_2 f_5 h_4 h_6\right),\ 3 P^+\right]$

1. $\mathbb{O}\left(\mathcal{P}(l,e)\, e^{\gamma l+\beta e} \mid l e\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^\gamma \beta e} \mid e l\right)$,
2. $\mathbb{O}\left(\mathcal{P}(l,f)\, e^{\gamma l+\beta f} \mid f l\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^\gamma \beta f} \mid l f\right)$.

```
NO_(x:f|e)_i_ l_j_ [E1n[ω_, L_, Q_, P_]] := With[{q = e^γ β x_i + γ l_j},
   CF[E1n[ω, L,
      e^γ β x_i + (Q /. x_i → 0),
      e^-q DP_{l_j→D_γ, x_i→D_β}[P][e^q]
    ] /. {γ → ∂_{l_j} L, β → ∂_{x_i} Q}]];
```

3. $\mathbb{O}\left(e^{\beta e+\alpha f+\delta e f} \mid f e\right) = \mathbb{O}\left(\nu e^{\nu(-\alpha\beta h+\beta e+\alpha f+\delta e f)} \mid e f\right)$, with $\nu = (1+h\delta)^{-1}$:

```
NO_{f_i_ e_j_→k_}[E1n[ω_, L_, Q_, P_]] := With[{q = ν (-α β h_k + β e_k + α f_k + δ e_k f_k)},
   CF[E1n[ν ω, L,
       q + (Q /. f_i | e_j → 0),
       e^-q DP_{f_i→D_α, e_j→D_β}[P][e^q] + (Λ_1[h_k, e_k, l_k, f_k, α, β, δ] - 1 /. ϵ → 1)
     ] /. ν → (1 + h_k δ)^-1 /. {α → ∂_{f_i} Q /. e_j → 0, β → ∂_{e_j} Q /. f_i → 0, δ → ∂_{f_i,e_j} Q}]];
```

```
m_{i_,j_→k_}[Z_] := Module[{x, z}, CF[(Z // NO_{f_i e_j→x} // NO_{l_i e_x} // NO_{f_x l_j}) /. z_{i|j|x} → z_k]]
```

Meta-associativity

$\zeta = \mathbb{E}1n\left[\omega,\ \sum_{i=1}^{4}\sum_{j=1}^{4} a_{i,j} h_i l_j,\ \sum_{i=1}^{4}\sum_{j=1}^{4} b_{i,j} e_i f_j,\ 0\right]$

$\mathbb{E}1n[\omega,\ h_1 l_1 a_{1,1} + h_1 l_2 a_{1,2} + h_1 l_3 a_{1,3} + h_1 l_4 a_{1,4} + h_2 l_1 a_{2,1} + h_2 l_2 a_{2,2} + h_2 l_3 a_{2,3} + h_2 l_4 a_{2,4} + h_3 l_1 a_{3,1} + h_3 l_2 a_{3,2} + h_3 l_3 a_{3,3} + h_3 l_4 a_{3,4} + h_4 l_1 a_{4,1} + h_4 l_2 a_{4,2} + h_4 l_3 a_{4,3} + h_4 l_4 a_{4,4},$
$e_1 f_1 b_{1,1} + e_1 f_2 b_{1,2} + e_1 f_3 b_{1,3} + e_1 f_4 b_{1,4} + e_2 f_1 b_{2,1} + e_2 f_2 b_{2,2} + e_2 f_3 b_{2,3} + e_2 f_4 b_{2,4} + e_3 f_1 b_{3,1} + e_3 f_2 b_{3,2} + e_3 f_3 b_{3,3} + e_3 f_4 b_{3,4} + e_4 f_1 b_{4,1} + e_4 f_2 b_{4,2} + e_4 f_3 b_{4,3} + e_4 f_4 b_{4,4},\ 0]$

```
Short[ζ // m_{1,2→1}, 5] // Timing
```

