The $\mathfrak{g}_1$ Invariant

Reminder

Make sure that you have Mathematica and that you play with these programs!

Differential polynomials

```
Coefficient[(x + y)^3, x y^2]
```
3

```
CoefficientList[(x + y)^3, x]
```
$\{y^3, 3y^2, 3y, 1\}$

```
CoefficientList[(x + y)^3, {x, y}]
```
{{0, 0, 0, 1}, {0, 0, 3, 0}, {0, 3, 0, 0}, {1, 0, 0, 0}}

```
CoefficientList[(x + y)^3, {x, y}] // MatrixForm
```
$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

```
CoefficientRules[(x + y)^3, {x, y}]
```
{{3, 0} → 1, {2, 1} → 3, {1, 2} → 3, {0, 3} → 1}

```
CoefficientRules[(x + y)^3, {x, y}] /. ({m_, n_} → c_) :→ c T^m H^n
```
$\{T^3, 3HT^2, 3H^2T, H^3\}$

```
Total[CoefficientRules[(x + y)^3, {x, y}] /. ({m_, n_} → c_) :→ c T^m H^n]
```
$H^3 + 3H^2T + 3HT^2 + T^3$

```
DP_{x_→D_α_, y_→D_β_}[P_][f_] := Total[CoefficientRules[P, {x, y}] /. ({m_, n_} → c_) :→ c D[f, {α, m}, {β, n}]]
```

```
DP_{x→D_α, y→D_β}[(x + y)^3][e^(α u + β v)] // Simplify
```
$e^{u\alpha+v\beta}(u+v)^3$

The main $\mathfrak{g}_k$ lemma

In $\mathfrak{g}^\epsilon = \langle h, e, l, f\rangle/([e, l] = -e,\ [f, l] = f,\ [e, f] = h - 2\epsilon l,\ [h, *] = 0)$ and at $\epsilon^{k+1} = 0$, we have

1. $\mathbb{O}(e^{\gamma l + \beta e} \mid le) = \mathbb{O}(e^{\gamma l + e^\gamma \beta e} \mid e\,l)$,
2. $\mathbb{O}(e^{\gamma l + \beta f} \mid f\,l) = \mathbb{O}(e^{\gamma l + e^\gamma \beta f} \mid l\,f)$,
3. $\mathbb{O}(e^{\beta e + \alpha f + \delta e f} \mid f\,e) = \mathbb{O}(\nu e^{\nu(-\alpha\beta h + \beta e + \alpha f + \delta e f)} \Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta) \mid e\,f)$, with $\nu = (1 + h\,\delta)^{-1}$ and where for any fixed k, $\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)$ is a fixed polynomial of degree at most $4k$ in e, $\sqrt{l}$, f, α, β, with scalar coefficients.

Comment. Even better, $\log(\Lambda_k)$ is of degree at most $2k + 2$ in said variables.

Finding the Logos

```
Series[ⅇ^x, {x, 0, 5}]
```

$$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O[x]^6$$

```
Normal@Series[ⅇ^x, {x, 0, 5}]
```

$$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}$$

(Logos=Λόγος, “a principle of order an knowledge”)

```
(* "D" for Detailed *)
DΛ_k_[h_, e_, l_, f_, α_, β_, δ_] := Module[
  {ρh, ρe, ρl, ρf, eqn, a, b, c, sol, λ, q, ν},
  ρh = ( -1  0 ; 0  -1 ); ρe = ( 0  0 ; -ϵ  0 ); ρl = ( -(1 + 1/ϵ)/2  0 ; 0  (1 - 1/ϵ)/2 ); ρf = ( 0  1 ; 0  0 );
  eqn = MatrixExp[α ρf].MatrixExp[β ρe] == MatrixExp[a ρe].MatrixExp[c (ρh - 2 ϵ ρl)].MatrixExp[b ρf];
  Echo[sol = Solve[Thread[Flatten /@ eqn], {a, b, c}][[1]] /. C[1] → 0];
  λ = Simplify[ⅇ^(-f α-e β+h α β) Normal@Series[ⅇ^(c h + a e - 2 ϵ c l + b f) /. sol, {ϵ, 0, k}]];
  q = ⅇ^(ν (f α+e β-h α β+e f δ));
  Collect[q^-1 DP_(α→D_f, β→D_e)[λ][q] /. ν → (1 + h δ)^-1, ϵ, Simplify]];
```

```
DΛ_1[h, e, l, f, α, β, δ]
```

