Pensieve header: The main lemma for ${\mathfrak g}_1$ using gl_2.

Representing $\mathfrak{g}^\epsilon = \langle h, e, l, f\rangle / ([e, l] = -e,\ [f, l] = f,\ [e, f] = h - 2\epsilon l,\ [h, *] = 0)$

```
ρh = ( -1  0 ; 0  -1 ); ρe = ( 0  0 ; -ϵ  0 ); ρl = ( -(1 + 1/ϵ)/2  0 ; 0  (1 - 1/ϵ)/2 ); ρf = ( 0  1 ; 0  0 );
MB[x_?MatrixQ, y_?MatrixQ] := x.y - y.x;
Simplify@{MB[ρe, ρl] == -ρe, MB[ρf, ρl] == ρf, MB[ρe, ρf] == ρh - 2 ϵ ρl}
```

{True, True, True}

Some $\mathfrak{g}^\epsilon$ lemmas

1. $\mathbb{O}\left(e^{\gamma l+\beta e} \mid l e\right) = \mathbb{O}\left(e^{\gamma l+e^\gamma \beta e} \mid e l\right)$.
2. $\mathbb{O}\left(e^{\gamma l+\beta f} \mid f l\right) = \mathbb{O}\left(e^{\gamma l+e^\gamma \beta f} \mid l f\right)$.

Proofs.

```
MatrixForm /@ {MatrixExp[γ ρl].MatrixExp[β ρe], MatrixExp[e^γ β ρe].MatrixExp[γ ρl]}
```

$$\left\{\begin{pmatrix} e^{-\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} & 0 \\ -e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}}\beta\epsilon & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} \end{pmatrix}, \begin{pmatrix} e^{-\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} & 0 \\ -e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}}\beta\epsilon & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} \end{pmatrix}\right\}$$

```
MatrixForm /@ {MatrixExp[β ρf].MatrixExp[γ ρl], MatrixExp[γ ρl].MatrixExp[e^γ β ρf]}
```

$$\left\{\begin{pmatrix} e^{-\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}}\beta \\ 0 & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} \end{pmatrix}, \begin{pmatrix} e^{-\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}}\beta \\ 0 & e^{\frac{\gamma}{2}-\frac{\gamma}{2\epsilon}} \end{pmatrix}\right\}$$

Deriving "lemma 3 at $\delta = 0$":

```
MatrixExp[α ρf].MatrixExp[β ρe] // Simplify // MatrixForm
```

$$\begin{pmatrix} 1-\alpha\beta\epsilon & \alpha \\ -\beta\epsilon & 1 \end{pmatrix}$$

```
eqn = MatrixExp[α ρf].MatrixExp[β ρe] == MatrixExp[a ρe].MatrixExp[c (ρh - 2 ϵ ρl)].MatrixExp[b ρf]
```

$\{\{1-\alpha\beta\epsilon, \alpha\}, \{-\beta\epsilon, 1\}\} == \{\{e^{c\epsilon}, b\, e^{c\epsilon}\}, \{-a\, e^{c\epsilon}\epsilon, e^{-c\epsilon} - a\, b\, e^{c\epsilon}\epsilon\}\}$

```
sol = Solve[Thread[Flatten /@ eqn], {a, b, c}][[1]]
```

$\left\{a \to -\frac{\beta}{-1+\alpha\beta\epsilon}, b \to -\frac{\alpha}{-1+\alpha\beta\epsilon}, c \to \text{ConditionalExpression}\left[\frac{2 i \pi C[1] + \text{Log}[1-\alpha\beta\epsilon]}{\epsilon}, C[1] \in \text{Integers}\right]\right\}$

```
sol = sol /. C[1] → 0
```

$\left\{a \to -\frac{\beta}{-1+\alpha\beta\epsilon}, b \to -\frac{\alpha}{-1+\alpha\beta\epsilon}, c \to \frac{\text{Log}[1-\alpha\beta\epsilon]}{\epsilon}\right\}$

And so,

3. $\mathbb{O}\left(e^{\alpha f+\beta e} \mid f e\right) = \mathbb{O}\left(e^{c h+a e-2\epsilon c l+b f} \mid e l f\right)$, with $\left\{a \to -\frac{\beta}{-1+\alpha\beta\epsilon}, b \to -\frac{\alpha}{-1+\alpha\beta\epsilon}, c \to \frac{\text{Log}[1-\alpha\beta\epsilon]}{\epsilon}\right\}$.

