Pensieve header: Proof of the main ${\mathfrak g}_0$ lemma and a poly-time program to compute the ${\mathfrak g}_0$ invariant.

Reminder

Make sure that you have Mathematica and that you play with these programs!

Representing $\mathfrak{g}_0 = \langle h, e, l, f\rangle / ([e, l] = -e,\ [f, l] = f,\ [e, f] = h,\ [h, *] = 0)$

```
ρh = (0 0 -1; 0 0 0; 0 0 0); ρe = (0 0 0; 0 0 1; 0 0 0); ρl = (0 0 0; 0 1 0; 0 0 0); ρf = (0 1 0; 0 0 0; 0 0 0); ρ0 = (0 0 0; 0 0 0; 0 0 0);
B[x_?MatrixQ, y_?MatrixQ] := x.y - y.x;
```

```
{B[ρe, ρl] == -ρe, B[ρf, ρl] == ρf, B[ρe, ρf] == ρh, B[ρh, ρe] == ρ0, B[ρh, ρl] == ρ0, B[ρh, ρf] == ρ0}
{True, True, True, True, True, True}
```

The Main $\mathfrak{g}_0$ Theorem

Raw Version. The $\mathfrak{g}_0$ invariant of any S-component tangle T can be written in the form $Z(T) = \mathbb{O}\left(\omega e^{L+Q} \mid \prod_{i\in S} e_i l_i f_i\right)$, where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i = e^{h_i}$), where $L = \sum a_{ij} h_i l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij} and where $Q = \sum b_{ij} e_i f_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij}. Furthermore, after setting $h_i = h$ and $t_i = t$ for all i, the invariant $Z(T)$ is poly-time computable.

Proof. Indeed, as shown below,

0. $R^s = e^{s(h\otimes l + e\otimes f)} = \mathbb{O}\left(\exp\left(s\,\mathrm{hl} + \frac{e^{sh}-1}{h}\,\mathrm{ef}\right) \mid e\otimes \mathrm{lf}\right)$,

1. $\mathbb{O}\left(e^{\gamma l + \beta e} \mid \mathrm{le}\right) = \mathbb{O}\left(e^{\gamma l + e^{\gamma}\beta e} \mid \mathrm{el}\right)$,

2. $\mathbb{O}\left(e^{\gamma l + \beta f} \mid \mathrm{fl}\right) = \mathbb{O}\left(e^{\gamma l + e^{\gamma}\beta f} \mid \mathrm{lf}\right)$,

3. $\mathbb{O}\left(e^{\beta e + \alpha f + \delta ef} \mid \mathrm{fe}\right) = \mathbb{O}\left(\nu e^{\nu(-\alpha\beta h + \beta e + \alpha f + \delta ef)} \mid \mathrm{ef}\right)$, with $\nu = (1 + h\delta)^{-1}$,

and the rest is straight-forward.

Proofs of the $\mathfrak{g}_0$ lemmas

```
(* 0 *) MatrixForm /@ {MatrixExp[h ρl + e ρf], MatrixExp[h ρl].MatrixExp[(e^h - 1)/h e ρf]}
```

$$\left\{\begin{pmatrix}1 & \frac{e(-1+e^h)}{h} & 0\\ 0 & e^h & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & \frac{e(-1+e^h)}{h} & 0\\ 0 & e^h & 0\\ 0 & 0 & 1\end{pmatrix}\right\}$$

```
(* 1 *) MatrixForm /@ {MatrixExp[γ ρl].MatrixExp[β ρe], MatrixExp[e^γ β ρe].MatrixExp[γ ρl]}
```

$$\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & e^{\gamma} & e^{\gamma}\beta\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 0 & e^{\gamma} & e^{\gamma}\beta\\ 0 & 0 & 1\end{pmatrix}\right\}$$

```
(* 2 *) MatrixForm /@ {MatrixExp[β ρf].MatrixExp[γ ρl], MatrixExp[γ ρl].MatrixExp[e^γ β ρf]}
```

$$\left\{\begin{pmatrix}1 & e^{\gamma}\beta & 0\\ 0 & e^{\gamma} & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & e^{\gamma}\beta & 0\\ 0 & e^{\gamma} & 0\\ 0 & 0 & 1\end{pmatrix}\right\}$$

```
(* 3 at δ=0 *)
MatrixForm /@ {MatrixExp[α ρf].MatrixExp[β ρe], MatrixExp[-α β ρh].MatrixExp[β ρe].MatrixExp[α ρf]}
```

$$\left\{\begin{pmatrix}1 & \alpha & \alpha\beta\\ 0 & 1 & \beta\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & \alpha & \alpha\beta\\ 0 & 1 & \beta\\ 0 & 0 & 1\end{pmatrix}\right\}$$