$$\left\{a\$384\to -\frac{\beta}{-1+\alpha\beta\epsilon},\ b\$384\to -\frac{\alpha}{-1+\alpha\beta\epsilon},\ c\$384\to \frac{\mathrm{Log}[1-\alpha\beta\epsilon]}{\epsilon}\right\}$$

$$1+\frac{1}{2\,(1+h\,\delta)^4}\Big(2\,e\,\alpha\,\beta^2-h\,\alpha^2\,\beta^2+4\,e\,\beta\,\delta-4\,h\,\alpha\,\beta\,\delta+2\,e^2\,\beta^2\,\delta-2\,h\,\delta^2+4\,e\,h\,\beta\,\delta^2-4\,h^2\,\alpha\,\beta\,\delta^2+e^2\,h\,\beta^2\,\delta^2-4\,h^2\,\delta^3-2\,h^3\,\delta^4+4\,l\,(1+h\,\delta)^2\,\big(\alpha\,(\beta+f\,\delta)+\delta\,(1+e\,\beta+e\,f\,\delta+h\,\delta)\big)+f^2\,\delta\,(\alpha+e\,\delta)\,\big(\alpha\,(2+h\,\delta)+e\,\delta\,(4+3\,h\,\delta)\big)+2\,f\,\big(\alpha^2\,\beta+2\,\alpha\,\delta\,(1+h\,\delta+e\,\beta\,(2+h\,\delta))+e\,\delta^2\,(4+6\,h\,\delta+2\,h^2\,\delta^2+e\,\beta\,(3+2\,h\,\delta))\big)\Big)\,\epsilon$$

```
DΛ2[h, e, l, f, α, β, δ]
```

$$\left\{a\$439 \to -\frac{\beta}{-1+\alpha\beta\epsilon},\ b\$439 \to -\frac{\alpha}{-1+\alpha\beta\epsilon},\ c\$439 \to \frac{\text{Log}[1-\alpha\beta\epsilon]}{\epsilon}\right\}$$