$\{4.45313,$
$\mathbb{E}1n\left[\frac{\omega}{1+h_1 b_{2,1}},\ h_1 l_1 a_{1,1} + h_1 l_1 a_{1,2} + h_1 l_3 a_{1,3} + h_1 l_4 a_{1,4} + h_1 l_1 a_{2,1} + h_1 l_1 a_{2,2} + h_1 l_3 a_{2,3} + h_1 l_4 a_{2,4} + h_3 l_1 a_{3,1} + h_3 l_1 a_{3,2} + h_3 l_3 a_{3,3} + h_3 l_4 a_{3,4} + h_4 l_1 a_{4,1} + h_4 l_1 a_{4,2} + h_4 l_3 a_{4,3} + h_4 l_4 a_{4,4},\ \frac{e^{\ll 1\gg} e_1 f_1 b_{1,1} + \ll 41\gg + \ll 1\gg}{1+h_1 b_{2,1}},\right.$
$\left.\left(4 l_1 b_{2,1} + 4 e^{h_1 a_{1,2}+h_1 a_{2,2}+h_3 a_{3,2}+h_4 a_{4,2}} e_1 f_1 b_{1,1} b_{2,1} + 4 e^{h_1 a_{\ll 1\gg}+\ll 2\gg+h_4 \ll 1\gg} e_1 f_1 l_1 b_{1,1} b_{2,1} + \ll 270\gg\right) \Big/ \left(2\left(1+h_1 b_{2,1}\right)^4\right)\right]\}$

```
Short[lhs = ζ // m_{1,2→1} // m_{1,3→1}, 5] // Timing
```

$\{294.,\ \mathbb{E}1n\left[\omega \Big/ \left(1 + h_1 b_{2,1} + e^{h_1 a_{1,2}+h_1 a_{2,2}+h_1 a_{3,2}+h_4 a_{4,2}} h_1 b_{3,1} - h_1^2 b_{2,2} b_{3,1} + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2}\right),\right.$
$h_1 l_1 a_{1,1} + h_1 l_1 a_{1,2} + h_1 l_1 a_{1,3} + \ll 10\gg + h_4 l_1 a_{4,2} + h_4 l_1 a_{4,3} + h_4 l_4 a_{4,4},\ \frac{\ll 1\gg}{\ll 1\gg},$
$\left(4 l_1 b_{2,1} + 4 e^{h_1 a_{1,2}+\ll 6\gg+h_4 a_{4,3}} e_1 f_1 b_{1,1} b_{2,1} + 4 e^{\ll 1\gg} e_1 f_1 l_1 b_{1,1} b_{2,1} + \ll 11299\gg + \ll 1\gg + 2 \ll 7\gg b_{4,2}^2 - e_4^2 f_4^2 h_1^5 b_{2,2}^2 b_{3,1}^2 b_{3,4}^2 b_{4,2}^2\right) \Big/ \left(2\left(1 + h_1 b_{2,1} + e^{\ll 1\gg} h_1 b_{3,1} - h_1^2 \ll 1\gg b_{\ll 1\gg} + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2}\right)^4\right)\left.\right]\}$

```
Short[rhs = ξ // m2,3→2 // m1,2→1, 5] // Timing
```

$\{197., \mathbb{E}1n[\omega / (1 + h_1 b_{2,1} + e^{h_1 a_{1,2}+h_1 a_{2,2}+h_1 a_{3,2}+h_4 a_{4,2}} h_1 b_{3,1} - h_1^2 b_{2,2} b_{3,1} + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2}),$
$h_1 l_1 a_{1,1} + h_1 l_1 a_{1,2} + h_1 l_1 a_{1,3} + \ll 10 \gg + h_4 l_1 a_{4,2} + h_4 l_1 a_{4,3} + h_4 l_4 a_{4,4}, \frac{\ll 1 \gg}{\ll 1 \gg},$
$(4 l_1 b_{2,1} + 4 e^{h_1 a_{1,2}+\ll 6 \gg+h_4 a_{4,3}} e_1 f_1 b_{1,1} b_{2,1} + 4 e^{\ll 1 \gg} e_1 f_1 l_1 b_{1,1} b_{2,1} + \ll 11299 \gg + \ll 1 \gg + 2 \ll 7 \gg b_{4,2}^2 -$
$e_4^2 f_4^2 h_1^5 b_{2,2}^2 b_{3,1}^2 b_{3,4}^2 b_{4,2}^2) / (2 (1 + h_1 b_{2,1} + e^{\ll 1 \gg} h_1 b_{3,1} - h_1^2 \ll 1 \gg b_{\ll 1 \gg} + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2})^4)]\}$

```
lhs ≡ rhs
```

True

Profiling

```
<< "C:\\drorbn\\AcademicPensieve\\Projects\\Profile\\Profile.m"
```