At $\epsilon = 0$ get

```
Limit[{a, b, c} /. sol, ϵ → 0]
```

$\{\beta, \alpha, -\alpha\beta\}$

The full $\mathfrak{g}_1$ lemma

At $\epsilon^2 = 0$ get

```
Series[{a, b, c} /. sol, {ϵ, 0, 1}] // Normal
```

$\{\beta + \alpha\beta^2 \epsilon,\ \alpha + \alpha^2\beta\epsilon,\ -\alpha\beta - \frac{1}{2}\alpha^2\beta^2\epsilon\}$

```
Collect[Series[e^(c h + a e - 2 ϵ c l + b f) /. sol, {ϵ, 0, 1}] // Normal, e-, Expand]
```

$e^{f\alpha + e\beta - h\alpha\beta}\left(1 + 2 l \alpha\beta\epsilon + f\alpha^2\beta\epsilon + e\alpha\beta^2\epsilon - \frac{1}{2} h \alpha^2\beta^2\epsilon\right)$

Which means $\mathbb{O}(e^{\alpha f + \beta e} \mid f e) = \mathbb{O}(e^{-\alpha\beta h + \beta e + \alpha f}(1 + \epsilon(2 l \alpha\beta + f\alpha^2\beta + e\alpha\beta^2 - \frac{1}{2} h \alpha^2\beta^2)) \mid e l f)$.

And so,

3. $\mathbb{O}(e^{\alpha f + \beta e + \delta e f} \mid f e) = \mathbb{O}(\nu(1 + \epsilon\Lambda) e^{\nu(-h\alpha\beta + \alpha f + \beta e + \delta e f)} \mid e l f)$, with $\nu = (1 + h\delta)^{-1}$ and Λ, the "logos", as below.

```
Λ = With[{q = e^(ν (f α+e β-h α β+e f δ))}, Simplify[q^-1 (2 l ∂_{f,e} q + f ∂_{f,f,e} q + e ∂_{f,e,e} q - 1/2 h ∂_{f,f,e,e} q)]]
```

$-\frac{1}{2}\nu\left(-4 l\left(\delta + \alpha\beta\nu + f\alpha\delta\nu + e\beta\delta\nu + e f\delta^2\nu\right) + \nu\left(e^2\beta^2\delta\nu\left(-2 + h\delta\nu\right) + 2 e\beta\left(2\delta + \alpha\beta\nu\right)\left(-1 + h\delta\nu\right) + h\left(2\delta^2 + 4\alpha\beta\delta\nu + \alpha^2\beta^2\nu^2\right) + f^2\delta\left(\alpha + e\delta\right)\nu\left(e\delta\left(-4 + h\delta\nu\right) + \alpha\left(-2 + h\delta\nu\right)\right) + 2 f\left(\alpha^2\beta\nu\left(-1 + h\delta\nu\right) + e\delta^2\left(-4 + 2 h\delta\nu + e\beta\nu\left(-3 + h\delta\nu\right)\right) + 2\alpha\delta\left(-1 + h\delta\nu + e\beta\nu\left(-2 + h\delta\nu\right)\right)\right)\right)\right)$

(The below is private; don't look)

```
oldΛ = Simplify[
  1/(2 μ^4) (-b α^2 β^2 + u^2 β^2 δ (2 + b δ) + w^2 δ (α + u δ) (α (2 + b δ) + u δ (4 + 3 b δ)) - 4 b α β δ μ + 4 c α β μ^2 - 2 b δ^2 μ^2 +
     4 c δ μ^3 + 2 u β (α β + 2 δ μ (1 + c μ)) + 2 w (α^2 β + 2 α δ (u β (2 + b δ) + μ (1 + c μ)) +
        u δ^2 (u β (3 + 2 b δ) + 2 μ (2 + b δ + c μ)))) /. {μ → ν^-1, b → h, c → l, u → e, w → f}]
```