$$\begin{aligned}
&1 + \frac{1}{2\,(1+h\,\delta)^4}\Big(2\,e\,\alpha\,\beta^2 - h\,\alpha^2\,\beta^2 + 4\,e\,\beta\,\delta - 4\,h\,\alpha\,\beta\,\delta + 2\,e^2\,\beta^2\,\delta - 2\,h\,\delta^2 + 4\,e\,h\,\beta\,\delta^2 - 4\,h^2\,\alpha\,\beta\,\delta^2 + e^2\,h\,\beta^2\,\delta^2 - 4\,h^2\,\delta^3 - 2\,h^3\,\delta^4 + \\
&\quad 4\,l\,(1+h\,\delta)^2\,\big(\alpha\,(\beta+f\,\delta) + \delta\,(1+e\,\beta+e\,f\,\delta+h\,\delta)\big) + f^2\,\delta\,(\alpha+e\,\delta)\,\big(\alpha\,(2+h\,\delta) + e\,\delta\,(4+3\,h\,\delta)\big) + \\
&\quad 2\,f\,\big(\alpha^2\,\beta + 2\,\alpha\,\delta\,(1+h\,\delta+e\,\beta\,(2+h\,\delta)) + e\,\delta^2\,(4+6\,h\,\delta+2\,h^2\,\delta^2+e\,\beta\,(3+2\,h\,\delta))\big)\Big)\,\epsilon + \\
&\frac{1}{24\,(1+h\,\delta)^8}\Big(24\,l\,(1+h\,\delta)^4\,\big((\alpha+e\,\delta)^2\,(\beta+f\,\delta)^2 + 4\,\delta\,(\alpha+e\,\delta)\,(\beta+f\,\delta)\,(1+h\,\delta) + 2\,\delta^2\,(1+h\,\delta)^2\big) + \\
&\quad 48\,l^2\,(1+h\,\delta)^4\,\big((\alpha+e\,\delta)^2\,(\beta+f\,\delta)^2 + 4\,\delta\,(\alpha+e\,\delta)\,(\beta+f\,\delta)\,(1+h\,\delta) + 2\,\delta^2\,(1+h\,\delta)^2\big) + \\
&\quad 24\,f\,(\alpha+e\,\delta)\,(1+h\,\delta)^3\,\big((\alpha+e\,\delta)^2\,(\beta+f\,\delta)^2 + 6\,\delta\,(\alpha+e\,\delta)\,(\beta+f\,\delta)\,(1+h\,\delta) + 6\,\delta^2\,(1+h\,\delta)^2\big) + \\
&\quad 48\,f\,l\,(\alpha+e\,\delta)\,(1+h\,\delta)^3\,\big((\alpha+e\,\delta)^2\,(\beta+f\,\delta)^2 + 6\,\delta\,(\alpha+e\,\delta)\,(\beta+f\,\delta)\,(1+h\,\delta) + 6\,\delta^2\,(1+h\,\delta)^2\big) + \\
&\quad 24\,e\,(\beta+f\,\delta)\,(1+h\,\delta)^3\,\big((\alpha+e\,\delta)^2\,(\beta+f\,\delta)^2 + 6\,\delta\,(\alpha+e\,\delta)\,(\beta+f\,\delta)\,(1+h\,\delta) + 6\,\delta^2\,(1+h\,\delta)^2\big) + \\
&\quad 48\,e\,l\,(\beta+f\,\delta)\,(1+h\,\delta)^3\,\big((\alpha+e\,\delta)^2\,(\beta+f\,\delta)^2 + 6\,\delta\,(\alpha+e\,\delta)\,(\beta+f\,\delta)\,(1+h\,\delta) + 6\,\delta^2\,(1+h\,\delta)^2\big) + \\
&\quad 12\,(\beta+f\,\delta)^2\,(e+e\,h\,\delta)^2\,\big((\alpha+e\,\delta)^2\,(\beta+f\,\delta)^2 + 8\,\delta\,(\alpha+e\,\delta)\,(\beta+f\,\delta)\,(1+h\,\delta) + 12\,\delta^2\,(1+h\,\delta)^2\big) + \\
&\quad 12\,(\alpha+e\,\delta)^2\,(f+f\,h\,\delta)^2\,\big((\alpha+e\,\delta)^2\,(\beta+f\,\delta)^2 + 8\,\delta\,(\alpha+e\,\delta)\,(\beta+f\,\delta)\,(1+h\,\delta) + 12\,\delta^2\,(1+h\,\delta)^2\big) + 24\,e\,f\,(1+h\,\delta)^2 \\
&\qquad \big((\alpha+e\,\delta)^3\,(\beta+f\,\delta)^3 + 9\,\delta\,(\alpha+e\,\delta)^2\,(\beta+f\,\delta)^2\,(1+h\,\delta) + 18\,\delta^2\,(\alpha+e\,\delta)\,(\beta+f\,\delta)\,(1+h\,\delta)^2 + 6\,\delta^3\,(1+h\,\delta)^3\big) - \\
&\quad 8\,h\,(1+h\,\delta)^2\,\big((\alpha+e\,\delta)^3\,(\beta+f\,\delta)^3 + 9\,\delta\,(\alpha+e\,\delta)^2\,(\beta+f\,\delta)^2\,(1+h\,\delta) + \\
&\qquad 18\,\delta^2\,(\alpha+e\,\delta)\,(\beta+f\,\delta)\,(1+h\,\delta)^2 + 6\,\delta^3\,(1+h\,\delta)^3\big) - 24\,h\,l\,(1+h\,\delta)^2 \\
&\qquad \big((\alpha+e\,\delta)^3\,(\beta+f\,\delta)^3 + 9\,\delta\,(\alpha+e\,\delta)^2\,(\beta+f\,\delta)^2\,(1+h\,\delta) + 18\,\delta^2\,(\alpha+e\,\delta)\,(\beta+f\,\delta)\,(1+h\,\delta)^2 + 6\,\delta^3\,(1+h\,\delta)^3\big) - \\
&\quad 12\,f\,h\,(\alpha+e\,\delta)\,(1+h\,\delta)\,\big((\alpha+e\,\delta)^3\,(\beta+f\,\delta)^3 + 12\,\delta\,(\alpha+e\,\delta)^2\,(\beta+f\,\delta)^2\,(1+h\,\delta) + \\
&\qquad 36\,\delta^2\,(\alpha+e\,\delta)\,(\beta+f\,\delta)\,(1+h\,\delta)^2 + 24\,\delta^3\,(1+h\,\delta)^3\big) - 12\,e\,h\,(\beta+f\,\delta)\,(1+h\,\delta) \\
&\qquad \big((\alpha+e\,\delta)^3\,(\beta+f\,\delta)^3 + 12\,\delta\,(\alpha+e\,\delta)^2\,(\beta+f\,\delta)^2\,(1+h\,\delta) + 36\,\delta^2\,(\alpha+e\,\delta)\,(\beta+f\,\delta)\,(1+h\,\delta)^2 + 24\,\delta^3\,(1+h\,\delta)^3\big) + \\
&\quad 3\,h^2\,\big((\alpha+e\,\delta)^4\,(\beta+f\,\delta)^4 + 16\,\delta\,(\alpha+e\,\delta)^3\,(\beta+f\,\delta)^3\,(1+h\,\delta) + 72\,\delta^2\,(\alpha+e\,\delta)^2\,(\beta+f\,\delta)^2\,(1+h\,\delta)^2 + \\
&\qquad 96\,\delta^3\,(\alpha+e\,\delta)\,(\beta+f\,\delta)\,(1+h\,\delta)^3 + 24\,\delta^4\,(1+h\,\delta)^4\big)\Big)\,\epsilon^2
\end{aligned}$$

```
Λk_[h_, e_, l_, f_, α_, β_, δ_] := Λk[h, e, l, f, α, β, δ] = Module[{λ},
    λ = Normal@Series[ℯ^((f α+e β)/(1-α β ϵ)) (1 - α β ϵ)^(-2 l+h/ϵ), {ϵ, 0, k}] /. ℯ- → 1;
    Collect[DP_{α→D_f, β→D_e}[λ][ℯ^((f α+e β+e f δ)/(1+h δ))] /. ℯ- → 1, ϵ, Simplify]];
```

```
Simplify[DΛ2[h, e, l, f, α, β, δ] == Λ2[h, e, l, f, α, β, δ]]
```

