Pensieve header: The polished ${\mathfrak g}_0$ invariant.

Reminder

Make sure that you have Mathematica and that you play with these programs!

The Main $\mathfrak{g}_0$ Theorem

Raw Version. The $\mathfrak{g}_0$ invariant of any S-component tangle T can be written in the form $Z(T)=\mathbb{O}\left(\omega e^{L+Q} \mid \prod_{i\in S} e_i l_i f_i\right)$, where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i=e^{h_i}$), where $L=\sum a_{ij} h_i l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij} and where $Q=\sum b_{ij} e_i f_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij}. Furthermore, after setting $h_i=h$ and $t_i=t$ for all i, the invariant $Z(T)$ is poly-time computable.

Proof. Indeed, as shown below,

0. $R^s=e^{s(h\otimes l+e\otimes f)}=\mathbb{O}\left(\exp\left(s\,\mathrm{hl}+\frac{e^{sh}-1}{h}\mathrm{ef}\right) \mid e\otimes \mathrm{lf}\right)$,

1. $\mathbb{O}\left(e^{\gamma \mathrm{l}+\beta \mathrm{e}} \mid \mathrm{le}\right)=\mathbb{O}\left(e^{\gamma \mathrm{l}+e^{\gamma}\beta \mathrm{e}} \mid \mathrm{el}\right)$,

2. $\mathbb{O}\left(e^{\gamma \mathrm{l}+\beta \mathrm{f}} \mid \mathrm{fl}\right)=\mathbb{O}\left(e^{\gamma \mathrm{l}+e^{\gamma}\beta \mathrm{f}} \mid \mathrm{lf}\right)$,

3. $\mathbb{O}\left(e^{\beta \mathrm{e}+\alpha \mathrm{f}+\delta \mathrm{ef}} \mid \mathrm{fe}\right)=\mathbb{O}\left(\nu e^{\nu(-\alpha\beta \mathrm{h}+\beta \mathrm{e}+\alpha \mathrm{f}+\delta \mathrm{ef})} \mid \mathrm{ef}\right)$, with $\nu=(1+\mathrm{h}\delta)^{-1}$,

and the rest is straight-forward.

The Main $\mathfrak{g}_0$ Theorem, Polished Version

Polished Version. With $\bar{e}=\frac{(e^h-1)}{h}e$, the $\mathfrak{g}_0$ invariant of any S-component tangle T can be written in the form $Z(T)=\mathbb{E}[\omega, L, Q]=\mathbb{O}\left(\omega^{-1} e^{L+\omega^{-1}Q} \mid \prod_{i\in S} \bar{e}_i l_i f_i\right)$, where ω is a scalar (meaning, a polynomial in the variables $t_i=e^{h_i}$), where $L=\sum a_{ij} h_i l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij} and where $Q=\sum b_{ij} \bar{e}_i f_j$ is a balanced quadratic in the variables $\bar{e}_i$ and f_j with scalar coefficients b_{ij}. Furthermore, after setting $t_i=t$ for all i, the invariant $Z(T)$ is poly-time computable.

Partial Proof. Indeed,

0. $R^s=e^{s(h\otimes l+e\otimes f)}=\mathbb{O}\left(\exp\left(s\,\mathrm{hl}+s\,t^{(s-1)/2}\,\bar{e}\,f\right) \mid \bar{e}\otimes \mathrm{lf}\right)$,

1. $\mathbb{O}\left(e^{\gamma \mathrm{l}+\beta\bar{e}} \mid l\,\bar{e}\right)=\mathbb{O}\left(e^{\gamma \mathrm{l}+e^{\gamma}\beta\bar{e}} \mid \bar{e}\,l\right)$,

2. $\mathbb{O}\left(e^{\gamma \mathrm{l}+\beta \mathrm{f}} \mid \mathrm{fl}\right)=\mathbb{O}\left(e^{\gamma \mathrm{l}+e^{\gamma}\beta \mathrm{f}} \mid \mathrm{lf}\right)$,

3. $\mathbb{O}\left(e^{\beta\bar{e}+\alpha \mathrm{f}+\delta\bar{e}f} \mid f\,\bar{e}\right)=\mathbb{O}\left(\nu e^{\nu((1-t)\alpha\beta+\beta\bar{e}+\alpha \mathrm{f}+\delta\bar{e}f)} \mid \bar{e}\,f\right)$, with $\nu=(1+(t-1)\,\delta)^{-1}$,

and the rest is straight-forward.

