

Pensieve header: The polished \mathfrak{g}_0 invariant.

Reminder

Make sure that you have Mathematica and that you play with these programs!

The Main \mathfrak{g}_0 Theorem

Raw Version. The \mathfrak{g}_0 invariant of any S-component tangle T can be written in the form $Z(T) = \mathcal{O}(\omega e^{L+Q} \mid \prod_{i \in S} e_i l_i f_i)$, where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i = e^{h_i}$), where $L = \sum a_{ij} h_i l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij} and where $Q = \sum b_{ij} e_i f_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij} . Furthermore, after setting $h_i = h$ and $t_i = t$ for all i , the invariant $Z(T)$ is poly-time computable.

Proof. Indeed, as shown below,

$$0. R^s = e^{s(h \otimes l + e \otimes f)} = \mathcal{O}(\exp(s h l + \frac{e^{s h} - 1}{h} e f \mid e \otimes l f),$$

$$1. \mathcal{O}(e^{\gamma l + \beta e} \mid l e) = \mathcal{O}(e^{\gamma l + e^\gamma \beta e} \mid e l),$$

$$2. \mathcal{O}(e^{\gamma l + \beta f} \mid f l) = \mathcal{O}(e^{\gamma l + e^\gamma \beta f} \mid l f),$$

$$3. \mathcal{O}(e^{\beta e + \alpha f + \delta e f} \mid f e) = \mathcal{O}(v e^{v(-\alpha \beta h + \beta e + \alpha f + \delta e f)} \mid e f), \text{ with } v = (1 + h \delta)^{-1},$$

and the rest is straight-forward.

The Main \mathfrak{g}_0 Theorem, Polished Version

Polished Version. With $\bar{e} = \frac{(e^h - 1)}{h} e$, the \mathfrak{g}_0 invariant of any S-component tangle T can be written in the form $Z(T) = \mathbb{E}[\omega, L, Q] = \mathcal{O}(\omega^{-1} e^{L + \omega^{-1} Q} \mid \prod_{i \in S} \bar{e}_i l_i f_i)$, where ω is a scalar (meaning, a polynomial in the variables $t_i = e^{h_i}$), where $L = \sum a_{ij} h_i l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij} and where $Q = \sum b_{ij} \bar{e}_i f_j$ is a balanced quadratic in the variables \bar{e}_i and f_j with scalar coefficients b_{ij} . Furthermore, after setting $t_i = t$ for all i , the invariant $Z(T)$ is poly-time computable.

Partial Proof. Indeed,

$$0. R^s = e^{s(h \otimes l + e \otimes f)} = \mathcal{O}(\exp(s h l + s t^{(s-1)/2} \bar{e} f \mid \bar{e} \otimes l f),$$

$$1. \mathcal{O}(e^{\gamma l + \beta \bar{e}} \mid l \bar{e}) = \mathcal{O}(e^{\gamma l + e^\gamma \beta \bar{e}} \mid \bar{e} l),$$

$$2. \mathcal{O}(e^{\gamma l + \beta f} \mid f l) = \mathcal{O}(e^{\gamma l + e^\gamma \beta f} \mid l f),$$

$$3. \mathcal{O}(e^{\beta \bar{e} + \alpha f + \delta \bar{e} f} \mid f \bar{e}) = \mathcal{O}(v e^{v((1-t) \alpha \beta + \beta \bar{e} + \alpha f + \delta \bar{e} f)} \mid \bar{e} f), \text{ with } v = (1 + (t-1) \delta)^{-1},$$

and the rest is straight-forward.