$$\left\{a\$495 \to -\frac{\beta}{-1+\alpha\beta\epsilon},\ b\$495 \to -\frac{\alpha}{-1+\alpha\beta\epsilon},\ c\$495 \to \frac{\text{Log}[1-\alpha\beta\epsilon]}{\epsilon}\right\}$$

```
True
```

Testing the Logos

```
NotebookOpen["C:\\drorbn\\AcademicPensieve\\Classes\\17-1350-AKT\\170317-TestingTheLogos@.nb"]
```

```
NotebookObject[ 170317-TestingTheLogos@.nb ]
```

The Main $\mathfrak{g}_k$ Theorem

Raw Version. The $\mathfrak{g}_k$ invariant of any *S*-component tangle *T* can be written in the form $Z(T) = \mathbb{O}\left(\omega\, e^{L+Q+P} \mid \prod_{i\in S} e_i\, l_i\, f_i\right)$, where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i = e^{h_i}$), where $L = \sum a_{ij}\, h_i\, l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij}, where $Q = \sum b_{ij}\, e_i\, f_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij}, and where P is a polynomial in $\{\epsilon,\ e_i,\ l_i,\ f_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2d+2$ in $\left\{e_i, \sqrt{l_i}\ , f_i\right\}$. Furthermore, after setting $h_i = h$ and $t_i = t$ for all i, the invariant $Z(T)$ is poly-time computable.

