The $\mathfrak{g}_1$ Invariant

Reminder

Make sure that you have Mathematica and that you play with these programs!

Differential polynomials

```
DP_{x_→D_{α_}, y_→D_{β_}}[P_][f_] := Total[CoefficientRules[P, {x, y}] /. ({m_, n_} → c_) :→ c D[f, {α, m}, {β, n}]]
```

```
DP_{x→D_α, y→D_β}[(x + y)^3][e^{α u+β v}] // Simplify
```

$e^{u\alpha+v\beta}(u+v)^3$

The main $\mathfrak{g}_k$ lemma

In $\mathfrak{g}^\epsilon=\langle h, e, l, f\rangle/([e, l]=-e,\ [f, l]=f,\ [e, f]=h-2\epsilon l,\ [h, *]=0)$ and at $\epsilon^{k+1}=0$, we have

1. $\mathbb{O}\left(e^{\gamma l+\beta e} \mid le\right)=\mathbb{O}\left(e^{\gamma l+e^\gamma \beta e} \mid e\, l\right)$,
2. $\mathbb{O}\left(e^{\gamma l+\beta f} \mid f\, l\right)=\mathbb{O}\left(e^{\gamma l+e^\gamma \beta f} \mid l\, f\right)$,
3. $\mathbb{O}\left(e^{\beta e+\alpha f+\delta e f} \mid f\, e\right)=\mathbb{O}\left(\nu e^{\nu(-\alpha\beta h+\beta e+\alpha f+\delta e f)}\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta) \mid e\, f\right)$, with $\nu=(1+h\,\delta)^{-1}$ and where for any fixed k, $\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)$ is a fixed polynomial of degree at most $4\,k$ in e, $\sqrt{l}$, f, α, β, with scalar coefficients.

Comment. Even better, $\log(\Lambda_k)$ is of degree at most $2\,k+2$ in said variables.

Finding the Logos

(Logos=Λόγος, “a principle of order an knowledge”)

```
(* "D" for Detailed *)
DΛ_{k_}[h_, e_, l_, f_, α_, β_, δ_] := Module[
  {ρh, ρe, ρl, ρf, eqn, a, b, c, sol, λ, q, ν},
  ρh = (-1 0; 0 -1); ρe = (0 0; -ϵ 0); ρl = (-(1+1/ϵ)/2 0; 0 (1-1/ϵ)/2); ρf = (0 1; 0 0);
  eqn = MatrixExp[α ρf].MatrixExp[β ρe] == MatrixExp[a ρe].MatrixExp[c (ρh - 2 ϵ ρl)].MatrixExp[b ρf];
  Echo[sol = Solve[Thread[Flatten /@ eqn], {a, b, c}][[1]] /. C[1] → 0];
  λ = Simplify[e^{-f α-e β+h α β} Normal@Series[e^{c h + a e - 2 ϵ c l + b f} /. sol, {ϵ, 0, k}]];
  q = e^{ν (f α+e β-h α β+e f δ)};
  Collect[q^-1 DP_{α→D_f, β→D_e}[λ][q] /. ν → (1 + h δ)^-1, ϵ, Simplify]];
```

```
DΛ_1[h, e, l, f, α, β, δ]
```

» $\left\{a\$329\to-\frac{\beta}{-1+\alpha\beta\epsilon},\ b\$329\to-\frac{\alpha}{-1+\alpha\beta\epsilon},\ c\$329\to\frac{\text{Log}[1-\alpha\beta\epsilon]}{\epsilon}\right\}$

