Pensieve header: The main lemma for ${\mathfrak g}_1$ using gl_2 (aborted re-implementation of the UEA section).

Representing $\mathfrak{g}^\epsilon = \langle h, e, l, f\rangle / ([e, l] = -e,\ [f, l] = f,\ [e, f] = h - 2\epsilon l,\ [h, *] = 0)$

```
ρh = (-1 0; 0 -1); ρe = (0 0; -ϵ 0); ρl = (-(1+1/ϵ)/2 0; 0 (1-1/ϵ)/2); ρf = (0 1; 0 0);
MB[x_?MatrixQ, y_?MatrixQ] := x.y - y.x;
Simplify@{MB[ρe, ρl] == -ρe, MB[ρf, ρl] == ρf, MB[ρe, ρf] == ρh - 2 ϵ ρl}
```

{True, True, True}

Some $\mathfrak{g}^\epsilon$ lemmas

1. $\mathbb{O}(e^{\gamma l+\beta e} \mid l e) = \mathbb{O}(e^{\gamma l + e^\gamma \beta e} \mid e l)$.
2. $\mathbb{O}(e^{\gamma l+\beta f} \mid f l) = \mathbb{O}(e^{\gamma l + e^\gamma \beta f} \mid l f)$.

Proofs.

```
MatrixForm /@ {MatrixExp[γ ρl].MatrixExp[β ρe], MatrixExp[e^γ β ρe].MatrixExp[γ ρl]}
```

$$\left\{\begin{pmatrix} e^{-\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} & 0 \\ -e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}}\beta\epsilon & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} \end{pmatrix}, \begin{pmatrix} e^{-\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} & 0 \\ -e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}}\beta\epsilon & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} \end{pmatrix}\right\}$$

```
MatrixForm /@ {MatrixExp[β ρf].MatrixExp[γ ρl], MatrixExp[γ ρl].MatrixExp[e^γ β ρf]}
```

$$\left\{\begin{pmatrix} e^{-\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}}\beta \\ 0 & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} \end{pmatrix}, \begin{pmatrix} e^{-\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}}\beta \\ 0 & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} \end{pmatrix}\right\}$$

Deriving “lemma 3 at $\delta = 0$”:

```
MatrixExp[α ρf].MatrixExp[β ρe] // Simplify // MatrixForm
```

$$\begin{pmatrix} 1-\alpha\beta\epsilon & \alpha \\ -\beta\epsilon & 1 \end{pmatrix}$$

```
eqn = MatrixExp[α ρf].MatrixExp[β ρe] == MatrixExp[a ρe].MatrixExp[c (ρh - 2 ϵ ρl)].MatrixExp[b ρf]
```

$\{\{1-\alpha\beta\epsilon, \alpha\}, \{-\beta\epsilon, 1\}\} == \{\{e^{c\epsilon}, b\, e^{c\epsilon}\}, \{-a\, e^{c\epsilon}\epsilon, e^{-c\epsilon} - a\, b\, e^{c\epsilon}\epsilon\}\}$

```
{sol} = Solve[Thread[Flatten /@ eqn], {a, b, c}]
```

$\left\{\left\{a \to -\frac{\beta}{-1+\alpha\beta\epsilon}, b \to -\frac{\alpha}{-1+\alpha\beta\epsilon}, c \to \text{ConditionalExpression}\left[\frac{2 i \pi C[1] + \text{Log}[1-\alpha\beta\epsilon]}{\epsilon}, C[1] \in \text{Integers}\right]\right\}\right\}$

```
sol = sol /. C[1] → 0
```

$\left\{a \to -\frac{\beta}{-1+\alpha\beta\epsilon}, b \to -\frac{\alpha}{-1+\alpha\beta\epsilon}, c \to \frac{\text{Log}[1-\alpha\beta\epsilon]}{\epsilon}\right\}$

And so,

3. $\mathbb{O}(e^{\alpha f+\beta e} \mid f e) = \mathbb{O}(e^{c h + a e - 2\epsilon c l + b f} \mid e l f)$, with $\left\{a \to -\frac{\beta}{-1+\alpha\beta\epsilon}, b \to -\frac{\alpha}{-1+\alpha\beta\epsilon}, c \to \frac{\text{Log}[1-\alpha\beta\epsilon]}{\epsilon}\right\}$.