Partial Proof. Indeed,

0. $R^\pm = ?$, $n^\pm = ?$.
1. $\mathbb{O}\left(\mathcal{P}(l, e)\, e^{\gamma l+\beta e} \mid l e\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l+e^\gamma \beta e} \mid e l\right)$,
2. $\mathbb{O}\left(\mathcal{P}(l, f)\, e^{\gamma l+\beta f} \mid f l\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l+e^\gamma \beta f} \mid l f\right)$,
3. $\mathbb{O}\left(\mathcal{P}(e, f)\, e^{\beta e+\alpha f+\delta e f} \mid f e\right) = \mathbb{O}\left(\nu \mathcal{P}(\partial_\beta, \partial_\alpha)\, e^{\nu(-\alpha\beta h+\beta e+\alpha f+\delta e f)}\, \Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta) \mid e f\right)$, with $\nu = (1+h\delta)^{-1}$, and $\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)$ as above.

Implementation at $k = 1$

$\mathbb{E}$1n[ω, L, Q, P] stands for $\omega\, e^{L+Q}(1+\epsilon P)$.

```
ϵ /: ϵ^p_ /; p > 1 := 0;
```

```
CF[𝔼1n[ω_, L_, Q_, P_]] := 𝔼1n[Together[ω], Together[L], Together[Q], Together[P]];
𝔼1n /: 𝔼1n[ω1_, L1_, Q1_, P1_] 𝔼1n[ω2_, L2_, Q2_, P2_] := CF@𝔼1n[ω1 ω2, L1 + L2, Q1 + Q2, P1 + P2];
𝔼1n[ω1_, L1_, Q1_, P1_] ≡ 𝔼1n[ω2_, L2_, Q2_, P2_] := Simplify[ω1 == ω2 ∧ L1 == L2 ∧ Q1 == Q2 ∧ P1 == P2];
```

0. $R = \mathbb{O}\left(\exp\left(hl + \frac{e^h-1}{h} ef + P\right) \mid e \otimes lf\right)$:

```
𝔼1n[X⁺_{i_,j_}] := 𝔼1n[1, h_i l_j, h_i^-1 (e^{h_i} - 1) e_i f_j, P⁺];
𝔼1n[X⁻_{i_,j_}] := 𝔼1n[1, -h_i l_j, h_i^-1 (e^{-h_i} - 1) e_i f_j, P⁻];
𝔼1n[p_Times] := 𝔼1n /@ p;
```

```
𝔼1n[X⁺_{4,1} X⁺_{2,5} X⁺_{6,3}]
```

1. $\mathbb{O}\left(\mathcal{P}(l, e)\, e^{\gamma l+\beta e} \mid l e\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l+e^\gamma \beta e} \mid e l\right)$,
2. $\mathbb{O}\left(\mathcal{P}(l, f)\, e^{\gamma l+\beta f} \mid f l\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l+e^\gamma \beta f} \mid l f\right)$.

```
NO_{(x:f|e)_i_ 1_j_}[𝔼1n[ω_, L_, Q_, P_]] := With[{q = e^γ β x_i + γ 1_j},
   CF[𝔼1n[ω, L,
      e^γ β x_i + (Q /. x_i → 0),
      e^-q DP_{1_j→D_γ, x_i→D_β}[P][e^q]
    ] /. {γ → ∂_{1_j} L, β → ∂_{x_i} Q}]];
```