$$1+\frac{1}{2\,(1+h\,\delta)^4}\Big(2\,e\,\alpha\,\beta^2-h\,\alpha^2\,\beta^2+4\,e\,\beta\,\delta-4\,h\,\alpha\,\beta\,\delta+2\,e^2\,\beta^2\,\delta-2\,h\,\delta^2+4\,e\,h\,\beta\,\delta^2-4\,h^2\,\alpha\,\beta\,\delta^2+e^2\,h\,\beta^2\,\delta^2-4\,h^2\,\delta^3-2\,h^3\,\delta^4+4\,l\,(1+h\,\delta)^2\,(\alpha\,(\beta+f\,\delta)+\delta\,(1+e\,\beta+e\,f\,\delta+h\,\delta))+f^2\,\delta\,(\alpha+e\,\delta)\,(\alpha\,(2+h\,\delta)+e\,\delta\,(4+3\,h\,\delta))+2\,f\,(\alpha^2\,\beta+2\,\alpha\,\delta\,(1+h\,\delta+e\,\beta\,(2+h\,\delta))+e\,\delta^2\,(4+6\,h\,\delta+2\,h^2\,\delta^2+e\,\beta\,(3+2\,h\,\delta)))\Big)\,\epsilon$$

```
Λk_[h_, e_, l_, f_, α_, β_, δ_] := Λk[h, e, l, f, α, β, δ] = Module[{λ},
    λ = Normal@Series[e^((f α + e β)/(1 - α β ϵ)) (1 - α β ϵ)^(-2 l + h/ϵ), {ϵ, 0, k}] /. e- → 1;
    Collect[DP_{α→D_f, β→D_e}[λ][e^((f α + e β + e f δ)/(1 + h δ))] /. e- → 1, ϵ, Simplify]];
```

```
Simplify[DΛ2[h, e, l, f, α, β, δ] == Λ2[h, e, l, f, α, β, δ]]
```

» $\left\{a\$382 \to -\frac{\beta}{-1+\alpha\beta\epsilon},\ b\$382 \to -\frac{\alpha}{-1+\alpha\beta\epsilon},\ c\$382 \to \frac{\mathrm{Log}[1-\alpha\beta\epsilon]}{\epsilon}\right\}$

```
True
```

Testing the Logos

```
NotebookOpen["C:\\drorbn\\AcademicPensieve\\Classes\\17-1350-AKT\\170317-TestingTheLogos.nb"]
```

```
NotebookObject[ 170317–TestingTheLogos.nb ]
```

The Main $\mathfrak{g}_k$ Theorem

Raw Version. The $\mathfrak{g}_k$ invariant of any S-component tangle T can be written in the form $Z(T) = \mathbb{O}\left(\omega\, e^{L+Q+P} \mid \prod_{i\in S} e_i\, l_i\, f_i\right)$, where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i = e^{h_i}$), where $L = \sum a_{ij}\, h_i\, l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij}, where $Q = \sum b_{ij}\, e_i\, f_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij}, and where P is a polynomial in $\{\epsilon,\ e_i,\ l_i,\ f_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2d + 2$ in $\left\{e_i,\ \sqrt{l_i}\ ,\ f_i\right\}$. Furthermore, after setting $h_i = h$ and $t_i = t$ for all i, the invariant $Z(T)$ is poly-time computable.

Partial Proof. Indeed,

0. $R^\pm = ?$, $n^\pm = ?$.
1. $\mathbb{O}\left(\mathcal{P}(l, e)\, e^{\gamma l + \beta e} \mid l\, e\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^\gamma \beta e} \mid e\, l\right)$,
2. $\mathbb{O}\left(\mathcal{P}(l, f)\, e^{\gamma l + \beta f} \mid f\, l\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^\gamma \beta f} \mid l\, f\right)$,
3. $\mathbb{O}\left(\mathcal{P}(e, f)\, e^{\beta e + \alpha f + \delta e f} \mid f\, e\right) = \mathbb{O}\left(\nu\, \mathcal{P}(\partial_\beta, \partial_\alpha)\, e^{\nu(-\alpha\beta h + \beta e + \alpha f + \delta e f)}\, \Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta) \mid e\, f\right)$, with $\nu = (1 + \mathrm{h}\delta)^{-1}$, and $\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)$ as above.

Implementation at k = 1

$\mathbb{E}$1n[ω, L, Q, P] stands for $\omega\, e^{L+Q}(1+\epsilon\,\mathrm{P})$.

```
ϵ /: ϵ^p_ /; p > 1 := 0;
```

```
CF[E1n[ω_, L_, Q_, P_]] := E1n[Together[ω], Together[L], Together[Q], Together[P]];
E1n /: E1n[ω1_, L1_, Q1_, P1_] E1n[ω2_, L2_, Q2_, P2_] := CF@E1n[ω1 ω2, L1 + L2, Q1 + Q2, P1 + P2];
E1n[ω1_, L1_, Q1_, P1_] ≡ E1n[ω2_, L2_, Q2_, P2_] := Simplify[ω1 == ω2 ∧ L1 == L2 ∧ Q1 == Q2 ∧ P1 == P2];
```