Implementation

```
CF[E[ω, L, Q]] := Expand /@ Together /@ E[ω, L, Q];
E /: E[ω1, L1, Q1] E[ω2, L2, Q2] := CF@E[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2];
E[ω1, L1, Q1] ≡ E[ω2, L2, Q2] := Expand[ω1 == ω2 ∧ L1 == L2 ∧ Q1 == Q2];
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E[Xi,j+] := E[1, Log[ti] lj, ei fj];
E[Xi,j-] := E[1, -Log[ti] lj, -ti-1 ei fj];
E[p_Times] := E /@ p;
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NO(x:f|e)i lj → k[E[ω, L, Q]] := With[{q = eγ β xk + γ lk}, CF[
  E[ω, γ lk + (L / . lj → θ), ω eγ β xk + (Q / . xi → θ)] / . {γ → ∂lj L, β → ω-1 ∂xi Q}];
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$$\text{NO}_{f_i \rightarrow e_j \rightarrow k} [\mathbb{E} [\omega, L, Q]] := \text{With} \left[\left\{ q = \left((1 - t_k) \alpha \beta + \beta e_k + \delta e_k f_k + \alpha f_k \right) / \mu \right\}, \text{CF} \left[\begin{aligned} & \mathbb{E} \left[\mu \omega, L, \mu \omega q + \mu (Q / \cdot f_i \mid e_j \rightarrow \theta) \right] / \cdot \mu \rightarrow 1 + (t_k - 1) \delta / \cdot \\ & \left\{ \alpha \rightarrow \omega^{-1} (\partial_{f_i} Q / \cdot e_j \rightarrow \theta), \beta \rightarrow \omega^{-1} (\partial_{e_j} Q / \cdot f_i \rightarrow \theta), \delta \rightarrow \omega^{-1} \partial_{f_i, e_j} Q \right\} \right] \end{aligned} \right];$$

The Stitching Formula

$$m_{i \rightarrow j \rightarrow k} [Z] := \text{Module} \left[\{x, z\}, \text{CF} \left[\left(Z // \text{NO}_{f_i \rightarrow e_j \rightarrow x} // \text{NO}_{1_i \rightarrow e_x \rightarrow x} // \text{NO}_{f_x \rightarrow 1_j \rightarrow x} \right) / \cdot z_{-i|j|x} \rightarrow z_k \right] \right]$$

$$\mathbb{E} [X_{4,1}^+ X_{2,5}^+ X_{6,3}^+]$$

$$\mathbb{E} [1, \text{Log}[t_4] l_1 + \text{Log}[t_6] l_3 + \text{Log}[t_2] l_5, e_4 f_1 + e_6 f_3 + e_2 f_5]$$

$$\mathbb{E} [X_{4,1}^+ X_{2,5}^+ X_{6,3}^+] // m_{1,2 \rightarrow 1}$$

$$\mathbb{E} [1, \text{Log}[t_4] l_1 + \text{Log}[t_6] l_3 + \text{Log}[t_1] l_5, e_4 f_1 + e_6 f_3 + e_4 f_5 - e_4 f_5 t_1 + e_1 f_5 t_4]$$

$$\mathbb{E} [X_{4,1}^+ X_{2,5}^+ X_{6,3}^+] // m_{1,2 \rightarrow 1} // m_{1,3 \rightarrow 1}$$

$$\mathbb{E} [1, \text{Log}[t_4] l_1 + \text{Log}[t_6] l_1 + \text{Log}[t_1] l_5, e_6 f_1 + e_4 f_5 - e_4 f_5 t_1 + e_1 f_5 t_4 + e_4 f_1 t_6]$$