Implementation

```
CF[𝔼[ω_, L_, Q_]] := Expand /@ Together /@ 𝔼[ω, L, Q];
𝔼 /: 𝔼[ω1_, L1_, Q1_] 𝔼[ω2_, L2_, Q2_] := CF@𝔼[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2];
𝔼[ω1_, L1_, Q1_] ≡ 𝔼[ω2_, L2_, Q2_] := Expand[ω1 == ω2 ∧ L1 == L2 ∧ Q1 == Q2];
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```
𝔼[X⁺[i_,j_]] := 𝔼[1, Log[t_i] l_j, e_i f_j];
𝔼[X⁻[i_,j_]] := 𝔼[1, - Log[t_i] l_j, -t_i^-1 e_i f_j];
𝔼[p_Times] := 𝔼 /@ p;
```

```
NO_(x:f|e)_i_ l_j_→k_ [𝔼[ω_, L_, Q_]] := With[{q = ℯ^γ β x_k + γ l_k}, CF[
    𝔼[ω, γ l_k + (L /. l_j → 0), ω ℯ^γ β x_k + (Q /. x_i → 0)] /. {γ → ∂_(l_j) L, β → ω^-1 ∂_(x_i) Q}]];
```

```
NO_{f_i_ e_j_→k_}[𝔼[ω_, L_, Q_]] := With[{q = ((1 - t_k) α β + β e_k + δ e_k f_k + α f_k) / μ}, CF[
   𝔼[μ ω, L, μ ω q + μ (Q /. f_i | e_j → 0)] /. μ → 1 + (t_k - 1) δ /.
    {α → ω^-1 (∂_{f_i} Q /. e_j → 0), β → ω^-1 (∂_{e_j} Q /. f_i → 0), δ → ω^-1 ∂_{f_i,e_j} Q}]];
```

The Stitching Formula

```
m_{i_,j_→k_}[Z_] := Module[{x, z}, CF[(Z // NO_{f_i e_j→x} // NO1_{l_i e_x→x} // NO_{f_x l_j→x}) /. z_{_i|j|x} → z_k]]
```

$\mathbb{E}[X^+_{4,1} X^+_{2,5} X^+_{6,3}]$

$\mathbb{E}[1, \mathrm{Log}[t_4]\, l_1 + \mathrm{Log}[t_6]\, l_3 + \mathrm{Log}[t_2]\, l_5, e_4 f_1 + e_6 f_3 + e_2 f_5]$

$\mathbb{E}[X^+_{4,1} X^+_{2,5} X^+_{6,3}]$ // $m_{1,2\to 1}$

$\mathbb{E}[1, \mathrm{Log}[t_4]\, l_1 + \mathrm{Log}[t_6]\, l_3 + \mathrm{Log}[t_1]\, l_5, e_4 f_1 + e_6 f_3 + e_4 f_5 - e_4 f_5 t_1 + e_1 f_5 t_4]$

$\mathbb{E}[X^+_{4,1} X^+_{2,5} X^+_{6,3}]$ // $m_{1,2\to 1}$ // $m_{1,3\to 1}$

$\mathbb{E}[1, \mathrm{Log}[t_4]\, l_1 + \mathrm{Log}[t_6]\, l_1 + \mathrm{Log}[t_1]\, l_5, e_6 f_1 + e_4 f_5 - e_4 f_5 t_1 + e_1 f_5 t_4 + e_4 f_1 t_6]$

$\mathbb{E}[X^+_{4,1} X^+_{2,5} X^+_{6,3}]$ // $m_{1,2\to 1}$ // $m_{1,3\to 1}$ // $m_{1,4\to 1}$