0. $R = \mathbb{O}\left(\exp\left(\mathrm{hl} + \frac{e^h - 1}{h}\,\mathrm{ef} + P\right) \mid e \otimes \mathrm{lf}\right)$:

```
E1n[X⁺_{i_,j_}] := E1n[1, h_i l_j, h_i^-1 (e^(h_i) - 1) e_i f_j, P⁺];
E1n[X⁻_{i_,j_}] := E1n[1, -h_i l_j, h_i^-1 (e^(-h_i) - 1) e_i f_j, P⁻];
E1n[p_Times] := E1n /@ p;
```

```
E1n[X⁺₄,₁ X⁺₂,₅ X⁺₆,₃]
```

$\mathbb{E}1n\left[1,\ h_4 l_1 + h_6 l_3 + h_2 l_5,\ \frac{1}{h_2 h_4 h_6}\left(-e_6 f_3 h_2 h_4 + e^{h_6} e_6 f_3 h_2 h_4 - e_4 f_1 h_2 h_6 + e^{h_4} e_4 f_1 h_2 h_6 - e_2 f_5 h_4 h_6 + e^{h_2} e_2 f_5 h_4 h_6\right),\ 3 P^+\right]$

1. $\mathbb{O}\left(\mathcal{P}(l, e)\, e^{\gamma l+\beta e} \mid l e\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l+e^{\gamma}\beta e} \mid e l\right)$,
2. $\mathbb{O}\left(\mathcal{P}(l, f)\, e^{\gamma l+\beta f} \mid f l\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l+e^{\gamma}\beta f} \mid l f\right)$.

```
NO(x:f|e)i_ lj_[E1n[ω_, L_, Q_, P_]] := With[{q = e^γ β xi + γ lj},
   CF[E1n[ω, L,
       e^γ β xi + (Q /. xi → 0),
       e^-q DP_{lj→Dγ, xi→Dβ}[P][e^q]
     ] /. {γ → ∂_{lj} L, β → ∂_{xi} Q}]];
```

3. $\mathbb{O}\left(e^{\beta e+\alpha f+\delta ef} \mid fe\right) = \mathbb{O}\left(\nu e^{\nu(-\alpha\beta h+\beta e+\alpha f+\delta ef)} \mid ef\right)$, with $\nu = (1+h\delta)^{-1}$:

```
NOfi_ ej_→k_[E1n[ω_, L_, Q_, P_]] := With[{q = ν (-α β hk + β ek + α fk + δ ek fk)},
   CF[E1n[ν ω, L,
       q + (Q /. fi | ej → 0),
       e^-q DP_{fi→Dα, ej→Dβ}[P][e^q] + (Λ1[hk, ek, lk, fk, α, β, δ] - 1 /. ϵ → 1)
     ] /. ν → (1 + hk δ)^-1 /. {α → ∂_{fi} Q /. ej → 0, β → ∂_{ej} Q /. fi → 0, δ → ∂_{fi,ej} Q}]];
```

```
mi_,j_→k_[Z_] := Module[{x, z}, CF[(Z // NO_{fi ej→x} // NO_{li ex} // NO_{fx lj}) /. z_i|j|x → zk]]
```

Meta-associativity

```
ζ = E1n[ω, Σ_{i=1}^{4} Σ_{j=1}^{4} a_{i,j} hi lj, Σ_{i=1}^{4} Σ_{j=1}^{4} b_{i,j} ei fj, 0]
```

$\mathbb{E}1n[\omega,\ h_1 l_1 a_{1,1} + h_1 l_2 a_{1,2} + h_1 l_3 a_{1,3} + h_1 l_4 a_{1,4} + h_2 l_1 a_{2,1} + h_2 l_2 a_{2,2} + h_2 l_3 a_{2,3} + h_2 l_4 a_{2,4} + h_3 l_1 a_{3,1} + h_3 l_2 a_{3,2} + h_3 l_3 a_{3,3} + h_3 l_4 a_{3,4} + h_4 l_1 a_{4,1} + h_4 l_2 a_{4,2} + h_4 l_3 a_{4,3} + h_4 l_4 a_{4,4},\ e_1 f_1 b_{1,1} + e_1 f_2 b_{1,2} + e_1 f_3 b_{1,3} + e_1 f_4 b_{1,4} + e_2 f_1 b_{2,1} + e_2 f_2 b_{2,2} + e_2 f_3 b_{2,3} + e_2 f_4 b_{2,4} + e_3 f_1 b_{3,1} + e_3 f_2 b_{3,2} + e_3 f_3 b_{3,3} + e_3 f_4 b_{3,4} + e_4 f_1 b_{4,1} + e_4 f_2 b_{4,2} + e_4 f_3 b_{4,3} + e_4 f_4 b_{4,4},\ 0]$

```
Short[ζ // m1,2→1, 5] // Timing
```