At $\epsilon = 0$ get

```
Limit[{a, b, c} /. sol, ϵ → 0]
```

$\{\beta, \alpha, -\alpha\beta\}$

The full $\mathfrak{g}_1$ lemma

At $\epsilon^2 = 0$ get

```
Series[{a, b, c} /. sol, {ϵ, 0, 1}] // Normal
```

$\{\beta + \alpha\beta^2 \epsilon, \alpha + \alpha^2\beta\epsilon, -\alpha\beta - \frac{1}{2}\alpha^2\beta^2\epsilon\}$

```
Collect[Series[e^(c h + a e - 2 ϵ c l + b f) /. sol, {ϵ, 0, 1}] // Normal, e-, Expand]
```

$e^{f\alpha + e\beta - h\alpha\beta}\left(1 + 2 l \alpha\beta\epsilon + f\alpha^2\beta\epsilon + e\alpha\beta^2\epsilon - \frac{1}{2} h\alpha^2\beta^2\epsilon\right)$

Which means $\mathbb{O}(e^{\alpha f + \beta e} \mid f e) = \mathbb{O}(e^{-\alpha\beta h + \beta e + \alpha f}(1 + \epsilon(2 l\alpha\beta + f\alpha^2\beta + e\alpha\beta^2 - \frac{1}{2} h\alpha^2\beta^2)) \mid e l f)$.

And so,

3. $\mathbb{O}(e^{\alpha f + \beta e + \delta e f} \mid f e) = \mathbb{O}(\nu(1 + \epsilon\Lambda) e^{\nu(-h\alpha\beta + \alpha f + \beta e + \delta e f)} \mid e l f)$, with $\nu = (1 + h\delta)^{-1}$ and Λ, the "logos", as below.

```
Λ = With[{q = e^(ν (f α+e β-h α β+e f δ))}, Simplify[q^-1 (2 l ∂_{f,e} q + f ∂_{f,f,e} q + e ∂_{f,e,e} q - 1/2 h ∂_{f,f,e,e} q)]]
```

$-\frac{1}{2}\nu\left(-4 l\left(\delta + \alpha\beta\nu + f\alpha\delta\nu + e\beta\delta\nu + e f\delta^2\nu\right) + \nu\left(e^2\beta^2\delta\nu\left(-2 + h\delta\nu\right) + 2 e\beta\left(2\delta + \alpha\beta\nu\right)\left(-1 + h\delta\nu\right) + h\left(2\delta^2 + 4\alpha\beta\delta\nu + \alpha^2\beta^2\nu^2\right) + f^2\delta\left(\alpha + e\delta\right)\nu\left(e\delta\left(-4 + h\delta\nu\right) + \alpha\left(-2 + h\delta\nu\right)\right) + 2 f\left(\alpha^2\beta\nu\left(-1 + h\delta\nu\right) + e\delta^2\left(-4 + 2 h\delta\nu + e\beta\nu\left(-3 + h\delta\nu\right)\right) + 2\alpha\delta\left(-1 + h\delta\nu + e\beta\nu\left(-2 + h\delta\nu\right)\right)\right)\right)\right)$

(The below is private; don't look)

```
oldΛ = Simplify[
  1/(2 μ^4) (-b α^2 β^2 + u^2 β^2 δ (2 + b δ) + w^2 δ (α + u δ) (α (2 + b δ) + u δ (4 + 3 b δ)) - 4 b α β δ μ + 4 c α β μ^2 - 2 b δ^2 μ^2 +
     4 c δ μ^3 + 2 u β (α β + 2 δ μ (1 + c μ)) + 2 w (α^2 β + 2 α δ (u β (2 + b δ) + μ (1 + c μ)) +
        u δ^2 (u β (3 + 2 b δ) + 2 μ (2 + b δ + c μ)))) /. {μ → ν^-1, b → h, c → l, u → e, w → f}]
```

$\frac{1}{2}\nu\left(4 l\left(\delta + \alpha\beta\nu + f\alpha\delta\nu + e\beta\delta\nu + e f\delta^2\nu\right) + \nu\left(2\left(e\beta + f\left(\alpha + 2 e\delta\right)\right)\nu\left(\alpha\beta\nu + e f\delta^2\nu + \delta\left(2 + f\alpha\nu + e\beta\nu\right)\right) + h\left(-4\alpha\beta\delta\nu - \alpha^2\beta^2\nu^2 + 3 e^2 f^2\delta^4\nu^2 + 4 e f\delta^3\nu\left(1 + f\alpha\nu + e\beta\nu\right) + \delta^2\left(-2 + f^2\alpha^2\nu^2 + 4 e f\alpha\beta\nu^2 + e^2\beta^2\nu^2\right)\right)\right)\right)$

```
Simplify[Λ == oldΛ /. ν → (1 + h δ)^-1]
```