$\mathbb{E}[1 - t_6 + t_1 t_6, \mathrm{Log}[t_1]\, l_1 + \mathrm{Log}[t_6]\, l_1 + \mathrm{Log}[t_1]\, l_5, e_6 f_1 + e_6 f_5 + e_1 f_5 t_1 - 2 e_6 f_5 t_1 + e_6 f_5 t_1^2 + e_1 f_1 t_1 t_6^2]$

$\mathbb{E}[X^+_{4,1} X^+_{2,5} X^+_{6,3}]$ // $m_{1,2\to 1}$ // $m_{1,3\to 1}$ // $m_{1,4\to 1}$ // $m_{1,5\to 1}$

$\mathbb{E}[1 - t_6 + t_1 t_6, 2\,\mathrm{Log}[t_1]\, l_1 + \mathrm{Log}[t_6]\, l_1, e_6 f_1 + e_1 f_1 t_1 - e_6 f_1 t_1 + e_6 f_1 t_1^2 + e_1 f_1 t_1^2 t_6^2]$

$\mathbb{E}[X^+_{4,1} X^+_{2,5} X^+_{6,3}]$ // $m_{1,2\to 1}$ // $m_{1,3\to 1}$ // $m_{1,4\to 1}$ // $m_{1,5\to 1}$ // $m_{1,6\to 1}$

$\mathbb{E}[t_1 - t_1^2 + t_1^3, 3\,\mathrm{Log}[t_1]\, l_1, e_1 f_1 t_1 + e_1 f_1 t_1^3 + e_1 f_1 t_1^5]$

Independent Proof of Invariance

Meta-Associativity:

$$\zeta = \mathbb{E}\left[\omega, \sum_{i=1}^{4}\sum_{j=1}^{4} a_{i,j}\,\mathrm{Log}[t_i]\, l_j, \sum_{i=1}^{4}\sum_{j=1}^{4} b_{i,j}\, e_i f_j\right]$$

$\mathbb{E}[\omega, \mathrm{Log}[t_1]\, l_1 a_{1,1} + \mathrm{Log}[t_1]\, l_2 a_{1,2} + \mathrm{Log}[t_1]\, l_3 a_{1,3} + \mathrm{Log}[t_1]\, l_4 a_{1,4} +$
$\mathrm{Log}[t_2]\, l_1 a_{2,1} + \mathrm{Log}[t_2]\, l_2 a_{2,2} + \mathrm{Log}[t_2]\, l_3 a_{2,3} + \mathrm{Log}[t_2]\, l_4 a_{2,4} + \mathrm{Log}[t_3]\, l_1 a_{3,1} + \mathrm{Log}[t_3]\, l_2 a_{3,2} +$
$\mathrm{Log}[t_3]\, l_3 a_{3,3} + \mathrm{Log}[t_3]\, l_4 a_{3,4} + \mathrm{Log}[t_4]\, l_1 a_{4,1} + \mathrm{Log}[t_4]\, l_2 a_{4,2} + \mathrm{Log}[t_4]\, l_3 a_{4,3} + \mathrm{Log}[t_4]\, l_4 a_{4,4},$
$e_1 f_1 b_{1,1} + e_1 f_2 b_{1,2} + e_1 f_3 b_{1,3} + e_1 f_4 b_{1,4} + e_2 f_1 b_{2,1} + e_2 f_2 b_{2,2} + e_2 f_3 b_{2,3} + e_2 f_4 b_{2,4} +$
$e_3 f_1 b_{3,1} + e_3 f_2 b_{3,2} + e_3 f_3 b_{3,3} + e_3 f_4 b_{3,4} + e_4 f_1 b_{4,1} + e_4 f_2 b_{4,2} + e_4 f_3 b_{4,3} + e_4 f_4 b_{4,4}]$