$\frac{1}{2}\nu\left(4 l\left(\delta + \alpha\beta\nu + f\alpha\delta\nu + e\beta\delta\nu + e f\delta^2\nu\right) + \nu\left(2\left(e\beta + f\left(\alpha + 2 e\delta\right)\right)\nu\left(\alpha\beta\nu + e f\delta^2\nu + \delta\left(2 + f\alpha\nu + e\beta\nu\right)\right) + h\left(-4\alpha\beta\delta\nu - \alpha^2\beta^2\nu^2 + 3 e^2 f^2\delta^4\nu^2 + 4 e f\delta^3\nu\left(1 + f\alpha\nu + e\beta\nu\right) + \delta^2\left(-2 + f^2\alpha^2\nu^2 + 4 e f\alpha\beta\nu^2 + e^2\beta^2\nu^2\right)\right)\right)\right)$

```
Simplify[Λ == oldΛ /. ν → (1 + h δ)^-1]
```

True

Implementing $\mathfrak{g}_1$

```
PBWRule = {e → 1, l → 2, f → 3};
B[U@e, U@l] = -U@e; B[U@f, U@l] = U@f; B[U@e, U@f] = h U[] -2 ϵ U[l];
ϵ /: ϵ^2 = 0;
```

```
$TD = 3; ħ /: ħ^d_. /; d > $TD := 0;
```

```
x_ ≤ y_ := OrderedQ[{x, y} /. PBWRule]; x_ ≺ y_ := !OrderedQ[{y, x} /. PBWRule];
Simp[ℰ_] := Collect[ℰ, _U, Expand];
```

```
U_i_[ℰ_] := ℰ /. {h → h_i, t → t_i, u_U :→ Replace[u, x_ :→ x_i, 1]};
B[U[(x_)_i_], U[(y_)_i_]] := B[U[x_i], U[y_i]] = U_i[B[U@x, U@y]];
B[U[(x_)_i_], U[(y_)_j_]] /; i =!= j := 0;
B[x_, x_] = 0;
B[U[y_], U[x_]] := B[U[y], U[x]] = Simp[-B[U[x], U[y]]];
B[x_, y_] := x ** y - y ** x;
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
0 ** _ = _ ** 0 = 0;
x_ ** U[] := x; U[] ** x_ := x;
(a_ * x_U) ** (b_ * y_U) := If[a b === 0, 0, Simp[a b (x ** y)]];
(a_ * x_U) ** y_ := Simp[a (x ** y)]; x_ ** (a_ * y_U) := Simp[a (x ** y)];
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
```

```
U[xx___, x_] ** U[y_, yy___] := If[x ≤ y, U[xx, x, y, yy], U@xx ** (U@y ** U@x + B[U@x, U@y]) ** U@yy];
```

```
UU[l___, x_^n_, r___] := UU[l, Sequence @@ Table[x, {n}], r];
UU[l___, 1, r___] := UU[l, r];
UU[] = U[];
UU[l_, r___] := U[l] ** UU[r];
```

```
UProducts[{}, 0] = {UU[]};
UProducts[{}, n_Integer] /; n > 0 = {};
UProducts[{x_, xs___}, n_Integer] :=
  Sort@Flatten@Table[UU[x^k] ** u, {k, 0, n}, {u, UProducts[{xs}, n - k]}];
UProducts[xs_List, k_Integer, n_Integer] := UProducts[Flatten@Table[x_j, {x, xs}, {j, k}], n];
UProducts[any__, {n_}] := Flatten@Table[UProducts[any, k], {k, 0, n}];
```

```
m[i_, j_, k_][ℰ_] := Simp[ℰ /. {
    u_U :> UU @@ Join[DeleteCases[u, x_{i|j}], U @@ Cases[u, x_i :> x_k], U @@ Cases[u, x_j :> x_k]],
    h_{i|j} → h_k}]
```

```
𝕆[poly_, specs___] := Module[{vs, us, z},
  vs = Join @@ (First /@ {specs});
  us = Join @@ ({specs} /. (l_ → s_) :> (l /. x_i_ :> x_s));
  Simp@Total[CoefficientRules[Normal@Series[poly, {ℏ, 0, $TD}], vs] /. (p_ → c_) :> c UU @@ (us^p)]
 ]
```

Debt. How does this work?