This is Profile.m of http://drorbn.net/AcademicPensieve/Projects/Profile/.
This version: March 2017. Original version: July 1994.

```
CF[𝔼1n[ω_, L_, Q_, P_]] := PP_CF@𝔼1n[Together[ω], Together[L], Together[Q], Together[P]];
NO_(x:f|e)i_ 1j_[𝔼1n[ω_, L_, Q_, P_]] := PP_NOx1@With[{q = e^γ β x_i + γ 1_j},
    CF[𝔼1n[ω, L,
        e^γ β x_i + (Q /. x_i → 0),
        e^-q DP_{1_j→D_γ, x_i→D_β}[P][e^q]
    ] /. {γ → ∂_{1_j} L, β → ∂_{x_i} Q}]];
NO_{f_i_ e_j_→k_}[𝔼1n[ω_, L_, Q_, P_]] := PP_NOfe@With[{q = ν (-α β h_k + β e_k + α f_k + δ e_k f_k)},
    CF[𝔼1n[ν ω, L,
         q + (Q /. f_i | e_j → 0),
         e^-q DP_{f_i→D_α, e_j→D_β}[P][e^q] + (Λ_1[h_k, e_k, 1_k, f_k, α, β, δ] - 1 /. ϵ → 1)
     ] /. ν → (1 + h_k δ)^-1 /. {α → ∂_{f_i} Q /. e_j → 0, β → ∂_{e_j} Q /. f_i → 0, δ → ∂_{f_i,e_j} Q}]];
```

```
BeginProfile[];
Short[ξ // m1,2→1 // m1,3→1, 5]
EndProfile[];
PrintProfile[]
```

$\mathbb{E}1\text{n}\Big[\omega \Big/ \left(1 + h_1 b_{2,1} + e^{h_1 a_{1,2}+h_1 a_{2,2}+h_1 a_{3,2}+h_4 a_{4,2}} h_1 b_{3,1} - h_1^2 b_{2,2} b_{3,1} + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2}\right),$
$h_1 l_1 a_{1,1} + h_1 l_1 a_{1,2} + h_1 l_1 a_{1,3} + \ll 10 \gg + h_4 l_1 a_{4,2} + h_4 l_1 a_{4,3} + h_4 l_4 a_{4,4}, \frac{\ll 1 \gg}{\ll 1 \gg},$
$\big(4 l_1 b_{2,1} + 4 e^{h_1 a_{1,2}+\ll 6 \gg + h_4 a_{4,3}} e_1 f_1 b_{1,1} b_{2,1} + 4 e^{\ll 1 \gg} e_1 f_1 l_1 b_{1,1} b_{2,1} + \ll 11299 \gg + \ll 1 \gg + 2 \ll 7 \gg b_{4,2}^2 -$
$e_4^2 f_4^2 h_1^5 b_{2,2}^2 b_{3,1}^2 b_{3,4}^2 b_{4,2}^2\big) \Big/ \left(2 \left(1 + h_1 b_{2,1} + e^{\ll 1 \gg} h_1 b_{3,1} - h_1^2 \ll 1 \gg b_{\ll 1 \gg} + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2}\right)^4\right)\Big]$

```
CF: called 8 times, time in 220.75/220.75
  Parents:
    ( 2)   70.187/ 70.187 under NOfe
    ( 4)  124.880/ 124.880 under NOxl
    ( 2)   25.688/ 25.688 under ProfileRoot
NOxl: called 4 times, time in 19.766/144.641
  Parents:
    ( 4)   19.766/ 144.640 under ProfileRoot
  Children:
    ( 4)  124.880/ 124.880 above CF
NOfe: called 2 times, time in 0.344/70.531
  Parents:
    ( 2)    0.344/ 70.531 under ProfileRoot
  Children:
    ( 2)   70.187/ 70.187 above CF
ProfileRoot: called 0 times, time in 0./0.
  Children:
    ( 2)   25.688/ 25.688 above CF
    ( 2)    0.344/ 70.531 above NOfe
    ( 4)   19.766/ 144.640 above NOxl
```