$\{1.0625,\ \mathbb{E}1n\left[\frac{\omega}{1 + h_1 b_{2,1}},\ h_1 l_1 a_{1,1} + h_1 l_1 a_{1,2} + h_1 l_3 a_{1,3} + h_1 l_4 a_{1,4} + h_1 l_1 a_{2,1} + h_1 l_1 a_{2,2} + h_1 l_3 a_{2,3} + h_1 l_4 a_{2,4} + h_3 l_1 a_{3,1} + h_3 l_1 a_{3,2} + h_3 l_3 a_{3,3} + h_3 l_4 a_{3,4} + h_4 l_1 a_{4,1} + h_4 l_1 a_{4,2} + h_4 l_3 a_{4,3} + h_4 l_4 a_{4,4},\ \frac{e^{\ll 1\gg} e_1 f_1 b_{1,1} + \ll 41\gg + \ll 1\gg}{1 + h_1 b_{2,1}},\ \left(4 l_1 b_{2,1} + 4 e^{h_1 a_{1,2}+h_1 a_{2,2}+h_3 a_{3,2}+h_4 a_{4,2}} e_1 f_1 b_{1,1} b_{2,1} + 4 e^{h_1 a_{\ll 1\gg} + \ll 2\gg + h_4 \ll 1\gg} e_1 f_1 l_1 b_{1,1} b_{2,1} + \ll 270\gg\right) \Big/ \left(2 \left(1 + h_1 b_{2,1}\right)^4\right)\right]\}$

```
Short[lhs = ζ // m1,2→1 // m1,3→1, 5] // Timing
```

$\{144.063,\ \mathbb{E}1n\left[\omega \Big/ \left(1 + h_1 b_{2,1} + e^{h_1 a_{1,2}+h_1 a_{2,2}+h_1 a_{3,2}+h_4 a_{4,2}} h_1 b_{3,1} - h_1^2 b_{2,2} b_{3,1} + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2}\right),\ h_1 l_1 a_{1,1} + h_1 l_1 a_{1,2} + h_1 l_1 a_{1,3} + \ll 10\gg + h_4 l_1 a_{4,2} + h_4 l_1 a_{4,3} + h_4 l_4 a_{4,4},\ \frac{\ll 1\gg}{\ll 1\gg},\ \left(4 l_1 b_{2,1} + 4 e^{h_1 a_{1,2}+\ll 6\gg+h_4 a_{4,3}} e_1 f_1 b_{1,1} b_{2,1} + 4 e^{\ll 1\gg} e_1 f_1 l_1 b_{1,1} b_{2,1} + \ll 11299\gg + \ll 1\gg + 2 \ll 7\gg b_{4,2}^2 - e_4^2 f_4^2 h_1^5 b_{2,2}^2 b_{3,1}^2 b_{3,4}^2 b_{4,2}^2\right) \Big/ \left(2 \left(1 + h_1 b_{2,1} + e^{\ll 1\gg} h_1 b_{3,1} - h_1^2 \ll 1\gg b_{\ll 1\gg} + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2}\right)^4\right)\right]\}$

```
Short[rhs = ξ // m_{2,3→2} // m_{1,2→1}, 5] // Timing
```