For the full proof of 3, see the blackboard and then check:

```
With[{ψ = ν e^(ν (t e f-α β h+α f+β e)) /. ν → (1 + t h)^-1}, Simplify@{∂_t ψ - ∂_{α,β} ψ, ψ /. t → 0}]
```

$\{0,\ e^{f\alpha+e\beta-h\alpha\beta}\}$

Implementation

```
CF[𝔼[ω_, L_, Q_]] := 𝔼[Simplify[ω], Simplify[L], Simplify[Q]];
𝔼 /: 𝔼[ω1_, L1_, Q1_] 𝔼[ω2_, L2_, Q2_] := CF@𝔼[ω1 ω2, L1 + L2, Q1 + Q2];
𝔼[ω1_, L1_, Q1_] ≡ 𝔼[ω2_, L2_, Q2_] := Simplify[ω1 == ω2 ∧ L1 == L2 ∧ Q1 == Q2];
```

0. $R = e^{h\otimes l + e\otimes f} = \mathbb{O}\left(\exp\left(hl + \frac{e^h-1}{h} ef \;\middle|\; e\otimes lf\right)\right)$:

```
𝔼[X⁺_{i_,j_}] := 𝔼[1, h_i l_j, h_i^-1 (e^(h_i) - 1) e_i f_j];
𝔼[X⁻_{i_,j_}] := 𝔼[1, -h_i l_j, h_i^-1 (e^(-h_i) - 1) e_i f_j];
𝔼[p_Times] := 𝔼 /@ p;
```

```
𝔼[X⁺_{4,1} X⁺_{2,5} X⁺_{6,3}]
```

1. $\mathbb{O}\left(e^{\gamma l+\beta e} \mid le\right) = \mathbb{O}\left(e^{\gamma l+e^{\gamma}\beta e} \mid el\right)$,
2. $\mathbb{O}\left(e^{\gamma l+\beta f} \mid fl\right) = \mathbb{O}\left(e^{\gamma l+e^{\gamma}\beta f} \mid lf\right)$:

```
NO_{(x:f|e)_i_ l_j_}[𝔼[ω_, L_, Q_]] := CF[𝔼[ω, L, e^γ α x_i + (Q /. x_i → 0) /. {γ → ∂_{l_j} L, α → ∂_{x_i} Q}]];
ANO_{(x:f|e)_i_ l_j_}[𝔼[ω_, L_, Q_]] := CF[𝔼[ω, L, e^-γ α x_i + (Q /. x_i → 0) /. {γ → ∂_{l_j} L, α → ∂_{x_i} Q}]];
```

```
𝔼[X⁺_{4,1} X⁺_{2,5} X⁺_{6,3}]
𝔼[X⁺_{4,1} X⁺_{2,5} X⁺_{6,3}] // ANO_{e_2 l_3}
(𝔼[X⁺_{4,1} X⁺_{2,5} X⁺_{6,3}] // ANO_{e_2 l_3} // NO_{e_2 l_3}) ≡ 𝔼[X⁺_{4,1} X⁺_{2,5} X⁺_{6,3}]
```

3. $\mathbb{O}\left(e^{\beta e+\alpha f+\delta ef} \mid fe\right) = \mathbb{O}\left(\nu e^{\nu(-\alpha\beta h+\beta e+\alpha f+\delta ef)} \mid ef\right)$, with $\nu = (1+h\delta)^{-1}$:

```
NO_{f_i_ e_j_→k_}[𝔼[ω_, L_, Q_]] := CF[
   𝔼[ν ω, L, ν (-α β h_k + β e_k + α f_k + δ e_k f_k) + (Q /. f_i | e_j → 0)]
     /. ν → (1 + h_k δ)^-1 /. {α → ∂_{f_i} Q /. e_j → 0, β → ∂_{e_j} Q /. f_i → 0, δ → ∂_{f_i,e_j} Q}];
```

```
𝔼[X⁺_{4,1} X⁺_{2,5} X⁺_{6,3}] // NO_{f_3 e_4→7}
```