Pragmatic Simplifications

$\mathbb{E}$1n[ω, L, Q, P] stands for $\omega e^{L+Q}(1+\epsilon P)$; $\mathbb{E}$1p[ω, L, Q, P] stands for $\omega^{-1} e^{L+\omega^{-1}Q}\left(1+\epsilon\,\omega^{-4} P\right)$ /. $e_i \to \frac{t_i-1}{h_i} e_i$, all written in $t_i = e^{h_i}$.

```
E1n[E1p[ω_, L_, Q_, P_]] := CF[PowerExpand /@ CF[E1n[ω^-1, L, ω^-1 Q, ω^-4 P] /. e_i_ :> (t_i - 1)/h_i e_i /. t_i_ :> e^(h_i)]];
CF[E1p[ω_, L_, Q_, P_]] := E1p[Together[ω], Together[L], Together[Q], Together[P]];
E1p[E1n[ω_, L_, Q_, P_]] := CF[E1p[ω^-1, L, ω^-1 Q, ω^-4 P] /. e_i_ :> h_i/(t_i - 1) e_i /. h_i_ :> Log[t_i]];
E1p[ω1_, L1_, Q1_, P1_] ≡ E1p[ω2_, L2_, Q2_, P2_] := Simplify[ω1 == ω2 ∧ L1 == L2 ∧ Q1 == Q2 ∧ P1 == P2];
```

```
X+1,2 // E1n
```

$\mathbb{E}1\text{n}\left[1, h_1 l_2, \frac{\left(-1 + e^{h_1}\right) e_1 f_2}{h_1}, P^+\right]$

```
X+1,2 // E1n // E1p
```

$\mathbb{E}1\text{p}\left[1, \text{Log}[t_1]\, l_2, e_1 f_2, P^+\right]$

```
ξ = ReplacePart[ξ, 4 → P]
```

$\mathbb{E}1\text{n}\big[\omega, h_1 l_1 a_{1,1} + h_1 l_2 a_{1,2} + h_1 l_3 a_{1,3} + h_1 l_4 a_{1,4} + h_2 l_1 a_{2,1} + h_2 l_2 a_{2,2} + h_2 l_3 a_{2,3} +$
$h_2 l_4 a_{2,4} + h_3 l_1 a_{3,1} + h_3 l_2 a_{3,2} + h_3 l_3 a_{3,3} + h_3 l_4 a_{3,4} + h_4 l_1 a_{4,1} + h_4 l_2 a_{4,2} + h_4 l_3 a_{4,3} + h_4 l_4 a_{4,4},$
$e_1 f_1 b_{1,1} + e_1 f_2 b_{1,2} + e_1 f_3 b_{1,3} + e_1 f_4 b_{1,4} + e_2 f_1 b_{2,1} + e_2 f_2 b_{2,2} + e_2 f_3 b_{2,3} + e_2 f_4 b_{2,4} +$
$e_3 f_1 b_{3,1} + e_3 f_2 b_{3,2} + e_3 f_3 b_{3,3} + e_3 f_4 b_{3,4} + e_4 f_1 b_{4,1} + e_4 f_2 b_{4,2} + e_4 f_3 b_{4,3} + e_4 f_4 b_{4,4}, P\big]$

$$\textbf{Simplify /@ }\mathbb{E}\textbf{1p}[\zeta]$$

$$\mathbb{E}1p\Big[\frac{1}{\omega}, \mathrm{Log}[t_1]\, l_3\, a_{1,3} + \mathrm{Log}[t_1]\, l_4\, a_{1,4} + \mathrm{Log}[t_2]\, l_3\, a_{2,3} + \mathrm{Log}[t_2]\, l_4\, a_{2,4} + \mathrm{Log}[t_3]\, l_3\, a_{3,3} + \mathrm{Log}[t_3]\, l_4\, a_{3,4} + l_1 \left(\mathrm{Log}[t_1]\, a_{1,1} + \mathrm{Log}[t_2]\, a_{2,1} + \mathrm{Log}[t_3]\, a_{3,1} + \mathrm{Log}[t_4]\, a_{4,1}\right) + l_2 \left(\mathrm{Log}[t_1]\, a_{1,2} + \mathrm{Log}[t_2]\, a_{2,2} + \mathrm{Log}[t_3]\, a_{3,2} + \mathrm{Log}[t_4]\, a_{4,2}\right) + \mathrm{Log}[t_4]\, l_3\, a_{4,3} + \mathrm{Log}[t_4]\, l_4\, a_{4,4},$$
$$\Big(\mathrm{Log}[t_1]\, e_1 (-1+t_2)(-1+t_3)(-1+t_4)\left(f_1 b_{1,1} + f_2 b_{1,2} + f_3 b_{1,3} + f_4 b_{1,4}\right) + (-1+t_1)\Big(\mathrm{Log}[t_2]\, e_2 (-1+t_3)(-1+t_4)\left(f_1 b_{2,1} + f_2 b_{2,2} + f_3 b_{2,3} + f_4 b_{2,4}\right) + (-1+t_2)\Big(\mathrm{Log}[t_3]\, e_3 (-1+t_4)\left(f_1 b_{3,1} + f_2 b_{3,2} + f_3 b_{3,3} + f_4 b_{3,4}\right) + \mathrm{Log}[t_4]\, e_4 (-1+t_3)\left(f_1 b_{4,1} + f_2 b_{4,2} + f_3 b_{4,3} + f_4 b_{4,4}\right)\Big)\Big)\Big) \Big/ \left(\omega (-1+t_1)(-1+t_2)(-1+t_3)(-1+t_4)\right), \frac{P}{\omega^4}\Big]$$