Verifying the full $\mathfrak{g}_1$ lemma

3. $\mathbb{O}\left(e^{\alpha f+\beta e+\delta e f} \mid f e\right)=\mathbb{O}\left(\nu(1+\epsilon\Lambda)\, e^{\nu(-h\alpha\beta+\alpha f+\beta e+\delta e f)} \mid e\, l\, f\right)$, with $\nu=(1+h\delta)^{-1}$ and Λ, the "logos", as above.

```
$TD = 2; 𝕆[e^(ℏ (α f1 + β e1 + δ e1 f1)), {f1, e1} → 1]
```

$$\left(1-\delta\hbar h_1-\alpha\beta\hbar^2 h_1-\delta^2\epsilon\hbar^2 h_1+\delta^2\hbar^2 h_1^2\right)\mathrm{U}[\,]+\left(\beta\hbar+2\beta\delta\epsilon\hbar^2-2\beta\delta\hbar^2 h_1\right)\mathrm{U}[e_1]+$$
$$\left(\alpha\hbar+2\alpha\delta\epsilon\hbar^2-2\alpha\delta\hbar^2 h_1\right)\mathrm{U}[f_1]+\left(2\delta\epsilon\hbar+2\alpha\beta\epsilon\hbar^2-4\delta^2\epsilon\hbar^2 h_1\right)\mathrm{U}[l_1]+\frac{1}{2}\beta^2\hbar^2\,\mathrm{U}[e_1,e_1]+$$
$$\left(\delta\hbar+\alpha\beta\hbar^2+4\delta^2\epsilon\hbar^2-2\delta^2\hbar^2 h_1\right)\mathrm{U}[e_1,f_1]+4\beta\delta\epsilon\hbar^2\,\mathrm{U}[e_1,l_1]+\frac{1}{2}\alpha^2\hbar^2\,\mathrm{U}[f_1,f_1]+4\alpha\delta\epsilon\hbar^2\,\mathrm{U}[l_1,f_1]+$$
$$\beta\delta\hbar^2\,\mathrm{U}[e_1,e_1,f_1]+\alpha\delta\hbar^2\,\mathrm{U}[e_1,f_1,f_1]+4\delta^2\epsilon\hbar^2\,\mathrm{U}[e_1,l_1,f_1]+\frac{1}{2}\delta^2\hbar^2\,\mathrm{U}[e_1,e_1,f_1,f_1]$$

```
Λ1 = Λ /. {α → ℏ α, h → ℏ h1, e → ℏ e1, l → l1, f → f1}
```

$$-\frac{1}{2}\nu\left(\nu\left(\hbar\left(2\delta^2+4\alpha\beta\delta\nu\hbar+\alpha^2\beta^2\nu^2\hbar^2\right)h_1+\beta^2\delta\nu\hbar^2 e_1^2\left(-2+\delta\nu\hbar h_1\right)+\right.\right.$$
$$2\beta\hbar\left(2\delta+\alpha\beta\nu\hbar\right)e_1\left(-1+\delta\nu\hbar h_1\right)+\delta\nu\left(\alpha\hbar+\delta\hbar e_1\right)f_1^2\left(\delta\hbar e_1\left(-4+\delta\nu\hbar h_1\right)+\alpha\hbar\left(-2+\delta\nu\hbar h_1\right)\right)+$$
$$2f_1\left(\alpha^2\beta\nu\hbar^2\left(-1+\delta\nu\hbar h_1\right)+\delta^2\hbar e_1\left(-4+2\delta\nu\hbar h_1+\beta\nu\hbar e_1\left(-3+\delta\nu\hbar h_1\right)\right)+\right.$$
$$\left.\left.\left.2\alpha\delta\hbar\left(-1+\delta\nu\hbar h_1+\beta\nu\hbar e_1\left(-2+\delta\nu\hbar h_1\right)\right)\right)\right)-4\left(\delta+\alpha\beta\nu\hbar+\beta\delta\nu\hbar e_1+\alpha\delta\nu\hbar f_1+\delta^2\nu\hbar e_1 f_1\right)l_1\right)$$

```
$TD = 7; Simp[
  𝕆[𝕖^(ℏ (α f₁+β e₁+δ e₁ f₁)), {f₁, e₁} → 1] - 𝕆[ν (1 + ϵ ℏ Λ1) 𝕖^(ℏ ν (-α β ℏ h₁+α f₁+β e₁+δ e₁ f₁)) /. ν → (1 + ℏ h₁ δ)^-1, {e₁, l₁, f₁} → 1]
]
```

0