Pensieve header: Proof of the main ${\mathfrak g}_0$ lemma and a poly-time program to compute the ${\mathfrak g}_0$ invariant.

Reminder

Make sure that you have Mathematica and that you play with these programs!

Representing $\mathfrak{g}_0 = \langle h, e, l, f\rangle / ([e, l] = -e,\ [f, l] = f,\ [e, f] = h,\ [h, *] = 0)$

```
ρh = (0 0 -1; 0 0 0; 0 0 0); ρe = (0 0 0; 0 0 1; 0 0 0); ρl = (0 0 0; 0 1 0; 0 0 0); ρf = (0 1 0; 0 0 0; 0 0 0); ρ0 = (0 0 0; 0 0 0; 0 0 0);
B[x_?MatrixQ, y_?MatrixQ] := x.y - y.x;
```

```
{B[ρe, ρl] == -ρe, B[ρf, ρl] == ρf, B[ρe, ρf] == ρh, B[ρh, ρe] == ρ0, B[ρh, ρl] == ρ0, B[ρh, ρf] == ρ0}
{True, True, True, True, True, True}
```

The Main $\mathfrak{g}_0$ Theorem

Raw Version. The $\mathfrak{g}_0$ invariant of any S-component tangle T can be written in the form $Z(T) = \mathbb{O}\left(\omega e^{L+Q} \mid \prod_{i\in S} e_i\, l_i\, f_i\right)$, where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i = e^{h_i}$), where $L = \sum a_{ij}\, h_i\, l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij} and where $Q = \sum b_{ij}\, e_i\, f_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij}. Furthermore, after setting $h_i = h$ and $t_i = t$ for all i, the invariant $Z(T)$ is poly-time computable.

Proof. Indeed, as shown below,

0. $R^s = e^{s(h\otimes l + e\otimes f)} = \mathbb{O}\left(\exp\left(s\,\mathrm{hl} + \frac{e^{sh}-1}{h}\,\mathrm{ef}\right) \mid e\otimes \mathrm{lf}\right)$,

1. $\mathbb{O}\left(e^{\gamma \mathrm{l} + \beta \mathrm{e}} \mid \mathrm{le}\right) = \mathbb{O}\left(e^{\gamma \mathrm{l} + e^{\gamma}\beta \mathrm{e}} \mid \mathrm{el}\right)$,

2. $\mathbb{O}\left(e^{\gamma \mathrm{l} + \beta \mathrm{f}} \mid \mathrm{fl}\right) = \mathbb{O}\left(e^{\gamma \mathrm{l} + e^{\gamma}\beta \mathrm{f}} \mid \mathrm{lf}\right)$,

3. $\mathbb{O}\left(e^{\beta \mathrm{e} + \alpha \mathrm{f} + \delta \mathrm{ef}} \mid \mathrm{fe}\right) = \mathbb{O}\left(\nu e^{\nu(-\alpha\beta \mathrm{h} + \beta \mathrm{e} + \alpha \mathrm{f} + \delta \mathrm{ef})} \mid \mathrm{ef}\right)$, with $\nu = (1 + \mathrm{h}\delta)^{-1}$,

and the rest is straight-forward.

Proofs of the $\mathfrak{g}_0$ lemmas

```
(* 0 *) MatrixForm /@ {MatrixExp[h ρl + e ρf], MatrixExp[h ρl].MatrixExp[(e^h - 1)/h e ρf]}
```

$$\left\{\begin{pmatrix}1 & \frac{e\,(-1+e^h)}{h} & 0\\ 0 & e^h & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & \frac{e\,(-1+e^h)}{h} & 0\\ 0 & e^h & 0\\ 0 & 0 & 1\end{pmatrix}\right\}$$

```
(* 1 *) MatrixForm /@ {MatrixExp[γ ρl].MatrixExp[β ρe], MatrixExp[e^γ β ρe].MatrixExp[γ ρl]}
```

$$\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & e^\gamma & e^\gamma \beta\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 0 & e^\gamma & e^\gamma \beta\\ 0 & 0 & 1\end{pmatrix}\right\}$$

```
(* 2 *) MatrixForm /@ {MatrixExp[β ρf].MatrixExp[γ ρl], MatrixExp[γ ρl].MatrixExp[e^γ β ρf]}
```

$$\left\{\begin{pmatrix}1 & e^\gamma \beta & 0\\ 0 & e^\gamma & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & e^\gamma \beta & 0\\ 0 & e^\gamma & 0\\ 0 & 0 & 1\end{pmatrix}\right\}$$

```
(* 3 at δ=0 *)
MatrixForm /@ {MatrixExp[α ρf].MatrixExp[β ρe], MatrixExp[-α β ρh].MatrixExp[β ρe].MatrixExp[α ρf]}
```

$$\left\{\begin{pmatrix}1 & \alpha & \alpha\beta\\ 0 & 1 & \beta\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & \alpha & \alpha\beta\\ 0 & 1 & \beta\\ 0 & 0 & 1\end{pmatrix}\right\}$$