$$\zeta \equiv \left(\mathbb{E}\textbf{1p}[\zeta] \;\textbf{//}\; \mathbb{E}\textbf{1n}\right)$$

True

$$\textbf{Simplify /@ }\mathbb{E}\textbf{1n}[\mathbb{E}\textbf{1p @@ }\zeta]$$

$$\mathbb{E}1n\Big[\frac{1}{\omega}, h_1 \left(l_1 a_{1,1} + l_2 a_{1,2} + l_3 a_{1,3} + l_4 a_{1,4}\right) + h_2 \left(l_1 a_{2,1} + l_2 a_{2,2} + l_3 a_{2,3} + l_4 a_{2,4}\right) + h_3 l_1 a_{3,1} + h_3 l_2 a_{3,2} + h_3 l_3 a_{3,3} + h_3 l_4 a_{3,4} + h_4 l_1 a_{4,1} + h_4 l_2 a_{4,2} + h_4 l_3 a_{4,3} + h_4 l_4 a_{4,4},$$
$$\frac{1}{\omega h_1 h_2 h_3 h_4}\Big(\left(-1+e^{h_1}\right) e_1 h_2 h_3 h_4 \left(f_1 b_{1,1} + f_2 b_{1,2} + f_3 b_{1,3} + f_4 b_{1,4}\right) + h_1 \Big(\left(-1+e^{h_2}\right) e_2 h_3 h_4 \left(f_1 b_{2,1} + f_2 b_{2,2} + f_3 b_{2,3} + f_4 b_{2,4}\right) + h_2 \Big(\left(-1+e^{h_3}\right) e_3 h_4 \left(f_1 b_{3,1} + f_2 b_{3,2} + f_3 b_{3,3} + f_4 b_{3,4}\right) + \left(-1+e^{h_4}\right) e_4 h_3 \left(f_1 b_{4,1} + f_2 b_{4,2} + f_3 b_{4,3} + f_4 b_{4,4}\right)\Big)\Big)\Big), \frac{P}{\omega^4}\Big]$$

$$\left(\mathbb{E}\textbf{1p @@ }\zeta \;\textbf{/. } h_{i\_} \mapsto \textbf{Log}[t_i]\right) \equiv \left(\mathbb{E}\textbf{1n}[\mathbb{E}\textbf{1p @@ }\zeta] \;\textbf{//}\; \mathbb{E}\textbf{1p}\right)$$

True

$$\mathbb{E}\textbf{1p}[\omega, L, e_1 f_2 + f_1 e_2 + \delta\, e_1 f_1, 0] \;\textbf{//}\; \mathbb{E}\textbf{1n}$$

$$\mathbb{E}1n\Big[\frac{1}{\omega}, L, \frac{1}{\omega h_1 h_2}\left(-e_2 f_1 h_1 + e^{h_2} e_2 f_1 h_1 - \delta\, e_1 f_1 h_2 + e^{h_1} \delta\, e_1 f_1 h_2 - e_1 f_2 h_2 + e^{h_1} e_1 f_2 h_2\right), 0\Big]$$