3. $\mathbb{O}\left(e^{\beta e+\alpha f+\delta e f} \mid fe\right) = \mathbb{O}\left(\nu e^{\nu(-\alpha\beta h+\beta e+\alpha f+\delta e f)} \mid ef\right)$, with $\nu = (1+h\delta)^{-1}$:

```
NO_{f_i_ e_j_→k_}[𝔼1n[ω_, L_, Q_, P_]] := With[{q = ν (-α β h_k + β e_k + α f_k + δ e_k f_k)},
   CF[𝔼1n[ν ω, L,
       q + (Q /. f_i | e_j → 0),
       e^-q DP_{f_i→D_α, e_j→D_β}[P][e^q] + (Λ1[h_k, e_k, 1_k, f_k, α, β, δ] - 1 /. ϵ → 1)
     ] /. ν → (1 + h_k δ)^-1 /. {α → ∂_{f_i} Q /. e_j → 0, β → ∂_{e_j} Q /. f_i → 0, δ → ∂_{f_i,e_j} Q}]];
```

```
m_{i_,j_→k_}[Z_] := Module[{x, z}, CF[(Z // NO_{f_i e_j→x} // NO_{1_i e_x} // NO_{f_x 1_j}) /. z_{i|j|x} → z_k]]
```

Meta-associativity

$$\zeta = \mathbb{E}1n\left[\omega, \sum_{i=1}^{4}\sum_{j=1}^{4} a_{i,j}\, h_i\, l_j,\ \sum_{i=1}^{4}\sum_{j=1}^{4} b_{i,j}\, e_i\, f_j,\ 0\right]$$

```
Short[ζ // m_{1,2→1}, 5] // Timing
```

```
Short[lhs = ζ // m_{1,2→1} // m_{1,3→1}, 5] // Timing
```

$$\Big\{144.063,\ \mathbb{E}1n\Big[\omega \Big/ \left(1 + h_1 b_{2,1} + e^{h_1 a_{1,2}+h_1 a_{2,2}+h_1 a_{3,2}+h_4 a_{4,2}} h_1 b_{3,1} - h_1^2 b_{2,2} b_{3,1} + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2}\right),$$
$$h_1 l_1 a_{1,1} + h_1 l_1 a_{1,2} + h_1 l_1 a_{1,3} + \ll 10 \gg + h_4 l_1 a_{4,2} + h_4 l_1 a_{4,3} + h_4 l_4 a_{4,4},\ \frac{\ll 1 \gg}{\ll 1 \gg},$$
$$\left(4\, l_1 b_{2,1} + 4\, e^{h_1 a_{1,2}+\ll 6 \gg + h_4 a_{4,3}} e_1 f_1 b_{1,1} b_{2,1} + 4\, e^{\ll 1 \gg} e_1 f_1 l_1 b_{1,1} b_{2,1} + \ll 11299 \gg + \ll 1 \gg + 2 \ll 7 \gg b_{4,2}^2 -\right.$$
$$\left. e_4^2 f_4^2 h_1^5 b_{2,2}^2 b_{3,1}^2 b_{3,4}^2 b_{4,2}^2\right) \Big/ \left(2\left(1 + h_1 b_{2,1} + e^{\ll 1 \gg} h_1 b_{3,1} - h_1^2 \ll 1 \gg b_{\ll 1 \gg} + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2}\right)^4\right)\Big]\Big\}$$

```
Short[rhs = ζ // m_{2,3→2} // m_{1,2→1}, 5] // Timing
```