For the full proof of 3, see the blackboard and then check:

```
With[{ψ = ν e^(ν (t e f - α β h + α f + β e)) /. ν → (1 + t h)^-1}, Simplify@{∂_t ψ - ∂_{α,β} ψ, ψ /. t → 0}]
```

$\{0, e^{f\alpha+e\beta-h\alpha\beta}\}$

Implementation

```
CF[E[ω_, L_, Q_]] := E[Simplify[ω], Simplify[L], Simplify[Q]];
E /: E[ω1_, L1_, Q1_] E[ω2_, L2_, Q2_] := CF@E[ω1 ω2, L1 + L2, Q1 + Q2];
E[ω1_, L1_, Q1_] ≡ E[ω2_, L2_, Q2_] := Simplify[ω1 == ω2 ∧ L1 == L2 ∧ Q1 == Q2];
```

0. $R = e^{h\otimes l + e\otimes f} = \mathbb{O}\left(\exp\left(hl + \frac{e^h-1}{h} ef\right) \mid e \otimes lf\right)$:

```
E[X^+_{i_,j_}] := E[1, h_i l_j, h_i^-1 (e^(h_i) - 1) e_i f_j];
E[X^-_{i_,j_}] := E[1, -h_i l_j, h_i^-1 (e^(-h_i) - 1) e_i f_j];
E[p_Times] := E /@ p;
```

```
E[X^+_{4,1} X^+_{2,5} X^+_{6,3}]
```

$\mathbb{E}\left[1, h_4 l_1 + h_6 l_3 + h_2 l_5, \frac{(-1+e^{h_2}) e_2 f_5}{h_2} + \frac{(-1+e^{h_4}) e_4 f_1}{h_4} + \frac{(-1+e^{h_6}) e_6 f_3}{h_6}\right]$

1. $\mathbb{O}\left(e^{\gamma l+\beta e} \mid le\right) = \mathbb{O}\left(e^{\gamma l + e^{\gamma}\beta e} \mid el\right)$,
2. $\mathbb{O}\left(e^{\gamma l+\beta f} \mid fl\right) = \mathbb{O}\left(e^{\gamma l + e^{\gamma}\beta f} \mid lf\right)$:

```
NO_{(x:f|e)_i_ l_j_}[E[ω_, L_, Q_]] := CF[E[ω, L, e^γ α x_i + (Q /. x_i → 0) /. {γ → ∂_{l_j} L, α → ∂_{x_i} Q}]];
ANO_{(x:f|e)_i_ l_j_}[E[ω_, L_, Q_]] := CF[E[ω, L, e^-γ α x_i + (Q /. x_i → 0) /. {γ → ∂_{l_j} L, α → ∂_{x_i} Q}]];
```

```
E[X^+_{4,1} X^+_{2,5} X^+_{6,3}]
```

$\mathbb{E}\left[1, h_4 l_1 + h_6 l_3 + h_2 l_5, \frac{(-1+e^{h_2}) e_2 f_5}{h_2} + \frac{(-1+e^{h_4}) e_4 f_1}{h_4} + \frac{(-1+e^{h_6}) e_6 f_3}{h_6}\right]$

```
E[X^+_{4,1} X^+_{2,5} X^+_{6,3}] // ANO_{e_2 l_3}
```

$\mathbb{E}\left[1, h_4 l_1 + h_6 l_3 + h_2 l_5, \frac{e^{-h_6}(-1+e^{h_2}) e_2 f_5}{h_2} + \frac{(-1+e^{h_4}) e_4 f_1}{h_4} + \frac{(-1+e^{h_6}) e_6 f_3}{h_6}\right]$

```
(E[X^+_{4,1} X^+_{2,5} X^+_{6,3}] // ANO_{e_2 l_3} // NO_{e_2 l_3}) == E[X^+_{4,1} X^+_{2,5} X^+_{6,3}]
```