$$\textbf{Simplify /@}$$
$$\left(\left(\mathbb{E}\textbf{1p}[\omega, L, e_1 f_2 + f_1 e_2 + \delta\, e_1 f_1, 0] \;\textbf{//}\; \mathbb{E}\textbf{1n} \;\textbf{//}\; \textbf{NO}_{f_1 e_1 \to 0} \;\textbf{//}\; \mathbb{E}\textbf{1p}\right) \textbf{/. } \{t_0 \to t_1, h_0 \to h_1, l_0 \to l_1\} \textbf{/. } \{e_2 \to \alpha, f_2 \to \beta\}\right)$$

$$\mathbb{E}1p\Big[-\delta + \omega + \delta t_1, L, \frac{\omega e_0 \left(\beta + \delta f_0\right) + \alpha\left(\beta + \omega f_0 - \beta t_1\right)}{\omega}, \frac{1}{2\,\mathrm{Log}[t_1]}$$
$$(-1+t_1)\Big(\alpha^2 \beta^2 - 4\alpha\beta\delta^2 + 2\delta^4 + 4\alpha\beta\delta\omega - 4\delta^3\omega + 2\delta^2\omega^2 + 4\alpha\beta\delta^2 l_1 - 4\delta^4 l_1 - 8\alpha\beta\delta\omega l_1 + 12\delta^3\omega l_1 + 4\alpha\beta\omega^2 l_1 - 12\delta^2\omega^2 l_1 + 4\delta\omega^3 l_1 - \alpha^2\beta^2 t_1 + 8\alpha\beta\delta^2 t_1 - 6\delta^4 t_1 - 4\alpha\beta\delta\omega t_1 + 8\delta^3\omega t_1 - 2\delta^2\omega^2 t_1 - 8\alpha\beta\delta^2 l_1 t_1 + 12\delta^4 l_1 t_1 + 8\alpha\beta\delta\omega l_1 t_1 - 24\delta^3\omega l_1 t_1 + 12\delta^2\omega^2 l_1 t_1 - 4\alpha\beta\delta^2 t_1^2 + 6\delta^4 t_1^2 - 4\delta^3\omega t_1^2 + 4\alpha\beta\delta^2 l_1 t_1^2 - 12\delta^4 l_1 t_1^2 + 12\delta^3\omega l_1 t_1^2 - 2\delta^4 t_1^3 + 4\delta^4 l_1 t_1^3 + \alpha^2\delta f_0^2\left(-\delta + 2\omega + \delta t_1\right) + \delta e_0^2\left(\beta + \delta f_0\right)\left(\beta\left(-\delta + 2\omega + \delta t_1\right) + \delta f_0\left(-3\delta + 4\omega + 3\delta t_1\right)\right) +$$
$$2\alpha f_0\left(2\delta l_1\left(-\delta + \omega + \delta t_1\right)^2 + \omega\left(\alpha\beta + 2\delta(-\delta + \omega) + 2\delta^2 t_1\right)\right) +$$
$$2 e_0\Big(\alpha\delta^2 f_0^2\left(-2\delta + 3\omega + 2\delta t_1\right) + 2\delta f_0\left(\delta l_1\left(-\delta + \omega + \delta t_1\right)^2 + \left(-\delta + 2\omega + \delta t_1\right)\left(\alpha\beta + \delta(-\delta + \omega) + \delta^2 t_1\right)\right) + \beta\left(2\delta l_1\left(-\delta + \omega + \delta t_1\right)^2 + \omega\left(\alpha\beta + 2\delta(-\delta + \omega) + 2\delta^2 t_1\right)\right)\Big)\Big)\Big]$$

```
Simplify /@ ((𝔼1p[ω, L, e1 f2 + f1 e2 + δ e1 f1, 0] // 𝔼1n // NO_{f1 e1→0} // 𝔼1p) /. {e2 → α, f2 → β})
```