ξ // $m_{1,2\to 1}$

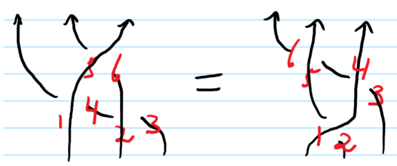
$$\mathbb{E}\Big[\omega - b_{2,1} + t_1 b_{2,1},\ \mathrm{Log}[t_1]\, l_1 a_{1,1} + \mathrm{Log}[t_1]\, l_1 a_{1,2} + \mathrm{Log}[t_1]\, l_3 a_{1,3} + \mathrm{Log}[t_1]\, l_4 a_{1,4} + \mathrm{Log}[t_1]\, l_1 a_{2,1} + \mathrm{Log}[t_1]\, l_1 a_{2,2} + \mathrm{Log}[t_1]\, l_3 a_{2,3} + \mathrm{Log}[t_1]\, l_4 a_{2,4} + \mathrm{Log}[t_3]\, l_1 a_{3,1} + \mathrm{Log}[t_3]\, l_1 a_{3,2} + \mathrm{Log}[t_3]\, l_3 a_{3,3} + \mathrm{Log}[t_3]\, l_4 a_{3,4} + \mathrm{Log}[t_4]\, l_1 a_{4,1} + \mathrm{Log}[t_4]\, l_1 a_{4,2} + \mathrm{Log}[t_4]\, l_3 a_{4,3} + \mathrm{Log}[t_4]\, l_4 a_{4,4},$$
$$e_1 f_1 t_1^{a_{1,2}+a_{2,2}} t_3^{a_{3,2}} t_4^{a_{4,2}} b_{1,1} + e_1 f_1 b_{1,2} + e_1 f_3 b_{1,3} + e_1 f_4 b_{1,4} + e_1 f_1 t_1^{a_{1,1}+a_{1,2}+a_{2,1}+a_{2,2}} t_3^{a_{3,1}+a_{3,2}} t_4^{a_{4,1}+a_{4,2}} b_{2,1} -$$
$$\frac{e_1 f_1 b_{1,2} b_{2,1}}{\omega} + \frac{e_1 f_1 t_1 b_{1,2} b_{2,1}}{\omega} - \frac{e_1 f_3 b_{1,3} b_{2,1}}{\omega} + \frac{e_1 f_3 t_1 b_{1,3} b_{2,1}}{\omega} - \frac{e_1 f_4 b_{1,4} b_{2,1}}{\omega} + \frac{e_1 f_4 t_1 b_{1,4} b_{2,1}}{\omega} +$$
$$e_1 f_1 t_1^{a_{1,1}+a_{2,1}} t_3^{a_{3,1}} t_4^{a_{4,1}} b_{2,2} + \frac{e_1 f_1 b_{1,1} b_{2,2}}{\omega} - \frac{e_1 f_1 t_1 b_{1,1} b_{2,2}}{\omega} + e_1 f_3 t_1^{a_{1,1}+a_{2,1}} t_3^{a_{3,1}} t_4^{a_{4,1}} b_{2,3} + \frac{e_1 f_3 b_{1,1} b_{2,3}}{\omega} -$$
$$\frac{e_1 f_3 t_1 b_{1,1} b_{2,3}}{\omega} + e_1 f_4 t_1^{a_{1,1}+a_{2,1}} t_3^{a_{3,1}} t_4^{a_{4,1}} b_{2,4} + \frac{e_1 f_4 b_{1,1} b_{2,4}}{\omega} - \frac{e_1 f_4 t_1 b_{1,1} b_{2,4}}{\omega} + e_3 f_1 t_1^{a_{1,2}+a_{2,2}} t_3^{a_{3,2}} t_4^{a_{4,2}} b_{3,1} +$$
$$\frac{e_3 f_1 b_{2,2} b_{3,1}}{\omega} - \frac{e_3 f_1 t_1 b_{2,2} b_{3,1}}{\omega} + \frac{e_3 f_3 b_{2,3} b_{3,1}}{\omega} - \frac{e_3 f_3 t_1 b_{2,3} b_{3,1}}{\omega} + \frac{e_3 f_4 b_{2,4} b_{3,1}}{\omega} - \frac{e_3 f_4 t_1 b_{2,4} b_{3,1}}{\omega} +$$
$$e_3 f_1 b_{3,2} - \frac{e_3 f_1 b_{2,1} b_{3,2}}{\omega} + \frac{e_3 f_1 t_1 b_{2,1} b_{3,2}}{\omega} + e_3 f_3 b_{3,3} - \frac{e_3 f_3 b_{2,1} b_{3,3}}{\omega} + \frac{e_3 f_3 t_1 b_{2,1} b_{3,3}}{\omega} + e_3 f_4 b_{3,4} -$$
$$\frac{e_3 f_4 b_{2,1} b_{3,4}}{\omega} + \frac{e_3 f_4 t_1 b_{2,1} b_{3,4}}{\omega} + e_4 f_1 t_1^{a_{1,2}+a_{2,2}} t_3^{a_{3,2}} t_4^{a_{4,2}} b_{4,1} + \frac{e_4 f_1 b_{2,2} b_{4,1}}{\omega} - \frac{e_4 f_1 t_1 b_{2,2} b_{4,1}}{\omega} +$$
$$\frac{e_4 f_3 b_{2,3} b_{4,1}}{\omega} - \frac{e_4 f_3 t_1 b_{2,3} b_{4,1}}{\omega} + \frac{e_4 f_4 b_{2,4} b_{4,1}}{\omega} - \frac{e_4 f_4 t_1 b_{2,4} b_{4,1}}{\omega} + e_4 f_1 b_{4,2} - \frac{e_4 f_1 b_{2,1} b_{4,2}}{\omega} +$$
$$\frac{e_4 f_1 t_1 b_{2,1} b_{4,2}}{\omega} + e_4 f_3 b_{4,3} - \frac{e_4 f_3 b_{2,1} b_{4,3}}{\omega} + \frac{e_4 f_3 t_1 b_{2,1} b_{4,3}}{\omega} + e_4 f_4 b_{4,4} - \frac{e_4 f_4 b_{2,1} b_{4,4}}{\omega} + \frac{e_4 f_4 t_1 b_{2,1} b_{4,4}}{\omega}\Big]$$