$$\{125.719,\ \mathbb{E}1n[\omega / (1 + h_1 b_{2,1} + e^{h_1 a_{1,2} + h_1 a_{2,2} + h_1 a_{3,2} + h_4 a_{4,2}} h_1 b_{3,1} - h_1^2 b_{2,2} b_{3,1} + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2}),$$
$$h_1 l_1 a_{1,1} + h_1 l_1 a_{1,2} + h_1 l_1 a_{1,3} + \ll 10 \gg + h_4 l_1 a_{4,2} + h_4 l_1 a_{4,3} + h_4 l_4 a_{4,4},\ \frac{\ll 1 \gg}{\ll 1 \gg},$$
$$(4 l_1 b_{2,1} + 4 e^{h_1 a_{1,2} + \ll 6 \gg + h_4 a_{4,3}} e_1 f_1 b_{1,1} b_{2,1} + 4 e^{\ll 1 \gg} e_1 f_1 l_1 b_{1,1} b_{2,1} + \ll 11299 \gg + \ll 1 \gg + 2 \ll 7 \gg b_{4,2}^2 -$$
$$e_4^2 f_4^2 h_1^5 b_{2,2}^2 b_{3,1}^2 b_{3,4}^2 b_{4,2}^2) / (2 (1 + h_1 b_{2,1} + e^{\ll 1 \gg} h_1 b_{3,1} - h_1^2 \ll 1 \gg b_{\ll 1 \gg} + h_1 b_{3,2} + h_1^2 b_{2,1} b_{3,2})^4)]\}$$

```
lhs ≡ rhs
```

True

Profiling

```
<< "C:\\drorbn\\AcademicPensieve\\Projects\\Profile\\Profile.m"
```

This is Profile.m of http://drorbn.net/AcademicPensieve/Projects/Profile/.
This version: March 2017. Original version: July 1994.

```
CF[𝔼1n[ω_, L_, Q_, P_]] := PP_CF@𝔼1n[Together[ω], Together[L], Together[Q], Together[P]];
NO_{(x:f|e)_i_ l_j_}[𝔼1n[ω_, L_, Q_, P_]] := PP_NOxl@With[{q = e^γ β x_i + γ l_j},
    CF[𝔼1n[ω, L,
        e^γ β x_i + (Q /. x_i → 0),
        e^-q DP_{l_j→D_γ, x_i→D_β}[P][e^q]
      ] /. {γ → ∂_{l_j} L, β → ∂_{x_i} Q}]];
NO_{f_i_ e_j_→k_}[𝔼1n[ω_, L_, Q_, P_]] := PP_NOfe@With[{q = ν (-α β h_k + β e_k + α f_k + δ e_k f_k)},
    CF[𝔼1n[ν ω, L,
         q + (Q /. f_i | e_j → 0),
         e^-q DP_{f_i→D_α, e_j→D_β}[P][e^q] + (Λ_1[h_k, e_k, l_k, f_k, α, β, δ] - 1 /. ϵ → 1)
      ] /. ν → (1 + h_k δ)^-1 /. {α → ∂_{f_i} Q /. e_j → 0, β → ∂_{e_j} Q /. f_i → 0, δ → ∂_{f_i,e_j} Q}]];
```

```
BeginProfile[];
Short[ξ // m_{1,2→1} // m_{1,3→1}]
EndProfile[];
PrintProfile[]
```

$$\mathbb{E}1n\left[\frac{\omega}{1 + h_1 b_{2,1} + \ll 1 \gg - \ll 1 \gg + h_1 b_{\ll 1 \gg} + h_1^2 b_{2,1} b_{3,2}},\ h_1 l_1 a_{1,1} + \ll 14 \gg + h_4 l_4 a_{4,4},\ \frac{\ll 1 \gg}{\ll 1 \gg},\ \frac{\ll 1 \gg}{2 (\ll 1 \gg)^4}\right]$$

```
CF: called 8 times, time in 133.377/133.377
  Parents:
    ( 2)  47.282/ 47.282 under NOfe
    ( 4)  78.532/ 78.532 under NOxl
    ( 2)   7.563/  7.563 under ProfileRoot
NOxl: called 4 times, time in 9.749/88.281
  Parents:
    ( 4)   9.749/ 88.281 under ProfileRoot
  Children:
    ( 4)  78.532/ 78.532 above CF
NOfe: called 2 times, time in 0.265/47.547
  Parents:
    ( 2)   0.265/ 47.547 under ProfileRoot
  Children:
    ( 2)  47.282/ 47.282 above CF
ProfileRoot: called 0 times, time in 0./0.
  Children:
    ( 2)   7.563/  7.563 above CF
    ( 2)   0.265/ 47.547 above NOfe
    ( 4)   9.749/ 88.281 above NOxl
```