True

3. $\mathbb{O}\left(e^{\beta e+\alpha f+\delta ef} \mid fe\right) = \mathbb{O}\left(\nu e^{\nu(-\alpha\beta h+\beta e+\alpha f+\delta ef)} \mid ef\right)$, with $\nu = (1+h\delta)^{-1}$:

```
NO_{f_i_ e_j_→k_}[E[ω_, L_, Q_]] := CF[
   E[ν ω, L, ν (-α β h_k + β e_k + α f_k + δ e_k f_k) + (Q /. f_i | e_j → 0)]
     /. ν → (1 + h_k δ)^-1 /. {α → ∂_{f_i} Q /. e_j → 0, β → ∂_{e_j} Q /. f_i → 0, δ → ∂_{f_i,e_j} Q}];
```

```
E[X^+_{4,1} X^+_{2,5} X^+_{6,3}] // NO_{f_3 e_4→7}
```

$\mathbb{E}\left[1, h_4 l_1 + h_6 l_3 + h_2 l_5, \frac{1}{h_2 h_4 h_6} \left(\left(\left(-1+e^{h_4}\right) e_7 f_1 h_2 + \left(-1+e^{h_2}\right) e_2 f_5 h_4\right) h_6 + \left(-1+e^{h_6}\right) e_6 h_2 \left(f_7 h_4 - \left(-1+e^{h_4}\right) f_1 h_7\right)\right)\right]$

The Stitching Formula

```
m_{i_,j_→k_}[Z_] := Module[{x, z}, CF[(Z // NO_{f_i e_j→x} // NO_{l_i e_x} // NO_{f_x l_j}) /. z_{i|j|x} → z_k]]
```

$$\mathbb{E}[X^+_{4,1}\, X^+_{2,5}\, X^+_{6,3}] \;//\; m_{1,2\to 1}$$

$$\mathbb{E}\Big[1,\ h_4 l_1 + h_6 l_3 + h_1 l_5,\ \frac{1}{h_1 h_4 h_6}\Big(\big(-1+e^{h_6}\big) e_6 f_3 h_1 h_4 + \Big(\big(-1+e^{h_4}\big) e_4 \big(f_1 - \big(-1+e^{h_1}\big) f_5\big) h_1 + e^{h_4}\big(-1+e^{h_1}\big) e_1 f_5 h_4\Big) h_6\Big)\Big]$$

$$\mathbb{E}[X^+_{4,1}\, X^+_{2,5}\, X^+_{6,3}] \;//\; m_{1,2\to 1} \;//\; m_{1,3\to 1}$$

$$\mathbb{E}\Big[1,\ h_4 l_1 + h_6 l_1 + h_1 l_5,\ \frac{1}{h_1 h_4 h_6}\Big(\big(-1+e^{h_6}\big) e_6 f_1 h_1 h_4 + \Big(\big(-1+e^{h_4}\big) e_4 \big(e^{h_6} f_1 - \big(-1+e^{h_1}\big) f_5\big) h_1 + e^{h_4}\big(-1+e^{h_1}\big) e_1 f_5 h_4\Big) h_6\Big)\Big]$$

$$\mathbb{E}[X^+_{4,1}\, X^+_{2,5}\, X^+_{6,3}] \;//\; m_{1,2\to 1} \;//\; m_{1,3\to 1} \;//\; m_{1,4\to 1}$$

$$\mathbb{E}\Big[\frac{1}{1+e^{h_6}\big(-1+e^{h_1}\big)},\ h_6 l_1 + h_1 (l_1 + l_5),\ \frac{\big(-1+e^{h_6}\big) e_6 \big(f_1 + \big(-1+e^{h_1}\big)^2 f_5\big) h_1 + e^{h_1}\big(-1+e^{h_1}\big) e_1 \big(e^{2 h_6} f_1 + f_5\big) h_6}{\big(1 - e^{h_6} + e^{h_1+h_6}\big) h_1 h_6}\Big]$$

$$\mathbb{E}[X^+_{4,1}\, X^+_{2,5}\, X^+_{6,3}] \;//\; m_{1,2\to 1} \;//\; m_{1,3\to 1} \;//\; m_{1,4\to 1} \;//\; m_{1,5\to 1} \;//\; m_{1,6\to 1}$$

$$\mathbb{E}\Big[\frac{e^{-h_1}}{1 - e^{h_1} + e^{2 h_1}},\ 3 h_1 l_1,\ \frac{\big(-1+e^{3 h_1}\big) e_1 f_1}{h_1}\Big]$$

