

# Testing the Logos

## Finding the Logos

(Logos= $\Lambda\acute{o}\gamma\omicron\varsigma$ , "a principle of order an knowledge")

```
DPx→Dα, y→Dβ[P-][f-] := Total[CoefficientRules[P, {x, y}] /. ({m-, n-} → c-) ⇒ c D[f, {α, m}, {β, n}]]
```

```
Δk[h-, e-, l-, f-, α-, β-, δ-] := Δk[h, e, l, f, α, β, δ] = Module[{λ},
  λ = Normal@Series[e $\frac{f\alpha+e\beta}{1-\alpha\beta e}$  (1 - αβe)-2L +  $\frac{h}{e}$ , {e, 0, k}] /. e → 1;
  Collect[DPα→Df, β→De[λ][e $(f\alpha+e\beta+ef\delta)/(1+h\delta)$ ] /. e → 1, e, Simplify];
```

Implementing  $g^\epsilon = \langle h, e, l, f \rangle / ([e, l] = -e, [f, l] = f, [e, f] = h - 2\epsilon l, [h, *] = 0)$

```
PBWRule = {e → 1, l → 2, f → 3};
B[U@e, U@l] = -U@e; B[U@f, U@l] = U@f; B[U@e, U@f] = h U[] - 2 ε U[1];
```

```
$TD = 3; h /: hd. /; d > $TD := 0;
```

```
x- ≤ y- := OrderedQ[{x, y} /. PBWRule]; x- < y- := ! OrderedQ[{y, x} /. PBWRule];
Simp[ε-] := Collect[ε, _U, Expand];
```

```
Ui[ε-] := ε /. {h → hi, t → ti, u U ⇒ Replace[u, x- ⇒ xi, 1]};
B[U[(x-)i], U[(y-)i]] := B[U[xi], U[yi]] = Ui[B[U@x, U@y]];
B[U[(x-)i], U[(y-)j]] /; i != j := 0;
B[x-, x-] = 0;
B[U[y-], U[x-]] := B[U[y], U[x]] = Simp[-B[U[x], U[y]]];
B[x-, y-] := x**y - y**x;
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x-] := x;
0**_ = _**0 = 0;
x-**U[] := x; U[]**x- := x;
(a-**x-U)**(b-**y-U) := If[ab === 0, 0, Simp[ab(x**y)]];
(a-**x-U)**y- := Simp[a(x**y)]; x-** (a-**y-U) := Simp[a(x**y)];
(x-Plus)**y- := (#**y) & /@ x; x-** (y-Plus) := (x**#) & /@ y;
```

```
U[xx-, x-] ** U[y-, yy-] := If[x ≤ y, U[xx, x, y, yy], U@xx ** (U@y ** U@x + B[U@x, U@y]) ** U@yy];
```

```
UU[L-, x-n, r-] := UU[L, Sequence@@Table[x, {n}], r];
UU[L-, 1, r-] := UU[L, r];
UU[] = U[];
UU[L-, r-] := U[L] ** UU[r];
```

```

O[poly_, specs___] := Module[{vs, us, z},
  vs = Join@@(First /@ {specs});
  us = Join@@({specs} /. (l_ -> s_) -> (l /. x_{i_} -> x_s));
  Simp@Total[CoefficientRules[Normal@Series[poly, {h, 0, $TD}], vs] /. (p_ -> c_) -> c UU@@(us^p)]
]

```

## Aside

```

r_{i_,j_} := h_i UU[l_j] - e UU[l_i, l_j] + UU[e_i, f_j];
B[r_{1,2}, r_{1,3}] + B[r_{1,2}, r_{2,3}] + B[r_{1,3}, r_{2,3}]
0

```

## The main $g_k$ lemma, hardest part only

In  $g^e = \langle h, e, l, f \rangle / ([e, l] = -e, [f, l] = f, [e, f] = h - 2e, [h, *] = 0)$  and at  $e^{k+1} = 0$ , we have

$$\mathcal{O}(e^{\beta e + \alpha f + \delta e f} \mid fe) = \mathcal{O}(v e^{v(-\alpha\beta h + \beta e + \alpha f + \delta e f)} \Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta) \mid ef),$$

with  $v = (1 + h\delta)^{-1}$  and where  $\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)$  is as above.

```
degrule = {alpha -> h alpha, h -> h h_1, e -> h e_1, l -> l_1, f -> f_1, epsilon -> h epsilon};
```

```

k = 1; $TD = 6; Clear[epsilon]; epsilon /: epsilon^{k+1} = 0;
Simp[O[e^{\alpha f + \beta e + \delta e f} /. degrule, {f_1, e_1} -> 1] -
  O[v \Lambda_k[h, e, l, f, alpha, beta, delta] e^{v(-\alpha\beta h + \alpha f + \beta e + \delta e f)} /. v -> (1 + h delta)^{-1} /. degrule, {e_1, l_1, f_1} -> 1]]
0

```

```

k = 2; $TD = 10; Clear[epsilon]; epsilon /: epsilon^{k+1} = 0;
Simp[O[e^{\alpha f + \beta e + \delta e f} /. degrule, {f_1, e_1} -> 1] -
  O[v \Lambda_k[h, e, l, f, alpha, beta, delta] e^{v(-\alpha\beta h + \alpha f + \beta e + \delta e f)} /. v -> (1 + h delta)^{-1} /. degrule, {e_1, l_1, f_1} -> 1]]
0

```

```

k = 3; $TD = 12; Clear[epsilon]; epsilon /: epsilon^{k+1} = 0;
Simp[O[e^{\alpha f + \beta e + \delta e f} /. degrule, {f_1, e_1} -> 1] -
  O[v \Lambda_k[h, e, l, f, alpha, beta, delta] e^{v(-\alpha\beta h + \alpha f + \beta e + \delta e f)} /. v -> (1 + h delta)^{-1} /. degrule, {e_1, l_1, f_1} -> 1]]
0

```

```

k = 2; $TD = 8; Clear[epsilon]; epsilon /: epsilon^{k+1} = 0;
Print[Short[lhs = Timing[O[e^{\alpha f + \beta e + \delta e f} /. degrule, {f_1, e_1} -> 1]]]];
Print[Short[
  rhs = Timing[O[v \Lambda_k[h, e, l, f, alpha, beta, delta] e^{v(-\alpha\beta h + \alpha f + \beta e + \delta e f)} /. v -> (1 + h delta)^{-1} /. degrule, {e_1, l_1, f_1} -> 1]]]];
lhs -
rhs

```

$$\left\{ 15.75, \left( \langle\langle 99 \rangle\rangle + \delta^8 \hbar^8 h_1^8 \right) U[] + (\beta \hbar + \langle\langle 82 \rangle\rangle) U[e_1] + \langle\langle 190 \rangle\rangle + \langle\langle 1 \rangle\rangle + \frac{\delta^8 \langle\langle 1 \rangle\rangle U[e_1, e_1, e_1, \langle\langle 11 \rangle\rangle, f_1, f_1]}{40320} \right\}$$

$$\left\{ 1.40625, \left( \langle\langle 99 \rangle\rangle + \delta^8 \hbar^8 h_1^8 \right) U[] + \langle\langle 192 \rangle\rangle + \frac{\delta^8 \hbar^8 U[e_1, e_1, \langle\langle 12 \rangle\rangle, f_1, f_1]}{40320} \right\}$$

```
{14.3438, 0}
```