$$\mathbb{E} [X_{4,1}^+ X_{2,5}^+ X_{6,3}^+] // m_{1,2 \rightarrow 1} // m_{1,3 \rightarrow 1} // m_{1,4 \rightarrow 1}$$

$$\mathbb{E} [1 - t_6 + t_1 t_6, \text{Log}[t_1] l_1 + \text{Log}[t_6] l_1 + \text{Log}[t_1] l_5, e_6 f_1 + e_6 f_5 + e_1 f_5 t_1 - 2 e_6 f_5 t_1 + e_6 f_5 t_1^2 + e_1 f_1 t_1 t_6^2]$$

$$\mathbb{E} [X_{4,1}^+ X_{2,5}^+ X_{6,3}^+] // m_{1,2 \rightarrow 1} // m_{1,3 \rightarrow 1} // m_{1,4 \rightarrow 1} // m_{1,5 \rightarrow 1}$$

$$\mathbb{E} [1 - t_6 + t_1 t_6, 2 \text{Log}[t_1] l_1 + \text{Log}[t_6] l_1, e_6 f_1 + e_1 f_1 t_1 - e_6 f_1 t_1 + e_6 f_1 t_1^2 + e_1 f_1 t_1^2 t_6^2]$$

$$\mathbb{E} [X_{4,1}^+ X_{2,5}^+ X_{6,3}^+] // m_{1,2 \rightarrow 1} // m_{1,3 \rightarrow 1} // m_{1,4 \rightarrow 1} // m_{1,5 \rightarrow 1} // m_{1,6 \rightarrow 1}$$

$$\mathbb{E} [t_1 - t_1^2 + t_1^3, 3 \text{Log}[t_1] l_1, e_1 f_1 t_1 + e_1 f_1 t_1^3 + e_1 f_1 t_1^5]$$

Independent Proof of Invariance

Meta-Associativity:

$$\zeta = \mathbb{E} \left[\omega, \sum_{i=1}^4 \sum_{j=1}^4 a_{i,j} \text{Log}[t_i] l_j, \sum_{i=1}^4 \sum_{j=1}^4 b_{i,j} e_i f_j \right]$$

$$\begin{aligned} & \mathbb{E} [\omega, \text{Log}[t_1] l_1 a_{1,1} + \text{Log}[t_1] l_2 a_{1,2} + \text{Log}[t_1] l_3 a_{1,3} + \text{Log}[t_1] l_4 a_{1,4} + \\ & \quad \text{Log}[t_2] l_1 a_{2,1} + \text{Log}[t_2] l_2 a_{2,2} + \text{Log}[t_2] l_3 a_{2,3} + \text{Log}[t_2] l_4 a_{2,4} + \text{Log}[t_3] l_1 a_{3,1} + \text{Log}[t_3] l_2 a_{3,2} + \\ & \quad \text{Log}[t_3] l_3 a_{3,3} + \text{Log}[t_3] l_4 a_{3,4} + \text{Log}[t_4] l_1 a_{4,1} + \text{Log}[t_4] l_2 a_{4,2} + \text{Log}[t_4] l_3 a_{4,3} + \text{Log}[t_4] l_4 a_{4,4}, \\ & \quad e_1 f_1 b_{1,1} + e_1 f_2 b_{1,2} + e_1 f_3 b_{1,3} + e_1 f_4 b_{1,4} + e_2 f_1 b_{2,1} + e_2 f_2 b_{2,2} + e_2 f_3 b_{2,3} + e_2 f_4 b_{2,4} + \\ & \quad e_3 f_1 b_{3,1} + e_3 f_2 b_{3,2} + e_3 f_3 b_{3,3} + e_3 f_4 b_{3,4} + e_4 f_1 b_{4,1} + e_4 f_2 b_{4,2} + e_4 f_3 b_{4,3} + e_4 f_4 b_{4,4}] \end{aligned}$$

$\xi // m_{1,2 \rightarrow 1}$

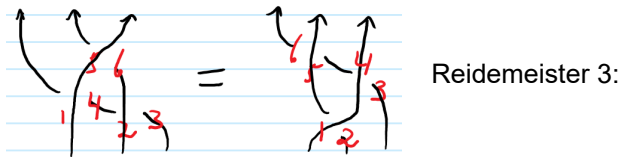
