Testing the Logos

Finding the Logos

(Logos=Λὀγος, “a principle of order an knowledge”)

```
DP_{x_→D_α_, y_→D_β_}[P_][f_] := Total[CoefficientRules[P, {x, y}] /. ({m_, n_} → c_) ⧴ c D[f, {α, m}, {β, n}]]
```

```
Λ_k_[h_, e_, l_, f_, α_, β_, δ_] := Λ_k[h, e, l, f, α, β, δ] = Module[{λ},
   λ = Normal@Series[ⅇ^((f α+e β)/(1-α β ϵ)) (1 - α β ϵ)^(-2 l+h/ϵ), {ϵ, 0, k}] /. ⅇ- → 1;
   Collect[DP_{α→D_f, β→D_e}[λ][ⅇ^((f α+e β+e f δ)/(1+h δ))] /. ⅇ- → 1, ϵ, Simplify]];
```

Implementing $\mathfrak{g}^\epsilon = \langle h, e, l, f\rangle / ([e, l] = -e,\ [f, l] = f,\ [e, f] = h - 2\,\epsilon\, l,\ [h, *] = 0)$

```
PBWRule = {e → 1, l → 2, f → 3};
B[U@e, U@l] = -U@e; B[U@f, U@l] = U@f; B[U@e, U@f] = h U[] - 2 ϵ U[l];
```

```
$TD = 3; ℏ /: ℏ^d_. /; d > $TD := 0;
```

```
x_ ≤ y_ := OrderedQ[{x, y} /. PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. PBWRule];
Simp[ℰ_] := Collect[ℰ, _U, Expand];
```

```
U_i_[ℰ_] := ℰ /. {h → h_i, t → t_i, u_U ⧴ Replace[u, x_ ⧴ x_i, 1]};
B[U[(x_)_i_], U[(y_)_i_]] := B[U[x_i], U[y_i]] = U_i[B[U@x, U@y]];
B[U[(x_)_i_], U[(y_)_j_]] /; i =!= j := 0;
B[x_, x_] = 0;
B[U[y_], U[x_]] := B[U[y], U[x]] = Simp[-B[U[x], U[y]]];
B[x_, y_] := x ** y - y ** x;
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
0 ** _ = _ ** 0 = 0;
x_ ** U[] := x; U[] ** x_ := x;
(a_ * x_U) ** (b_ * y_U) := If[a b === 0, 0, Simp[a b (x ** y)]];
(a_ * x_U) ** y_ := Simp[a (x ** y)]; x_ ** (a_ * y_U) := Simp[a (x ** y)];
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
```

```
U[xx___, x_] ** U[y_, yy___] := If[x ≤ y, U[xx, x, y, yy], U@xx ** (U@y ** U@x + B[U@x, U@y]) ** U@yy];
```

```
UU[l___, x_^n_, r___] := UU[l, Sequence @@ Table[x, {n}], r];
UU[l___, 1, r___] := UU[l, r];
UU[] = U[];
UU[l_, r___] := U[l] ** UU[r];
```

```
𝕆[poly_, specs___] := Module[{vs, us, z},
  vs = Join @@ (First /@ {specs});
  us = Join @@ ({specs} /. (l_ → s_) ⧴ (l /. x_i_ ⧴ x_s));
  Simp@Total[CoefficientRules[Normal@Series[poly, {ħ, 0, $TD}], vs] /. (p_ → c_) ⧴ c UU @@ (us^p)]
 ]
```

Aside

```
r_{i_,j_} := h_i UU[l_j] - ϵ UU[l_i, l_j] + UU[e_i, f_j];
B[r_{1,2}, r_{1,3}] + B[r_{1,2}, r_{2,3}] + B[r_{1,3}, r_{2,3}]
```

```
0
```

The main $\mathfrak{g}_k$ lemma, hardest part only

In $\mathfrak{g}^\epsilon = \langle h, e, l, f\rangle / ([e, l] = -e,\ [f, l] = f,\ [e, f] = h - 2\epsilon l,\ [h, *] = 0)$ and at $\epsilon^{k+1} = 0$, we have

$\mathbb{O}(e^{\beta e + \alpha f + \delta e f} \mid fe) = \mathbb{O}(\nu e^{\nu(-\alpha\beta h + \beta e + \alpha f + \delta e f)} \Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta) \mid ef)$,

with $\nu = (1 + h\delta)^{-1}$ and where $\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)$ is as above.

```
degrule = {α → ħ α, h → ħ h_1, e → ħ e_1, l → l_1, f → f_1, ϵ → ħ ϵ};
```

```
k = 1; $TD = 6; Clear[ϵ]; ϵ /: ϵ^(k+1) = 0;
Simp[𝕆[e^(α f+β e+δ e f) /. degrule, {f_1, e_1} → 1] -
  𝕆[ν Λ_k[h, e, l, f, α, β, δ] e^(ν (-α β h+α f+β e+δ e f)) /. ν → (1 + h δ)^-1 /. degrule, {e_1, l_1, f_1} → 1]]
```

```
0
```

```
k = 2; $TD = 10; Clear[ϵ]; ϵ /: ϵ^(k+1) = 0;
Simp[𝕆[e^(α f+β e+δ e f) /. degrule, {f_1, e_1} → 1] -
  𝕆[ν Λ_k[h, e, l, f, α, β, δ] e^(ν (-α β h+α f+β e+δ e f)) /. ν → (1 + h δ)^-1 /. degrule, {e_1, l_1, f_1} → 1]]
```

```
k = 3; $TD = 12; Clear[ϵ]; ϵ /: ϵ^(k+1) = 0;
Simp[𝕆[e^(α f+β e+δ e f) /. degrule, {f_1, e_1} → 1] -
  𝕆[ν Λ_k[h, e, l, f, α, β, δ] e^(ν (-α β h+α f+β e+δ e f)) /. ν → (1 + h δ)^-1 /. degrule, {e_1, l_1, f_1} → 1]]
```

```
k = 2; $TD = 8; Clear[ϵ]; ϵ /: ϵ^(k+1) = 0;
Print[Short[lhs = Timing[𝕆[e^(α f+β e+δ e f) /. degrule, {f_1, e_1} → 1]]]];
Print[Short[
   rhs = Timing[𝕆[ν Λ_k[h, e, l, f, α, β, δ] e^(ν (-α β h+α f+β e+δ e f)) /. ν → (1 + h δ)^-1 /. degrule, {e_1, l_1, f_1} → 1]]]];
lhs -
 rhs
```

```
{14.6406, (<<99>> + δ^8 ħ^8 h_1^8) U[] + (<<1>>) U[<<1>>] + <<191>> + <<1>>/40320}

{2.39063, (<<99>> + δ^8 ħ^8 h_1^8) U[] + <<192>> + <<1>>/40320}

{12.25, 0}
```