Short[ξ // $m_{1,2\to 1}$ // $m_{1,3\to 1}$]

$$\mathbb{E}\Big[\omega - b_{2,1} + \ll 15 \gg + \frac{t_1^2 b_{2,1} b_{3,2}}{\omega},\ \ll 1 \gg + \ll 14 \gg + \ll 1 \gg,\ e_1 f_1 t_1^{\ll 1 \gg} t_4^{a_{\ll 1 \gg} + \ll 1 \gg} b_{1,1} + \ll 224 \gg + \frac{e_4 \ll 4 \gg b_{\ll 1 \gg}}{\omega^2}\Big]$$

FullSimplify[(ξ // $m_{1,2\to 1}$ // $m_{1,3\to 1}$) ≡ (ξ // $m_{2,3\to 2}$ // $m_{1,2\to 1}$)]

True



Reidemeister 3:

lhs = $\mathbb{E}\big[X^{+}_{1,4}\, X^{+}_{2,3}\, X^{+}_{5,6}\big]$ // $m_{1,5\to 1}$ // $m_{2,6\to 2}$ // $m_{3,4\to 3}$

$\mathbb{E}[1,\ \mathrm{Log}[t_1]\, l_2 + \mathrm{Log}[t_1]\, l_3 + \mathrm{Log}[t_2]\, l_3,\ e_1 f_2 + e_1 f_3 + e_2 f_3 t_1]$

rhs = $\mathbb{E}\big[X^{+}_{1,2}\, X^{+}_{4,3}\, X^{+}_{5,6}\big]$ // $m_{1,4\to 1}$ // $m_{2,5\to 2}$ // $m_{3,6\to 3}$

$\mathbb{E}[1,\ \mathrm{Log}[t_1]\, l_2 + \mathrm{Log}[t_1]\, l_3 + \mathrm{Log}[t_2]\, l_3,\ e_1 f_2 + e_1 f_3 + e_2 f_3 t_1]$

lhs ≡ rhs

True

Homework.

1. Use the same methodology to verify $m_{a,b\to c}$ // $m_{d,e\to f}$ == $m_{d,e\to f}$ // $m_{a,b\to c}$.
2. Likewise, verify the two types of R2 moves.
3. Make sure that R1 gives no trouble.