True

Implementing general universal enveloping algebras

```
B[0, _] = 0; B[_, 0] = 0;
B[c_ * x_, y_] /; MemberQ[$Basis, x] := Expand[c B[x, y]];
B[y_, c_ * x_] /; MemberQ[$Basis, x] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```
x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[ℰ_] := ℰ /. x_ /; MemberQ[$Basis, x] :→ U_i[x];
USimp[ℰ_] := Collect[ℰ, Times[U_[__] ..], Expand];
USimp[ℰ_] := Expand[ℰ];
```

```
m_s__[0] = 0; m_s__[x_Plus] := m_s /@ x;
m_i_→j_[ℰ_] := ℰ /. U_i → U_j;
```

```
m_{i_,j_→k_}[c_. U_{i_}[x___] U_{j_}[]] := c U_k[x];
m_{i_,j_→k_}[c_. U_{i_}[] U_{j_}[y___]] := c U_k[y];
m_{i_,j_→k_}[c_. U_{i_}[xx___, x_] U_{j_}[y_, yy___]] := If[x ≤ y,
   c U_k[xx, x, y, yy],
   ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_{i,j→i}) U_j[yy] // Expand // m_{i,j→k}) c // USimp
  ];
```

```
UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is___}, d_Integer] :=
  Sort@Flatten@Table[(U_i @@@ Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is}, d - j]}];
```

```
S[ℰ_] := Union@Cases[{ℰ}, U_{i_}[___] ⧴ i, ∞];
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[{is = S[x] ⋂ S[y], σ, z},
   z = x; Do[z = m_{i→σ@i}[z], {i, is}];
   z = Expand[y z]; Do[z = m_{σ@i,i→i}[z], {i, is}]; z];
UB[x_, y_] := USimp[x ** y - y ** x];
```

Implementing $\mathfrak{g}^\epsilon = \langle h, e, l, f\rangle / ([e, l] = -e,\ [f, l] = f,\ [e, f] = h - 2\epsilon l,\ [h, *] = 0)$

```
B[e, l] = -e; B[f, l] = f; B[e, f] = h - 2 ∈ l;  B[h, _] = 0;
$Basis = {h, e, l, f};
$PBWRule = {h → 1, e → 2, l → 3, f → 4};
```

```
Table[{x, y} → B[x, y], {x, $Basis}, {y, $Basis}] // MatrixForm
```

{h, h} → 0	{h, e} → 0	{h, l} → 0	{h, f} → 0
{e, h} → 0	{e, e} → 0	{e, l} → -e	{e, f} → h - 2 l ∈
{l, h} → 0	{l, e} → e	{l, l} → 0	{l, f} → -f
{f, h} → 0	{f, e} → -h + 2 l ∈	{f, l} → f	{f, f} → 0

```
Module[{x, y}, Union@Table[{x, y} = t; B[x, y] + B[y, x], {t, Tuples[$Basis, 2]}]]
```

{0}

```
Module[{x, y, z}, DeleteCases[Table[
   ({x, y, z} = t) → B[x, B[y, z]] + B[y, B[z, x]] + B[z, B[x, y]],
   {t, Tuples[$Basis, 3]}
  ], _ → 0]]
```

{}

```
Union[(u ↦ m_{1,3→1}[m_{1,2→1}[u]] - m_{1,2→1}[m_{2,3→2}[u]]) /@ UProducts[{1, 2, 3}, 3]]
```

{0}

```
r_{i_,j_} := USimp[U_i[e] U_j[f] + (U_i[h] - ∈ U_i[l]) U_j[l]];
r_{1,2}
```

$U_1[e]\, U_2[f] + U_1[h]\, U_2[l] - \epsilon\, U_1[l]\, U_2[l]$

```
UB[r_{1,2}, r_{1,3}] + UB[r_{1,2}, r_{2,3}] + UB[r_{1,3}, r_{2,3}]
```

0

```
$TD = 3;
©[poly_, specs_] := Module[{rules, vars, gens},
  rules = Cases[specs, _Rule, ∞];
  vars = First /@ rules;
  gens = Last /@ rules;
  (* Wrong/unfinished:
    USimp@Total[CoefficientRules[Normal@Series[poly, {ℏ,0,$TD}], vars] /. (p_→c_) :→ c UU@@(us^p)]*)
 ]
```

Debt. How does this work?

```
CoefficientRules[(x + y)^3, {x, y}]
```

```
{{3, 0} → 1, {2, 1} → 3, {1, 2} → 3, {0, 3} → 1}
```