$$\mathbb{E} \left[\omega - b_{2,1} + t_1 b_{2,1}, \text{Log}[t_1] l_1 a_{1,1} + \text{Log}[t_1] l_1 a_{1,2} + \text{Log}[t_1] l_3 a_{1,3} + \text{Log}[t_1] l_4 a_{1,4} + \right. \\ \left. \text{Log}[t_1] l_1 a_{2,1} + \text{Log}[t_1] l_1 a_{2,2} + \text{Log}[t_1] l_3 a_{2,3} + \text{Log}[t_1] l_4 a_{2,4} + \text{Log}[t_3] l_1 a_{3,1} + \text{Log}[t_3] l_1 a_{3,2} + \right. \\ \left. \text{Log}[t_3] l_3 a_{3,3} + \text{Log}[t_3] l_4 a_{3,4} + \text{Log}[t_4] l_1 a_{4,1} + \text{Log}[t_4] l_1 a_{4,2} + \text{Log}[t_4] l_3 a_{4,3} + \text{Log}[t_4] l_4 a_{4,4}, \right. \\ \left. e_1 f_1 t_1^{a_{1,2}+a_{2,2}} t_3^{a_{3,2}} t_4^{a_{4,2}} b_{1,1} + e_1 f_1 b_{1,2} + e_1 f_3 b_{1,3} + e_1 f_4 b_{1,4} + e_1 f_1 t_1^{a_{1,1}+a_{1,2}+a_{2,1}+a_{2,2}} t_3^{a_{3,1}+a_{3,2}} t_4^{a_{4,1}+a_{4,2}} b_{2,1} - \right. \\ \left. \frac{e_1 f_1 b_{1,2} b_{2,1}}{\omega} + \frac{e_1 f_1 t_1 b_{1,2} b_{2,1}}{\omega} - \frac{e_1 f_3 b_{1,3} b_{2,1}}{\omega} + \frac{e_1 f_3 t_1 b_{1,3} b_{2,1}}{\omega} - \frac{e_1 f_4 b_{1,4} b_{2,1}}{\omega} + \frac{e_1 f_4 t_1 b_{1,4} b_{2,1}}{\omega} + \right. \\ \left. e_1 f_1 t_1^{a_{1,1}+a_{2,1}} t_3^{a_{3,1}} t_4^{a_{4,1}} b_{2,2} + \frac{e_1 f_1 b_{1,1} b_{2,2}}{\omega} - \frac{e_1 f_1 t_1 b_{1,1} b_{2,2}}{\omega} + e_1 f_3 t_1^{a_{1,1}+a_{2,1}} t_3^{a_{3,1}} t_4^{a_{4,1}} b_{2,3} + \frac{e_1 f_3 b_{1,1} b_{2,3}}{\omega} - \right. \\ \left. \frac{e_1 f_3 t_1 b_{1,1} b_{2,3}}{\omega} + e_1 f_4 t_1^{a_{1,1}+a_{2,1}} t_3^{a_{3,1}} t_4^{a_{4,1}} b_{2,4} + \frac{e_1 f_4 b_{1,1} b_{2,4}}{\omega} - \frac{e_1 f_4 t_1 b_{1,1} b_{2,4}}{\omega} + e_3 f_1 t_1^{a_{1,2}+a_{2,2}} t_3^{a_{3,2}} t_4^{a_{4,2}} b_{3,1} + \right. \\ \left. \frac{e_3 f_1 b_{2,2} b_{3,1}}{\omega} - \frac{e_3 f_1 t_1 b_{2,2} b_{3,1}}{\omega} + \frac{e_3 f_3 b_{2,3} b_{3,1}}{\omega} - \frac{e_3 f_3 t_1 b_{2,3} b_{3,1}}{\omega} + \frac{e_3 f_4 b_{2,4} b_{3,1}}{\omega} - \frac{e_3 f_4 t_1 b_{2,4} b_{3,1}}{\omega} + \right. \\ \left. e_3 f_1 b_{3,2} - \frac{e_3 f_1 b_{2,1} b_{3,2}}{\omega} + \frac{e_3 f_1 t_1 b_{2,1} b_{3,2}}{\omega} + e_3 f_3 b_{3,3} - \frac{e_3 f_3 b_{2,1} b_{3,3}}{\omega} + \frac{e_3 f_3 t_1 b_{2,1} b_{3,3}}{\omega} + e_3 f_4 b_{3,4} - \right. \\ \left. \frac{e_3 f_4 b_{2,1} b_{3,4}}{\omega} + \frac{e_3 f_4 t_1 b_{2,1} b_{3,4}}{\omega} + e_4 f_1 