$$\mathbb{E}1p\Big[\frac{-\delta\,\mathrm{Log}[t_0]+\omega\,\mathrm{Log}[t_1]+\delta\,\mathrm{Log}[t_0]\,t_1}{\mathrm{Log}[t_1]},\ L,$$
$$\Big(-\alpha\,(-1+t_0)\,\big(-\omega\,\mathrm{Log}[t_1]\,f_0+\beta\,\mathrm{Log}[t_0]\,(-1+t_1)\big)+\omega\,\mathrm{Log}[t_0]\,e_0\,(\beta+\delta f_0)\,(-1+t_1)\Big)\Big/\Big(\omega\,\mathrm{Log}[t_1]\,(-1+t_0)\Big),$$
$$\frac{1}{2\,\mathrm{Log}[t_1]^4\,(-1+t_0)^2}\,(-1+t_1)\,\Big(\delta\,\mathrm{Log}[t_0]^2\,e_0^2\,(\beta+\delta f_0)\,(-1+t_1)^2$$
$$\big(\beta\,(-\delta\,\mathrm{Log}[t_0]+2\,\omega\,\mathrm{Log}[t_1]+\delta\,\mathrm{Log}[t_0]\,t_1)+\delta f_0\,(-3\,\delta\,\mathrm{Log}[t_0]+4\,\omega\,\mathrm{Log}[t_1]+3\,\delta\,\mathrm{Log}[t_0]\,t_1)\big)+$$
$$2\,\mathrm{Log}[t_0]\,e_0\,(-1+t_0)\,(-1+t_1)\,\Big(\alpha\,\delta^2\,\mathrm{Log}[t_1]\,f_0^2\,(-2\,\delta\,\mathrm{Log}[t_0]+3\,\omega\,\mathrm{Log}[t_1]+2\,\delta\,\mathrm{Log}[t_0]\,t_1)+$$
$$2\,\delta f_0\,\Big(\delta\,l_0\,(-\delta\,\mathrm{Log}[t_0]+\omega\,\mathrm{Log}[t_1]+\delta\,\mathrm{Log}[t_0]\,t_1)^2+(-\delta\,\mathrm{Log}[t_0]+2\,\omega\,\mathrm{Log}[t_1]+\delta\,\mathrm{Log}[t_0]\,t_1)$$
$$\big(-\delta^2\,\mathrm{Log}[t_0]+(\alpha\beta+\delta\omega)\,\mathrm{Log}[t_1]+\delta^2\,\mathrm{Log}[t_0]\,t_1\big)\Big)+\beta\,\Big(2\,\delta\,l_0\,(-\delta\,\mathrm{Log}[t_0]+\omega\,\mathrm{Log}[t_1]+\delta\,\mathrm{Log}[t_0]\,t_1)^2+$$
$$\omega\,\mathrm{Log}[t_1]\,\big(-2\,\delta^2\,\mathrm{Log}[t_0]+(\alpha\beta+2\,\delta\omega)\,\mathrm{Log}[t_1]+2\,\delta^2\,\mathrm{Log}[t_0]\,t_1\big)\Big)\Big)+$$
$$(-1+t_0)^2\,\Big(\alpha^2\,\delta\,\mathrm{Log}[t_1]^2\,f_0^2\,(-\delta\,\mathrm{Log}[t_0]+2\,\omega\,\mathrm{Log}[t_1]+\delta\,\mathrm{Log}[t_0]\,t_1)+$$
$$4\,l_0\,(-\delta\,\mathrm{Log}[t_0]+\omega\,\mathrm{Log}[t_1]+\delta\,\mathrm{Log}[t_0]\,t_1)^2\,\big(-\delta^2\,\mathrm{Log}[t_0]+(\alpha\beta+\delta\omega)\,\mathrm{Log}[t_1]+\delta^2\,\mathrm{Log}[t_0]\,t_1\big)-$$
$$\mathrm{Log}[t_0]\,(-1+t_1)\,\Big(2\,\delta^4\,\mathrm{Log}[t_0]^2-4\,\delta^2\,(\alpha\beta+\delta\omega)\,\mathrm{Log}[t_0]\,\mathrm{Log}[t_1]+(\alpha^2\beta^2+4\,\alpha\beta\delta\omega+2\,\delta^2\omega^2)\,\mathrm{Log}[t_1]^2+$$
$$4\,\delta^2\,\mathrm{Log}[t_0]\,\big(-\delta^2\,\mathrm{Log}[t_0]+(\alpha\beta+\delta\omega)\,\mathrm{Log}[t_1]\big)\,t_1+2\,\delta^4\,\mathrm{Log}[t_0]^2\,t_1^2\Big)+$$
$$2\,\alpha\,\mathrm{Log}[t_1]\,f_0\,\Big(2\,\delta\,l_0\,(-\delta\,\mathrm{Log}[t_0]+\omega\,\mathrm{Log}[t_1]+\delta\,\mathrm{Log}[t_0]\,t_1)^2+$$
$$\omega\,\mathrm{Log}[t_1]\,\big(-2\,\delta^2\,\mathrm{Log}[t_0]+(\alpha\beta+2\,\delta\omega)\,\mathrm{Log}[t_1]+2\,\delta^2\,\mathrm{Log}[t_0]\,t_1\big)\Big)\Big)\Big)\Big]$$

```
Simplify /@ ((𝔼1p[ω, L, e1 f2 + f1 e2 + δ e1 f1, 0] // 𝔼1n // NO_{f1 e1→1} // 𝔼1p) /. {e2 → α, f2 → β})
```