The Stitching Formula

```
m_{i_,j_→k_}[Z_] := Module[{x, z}, CF[(Z // NO_{f_i e_j→x} // NO_{l_i e_x} // NO_{f_x l_j}) /. z_i|j|x → z_k]]
```

```
𝔼[X⁺_{4,1} X⁺_{2,5} X⁺_{6,3}] // m_{1,2→1}
𝔼[X⁺_{4,1} X⁺_{2,5} X⁺_{6,3}] // m_{1,2→1} // m_{1,3→1}
𝔼[X⁺_{4,1} X⁺_{2,5} X⁺_{6,3}] // m_{1,2→1} // m_{1,3→1} // m_{1,4→1}
𝔼[X⁺_{4,1} X⁺_{2,5} X⁺_{6,3}] // m_{1,2→1} // m_{1,3→1} // m_{1,4→1} // m_{1,5→1} // m_{1,6→1}
```

Independent Proof of Invariance

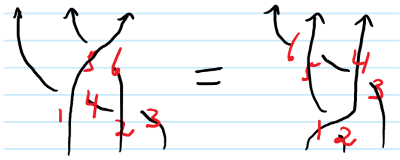
Meta-Associativity:

$$\zeta = \mathbb{E}\left[\omega,\ \sum_{i=1}^{4}\sum_{j=1}^{4} a_{i,j}\, h_i\, l_j,\ \sum_{i=1}^{4}\sum_{j=1}^{4} b_{i,j}\, e_i\, f_j\right]$$

$$\zeta\ //\ m_{1,2\to 1}$$

$$\zeta\ //\ m_{1,2\to 1}\ //\ m_{1,3\to 1}$$

$$(\zeta\ //\ m_{1,2\to 1}\ //\ m_{1,3\to 1}) \equiv (\zeta\ //\ m_{2,3\to 2}\ //\ m_{1,2\to 1})$$



Reidemeister 3:

$$\text{lhs} = \mathbb{E}\left[X^{+}_{1,4}\, X^{+}_{2,3}\, X^{+}_{5,6}\right]\ //\ m_{1,5\to 1}\ //\ m_{2,6\to 2}\ //\ m_{3,4\to 3}$$

$$\text{rhs} = \mathbb{E}\left[X^{+}_{1,2}\, X^{+}_{4,3}\, X^{+}_{5,6}\right]\ //\ m_{1,4\to 1}\ //\ m_{2,5\to 2}\ //\ m_{3,6\to 3}$$

$$\text{lhs} \equiv \text{rhs}$$

Homework.

1. Use the same methodology to verify $m_{a,b\to c}\ //\ m_{d,e\to f} == m_{d,e\to f}\ //\ m_{a,b\to c}$.
2. Likewise, verify the two types of R2 moves.
3. Make sure that R1 gives no trouble.
4. Implement the “polished version” of the main theorem below, and verify that everything works.

The Main $\mathfrak{g}_0$ Theorem, Polished Version

Polished Version. With $\overline{e} = \frac{(e^h - 1)}{h} e$, the $\mathfrak{g}_0$ invariant of any S-component tangle T can be written in the form $Z(T) = \mathbb{O}\left(\omega^{-1}\, e^{L + \omega^{-1} Q}\ \middle|\ \prod_{i\in S} \overline{e}_i\, l_i\, f_i\right)$, where ω is a scalar (meaning, a polynomial in the variables $t_i = e^{h_i}$), where $L = \sum a_{ij}\, h_i\, l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij} and where $Q = \sum b_{ij}\, \overline{e}_i\, f_j$ is a balanced quadratic in the variables $\overline{e}_i$ and f_j with scalar coefficients b_{ij}. Furthermore, after setting $t_i = t$ for all i, the invariant $Z(T)$ is poly-time computable.