t_1^{a_{1,2}+a_{2,2}} t_3^{a_{3,2}} t_4^{a_{4,2}} b_{4,1} + \frac{e_4 f_1 b_{2,2} b_{4,1}}{\omega} - \frac{e_4 f_1 t_1 b_{2,2} b_{4,1}}{\omega} + \right. \\ \left. \frac{e_4 f_3 b_{2,3} b_{4,1}}{\omega} - \frac{e_4 f_3 t_1 b_{2,3} b_{4,1}}{\omega} + \frac{e_4 f_4 b_{2,4} b_{4,1}}{\omega} - \frac{e_4 f_4 t_1 b_{2,4} b_{4,1}}{\omega} + e_4 f_1 b_{4,2} - \frac{e_4 f_1 b_{2,1} b_{4,2}}{\omega} + \right. \\ \left. \frac{e_4 f_1 t_1 b_{2,1} b_{4,2}}{\omega} + e_4 f_3 b_{4,3} - \frac{e_4 f_3 b_{2,1} b_{4,3}}{\omega} + \frac{e_4 f_3 t_1 b_{2,1} b_{4,3}}{\omega} + e_4 f_4 b_{4,4} - \frac{e_4 f_4 b_{2,1} b_{4,4}}{\omega} + \frac{e_4 f_4 t_1 b_{2,1} b_{4,4}}{\omega} \right]$$

Short [$\xi // m_{1,2 \rightarrow 1} // m_{1,3 \rightarrow 1}, 5$]

$$\mathbb{E} \left[\omega - b_{2,1} + t_1 b_{2,1} - t_1^{a_{1,2}+a_{2,2}+a_{3,2}} t_4^{a_{4,2}} b_{3,1} + t_1^{1+a_{1,2}+a_{2,2}+a_{3,2}} t_4^{a_{4,2}} b_{3,1} - \frac{b_{2,2} b_{3,1}}{\omega} + \frac{2 t_1 b_{2,2} b_{3,1}}{\omega} - \frac{t_1^2 b_{2,2} b_{3,1}}{\omega} - b_{3,2} + t_1 b_{3,2} + \right. \\ \left. \frac{b_{2,1} b_{3,2}}{\omega} - \frac{2 t_1 b_{2,1} b_{3,2}}{\omega} + \frac{t_1^2 b_{2,1} b_{3,2}}{\omega}, \ll 1 \gg, e_1 f_1 t_1^{a_{\ll 1 \gg} + \ll 5 \gg} t_4^{a_{\ll 1 \gg} \ll 1 \gg + \ll 1 \gg} b_{1,1} + \ll 224 \gg + \frac{e_4 \ll 4 \gg b_{\ll 1 \gg}}{\omega^2} \right]$$

FullSimplify[($\xi // m_{1,2 \rightarrow 1} // m_{1,3 \rightarrow 1}$) \equiv ($\xi // m_{2,3 \rightarrow 2} // m_{1,2 \rightarrow 1}$)]

True



lhs = $\mathbb{E} [X_{1,4}^+, X_{2,3}^+, X_{5,6}^+] // m_{1,5 \rightarrow 1} // m_{2,6 \rightarrow 2} // m_{3,4 \rightarrow 3}$

$\mathbb{E} [1, \text{Log}[t_1] l_2 + \text{Log}[t_1] l_3 + \text{Log}[t_2] l_3, e_1 f_2 + e_1 f_3 + e_2 f_3 t_1]$

rhs = $\mathbb{E} [X_{1,2}^+, X_{4,3}^+, X_{5,6}^+] // m_{1,4 \rightarrow 1} // m_{2,5 \rightarrow 2} // m_{3,6 \rightarrow 3}$

$\mathbb{E} [1, \text{Log}[t_1] l_2 + \text{Log}[t_1] l_3 + \text{Log}[t_2] l_3, e_1 f_2 + e_1 f_3 + e_2 f_3 t_1]$

lhs \equiv rhs

True

Homework.

1. Use the same methodology to verify $m_{a,b \rightarrow c} // m_{d,e \rightarrow f} == m_{d,e \rightarrow f} // m_{a,b \rightarrow c}$.
2. Likewise, verify the two types of R2 moves.
3. Make sure that R1 gives no trouble.