$$\mathbb{E}1p\Big[-\delta+\omega+\delta t_1,\ L,\ \frac{\omega\,e_1\,(\beta+\delta f_1)+\alpha\,(\beta+\omega f_1-\beta t_1)}{\omega},\ \frac{1}{2\,\mathrm{Log}[t_1]}$$
$$(-1+t_1)\,\Big(\alpha^2\beta^2-4\,\alpha\beta\delta^2+2\,\delta^4+4\,\alpha\beta\delta\omega-4\,\delta^3\omega+2\,\delta^2\omega^2+4\,\alpha\beta\delta^2\,l_1-4\,\delta^4\,l_1-8\,\alpha\beta\delta\omega\,l_1+12\,\delta^3\omega\,l_1+4\,\alpha\beta\omega^2\,l_1-$$
$$12\,\delta^2\omega^2\,l_1+4\,\delta\omega^3\,l_1-\alpha^2\beta^2\,t_1+8\,\alpha\beta\delta^2\,t_1-6\,\delta^4\,t_1-4\,\alpha\beta\delta\omega\,t_1+8\,\delta^3\omega\,t_1-2\,\delta^2\omega^2\,t_1-8\,\alpha\beta\delta^2\,l_1\,t_1+12\,\delta^4\,l_1\,t_1+$$
$$8\,\alpha\beta\delta\omega\,l_1\,t_1-24\,\delta^3\omega\,l_1\,t_1+12\,\delta^2\omega^2\,l_1\,t_1-4\,\alpha\beta\delta^2\,t_1^2+6\,\delta^4\,t_1^2-4\,\delta^3\omega\,t_1^2+4\,\alpha\beta\delta^2\,l_1\,t_1^2-12\,\delta^4\,l_1\,t_1^2+12\,\delta^3\omega\,l_1\,t_1^2-$$
$$2\,\delta^4\,t_1^3+4\,\delta^4\,l_1\,t_1^3+\alpha^2\,\delta\,f_1^2\,(-\delta+2\,\omega+\delta t_1)+\delta\,e_1^2\,(\beta+\delta f_1)\,\big(\beta\,(-\delta+2\,\omega+\delta t_1)+\delta f_1\,(-3\,\delta+4\,\omega+3\,\delta t_1)\big)+$$
$$2\,\alpha f_1\,\Big(2\,\delta\,l_1\,(-\delta+\omega+\delta t_1)^2+\omega\,\big(\alpha\beta+2\,\delta\,(-\delta+\omega)+2\,\delta^2\,t_1\big)\Big)+$$
$$2\,e_1\,\Big(\alpha\,\delta^2\,f_1^2\,(-2\,\delta+3\,\omega+2\,\delta t_1)+2\,\delta f_1\,\Big(\delta\,l_1\,(-\delta+\omega+\delta t_1)^2+(-\delta+2\,\omega+\delta t_1)\,\big(\alpha\beta+\delta\,(-\delta+\omega)+\delta^2\,t_1\big)\Big)+$$
$$\beta\,\Big(2\,\delta\,l_1\,(-\delta+\omega+\delta t_1)^2+\omega\,\big(\alpha\beta+2\,\delta\,(-\delta+\omega)+2\,\delta^2\,t_1\big)\Big)\Big)\Big)\Big]$$

```
𝔼1n[𝔼1p[1, Log[t1] l2, e1 f2, 7]]
```

$$\mathbb{E}1n\Big[1,\ h_1\,l_2,\ \frac{(-1+e^{h_1})\,e_1\,f_2}{h_1},\ 7\Big]$$

```
𝔼1n[𝔼1p[1, Log[t1] l2, e1 f2, 7]] // 𝔼1p
```

$$\mathbb{E}1p[1,\ \mathrm{Log}[t_1]\,l_2,\ e_1\,f_2,\ 7]$$