$$\Big\{125.719,\ \mathbb{E}1n\Big[\omega \Big/ \left(1 + h_1 b_{2,1} + e^{h_1 a_{1,2}+h_1 a_{2,2}+h_1 a_{3,2}+h_4 a_{4,2}} h_1 b_{3,1} - h_1^2 b_{2,2} b_{3,1} + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2}\right),$$
$$h_1 l_1 a_{1,1} + h_1 l_1 a_{1,2} + h_1 l_1 a_{1,3} + \ll 10 \gg + h_4 l_1 a_{4,2} + h_4 l_1 a_{4,3} + h_4 l_4 a_{4,4},\ \frac{\ll 1 \gg}{\ll 1 \gg},$$
$$\left(4\, l_1 b_{2,1} + 4\, e^{h_1 a_{1,2}+\ll 6 \gg + h_4 a_{4,3}} e_1 f_1 b_{1,1} b_{2,1} + 4\, e^{\ll 1 \gg} e_1 f_1 l_1 b_{1,1} b_{2,1} + \ll 11299 \gg + \ll 1 \gg + 2 \ll 7 \gg b_{4,2}^2 -\right.$$
$$\left. e_4^2 f_4^2 h_1^5 b_{2,2}^2 b_{3,1}^2 b_{3,4}^2 b_{4,2}^2\right) \Big/ \left(2\left(1 + h_1 b_{2,1} + e^{\ll 1 \gg} h_1 b_{3,1} - h_1^2 \ll 1 \gg b_{\ll 1 \gg} + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2}\right)^4\right)\Big]\Big\}$$

```
lhs ≡ rhs
```

True

Profiling

```
<< "C:\\drorbn\\AcademicPensieve\\Projects\\Profile\\Profile.m"

CF[𝔼1n[ω_, L_, Q_, P_]] := PP_CF@𝔼1n[Together[ω], Together[L], Together[Q], Together[P]];
NO_{(x:f|e)_{i_} l_{j_}}[𝔼1n[ω_, L_, Q_, P_]] := PP_NOxl@With[{q = e^γ β x_i + γ l_j},
    CF[𝔼1n[ω, L,
        e^γ β x_i + (Q /. x_i → 0),
        e^-q DP_{l_j→D_γ, x_i→D_β}[P][e^q]
      ] /. {γ → ∂_{l_j} L, β → ∂_{x_i} Q}]];
NO_{f_{i_} e_{j_}→k_}[𝔼1n[ω_, L_, Q_, P_]] := PP_NOfe@With[{q = ν (-α β h_k + β e_k + α f_k + δ e_k f_k)},
    CF[𝔼1n[ν ω, L,
        q + (Q /. f_i | e_j → 0),
        e^-q DP_{f_i→D_α, e_j→D_β}[P][e^q] + (Λ_1[h_k, e_k, l_k, f_k, α, β, δ] - 1 /. ϵ → 1)
      ] /. ν → (1 + h_k δ)^-1 /. {α → ∂_{f_i} Q /. e_j → 0, β → ∂_{e_j} Q /. f_i → 0, δ → ∂_{f_i,e_j} Q}]];
```

```
BeginProfile[];
Short[ξ // m1,2→1 // m1,3→1]
EndProfile[];
PrintProfile[]
```

$\mathbb{E}1n\left[\omega \middle/ \left(1 + h_1\, b_{2,1} + \ll 1 \gg - \ll 1 \gg + h_1\, b_{\ll 1 \gg} + h_1^2\, b_{2,1}\, b_{3,2}\right),\ h_1\, l_1\, a_{1,1} + \ll 14 \gg + h_4\, l_4\, a_{4,4},\ \frac{\ll 1 \gg}{\ll 1 \gg},\ \frac{\ll 1 \gg}{2\left(\ll 1 \gg\right)^4}\right]$

```
CF: called 8 times, time in 133.377/133.377
  Parents:
    ( 2)  47.282/ 47.282 under NOfe
    ( 4)  78.532/ 78.532 under NOxl
    ( 2)   7.563/  7.563 under ProfileRoot
NOxl: called 4 times, time in 9.749/88.281
  Parents:
    ( 4)   9.749/ 88.281 under ProfileRoot
  Children:
    ( 4)  78.532/ 78.532 above CF
NOfe: called 2 times, time in 0.265/47.547
  Parents:
    ( 2)   0.265/ 47.547 under ProfileRoot
  Children:
    ( 2)  47.282/ 47.282 above CF
ProfileRoot: called 0 times, time in 0./0.
  Children:
    ( 2)   7.563/  7.563 above CF
    ( 2)   0.265/ 47.547 above NOfe
    ( 4)   9.749/ 88.281 above NOxl
```