Independent Proof of Invariance

Meta-Associativity:

$$\zeta = \mathbb{E}\Big[\omega,\ \sum_{i=1}^{4}\sum_{j=1}^{4} a_{i,j} h_i l_j,\ \sum_{i=1}^{4}\sum_{j=1}^{4} b_{i,j} e_i f_j\Big]$$

$$\mathbb{E}[\omega,\ h_1 l_1 a_{1,1} + h_1 l_2 a_{1,2} + h_1 l_3 a_{1,3} + h_1 l_4 a_{1,4} + h_2 l_1 a_{2,1} + h_2 l_2 a_{2,2} + h_2 l_3 a_{2,3} + h_2 l_4 a_{2,4} + h_3 l_1 a_{3,1} + h_3 l_2 a_{3,2} + h_3 l_3 a_{3,3} + h_3 l_4 a_{3,4} + h_4 l_1 a_{4,1} + h_4 l_2 a_{4,2} + h_4 l_3 a_{4,3} + h_4 l_4 a_{4,4},$$
$$e_1 f_1 b_{1,1} + e_1 f_2 b_{1,2} + e_1 f_3 b_{1,3} + e_1 f_4 b_{1,4} + e_2 f_1 b_{2,1} + e_2 f_2 b_{2,2} + e_2 f_3 b_{2,3} + e_2 f_4 b_{2,4} + e_3 f_1 b_{3,1} + e_3 f_2 b_{3,2} + e_3 f_3 b_{3,3} + e_3 f_4 b_{3,4} + e_4 f_1 b_{4,1} + e_4 f_2 b_{4,2} + e_4 f_3 b_{4,3} + e_4 f_4 b_{4,4}]$$

$$\zeta \;//\; m_{1,2\to 1}$$

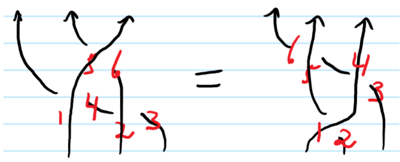
$$\mathbb{E}\Big[\frac{\omega}{1 + h_1 b_{2,1}},\ h_1 \big(l_1 (a_{1,1} + a_{1,2} + a_{2,1} + a_{2,2}) + l_3 (a_{1,3} + a_{2,3}) + l_4 (a_{1,4} + a_{2,4})\big) + h_3 \big(l_1 (a_{3,1} + a_{3,2}) + l_3 a_{3,3} + l_4 a_{3,4}\big) + h_4 \big(l_1 (a_{4,1} + a_{4,2}) + l_3 a_{4,3} + l_4 a_{4,4}\big),$$
$$e_1 f_1 b_{1,2} + e_1 f_3 b_{1,3} + e_1 f_4 b_{1,4} + \frac{e^{h_1 (a_{1,1}+a_{2,1}) + h_3 a_{3,1} + h_4 a_{4,1}} e_1 \big(f_1 b_{2,2} + f_3 b_{2,3} + f_4 b_{2,4}\big)}{1 + h_1 b_{2,1}} + e_3 f_1 b_{3,2} + e_3 f_3 b_{3,3} + e_3 f_4 b_{3,4} - \frac{h_1 \big(f_1 b_{2,2} + f_3 b_{2,3} + f_4 b_{2,4}\big)\big(e_1 b_{1,1} + e_3 b_{3,1} + e_4 b_{4,1}\big)}{1 + h_1 b_{2,1}} + \frac{1}{1 + h_1 b_{2,1}}$$
$$e^{h_1 (a_{1,2}+a_{2,2}) + h_3 a_{3,2} + h_4 a_{4,2}} f_1 \Big(e_1 \big(b_{1,1} + e^{h_1 (a_{1,1}+a_{2,1}) + h_3 a_{3,1} + h_4 a_{4,1}} b_{2,1}\big) + e_3 b_{3,1} + e_4 b_{4,1}\Big) + e_4 f_1 b_{4,2} + e_4 f_3 b_{4,3} + e_4 f_4 b_{4,4}\Big]$$

```
ζ // m1,2→1 // m1,3→1
```

$$
\begin{aligned}
\mathbb{E}\Big[& \frac{\omega}{1+h_1\left(b_{2,1}+\mathbb{e}^{h_1(a_{1,2}+a_{2,2}+a_{3,2})+h_4 a_{4,2}}\,b_{3,1}+b_{3,2}\right)+h_1^2\left(-b_{2,2}\,b_{3,1}+b_{2,1}\,b_{3,2}\right)}, \\
& h_1\left(l_1\left(a_{1,1}+a_{1,2}+a_{1,3}+a_{2,1}+a_{2,2}+a_{2,3}+a_{3,1}+a_{3,2}+a_{3,3}\right)+l_4\left(a_{1,4}+a_{2,4}+a_{3,4}\right)\right)+h_4\left(l_1\left(a_{4,1}+a_{4,2}+a_{4,3}\right)+l_4\,a_{4,4}\right), \\
& e_1\,f_1\,b_{1,3}+e_1\,f_4\,b_{1,4}+\frac{\mathbb{e}^{h_1(a_{1,1}+a_{2,1}+a_{3,1})+h_4 a_{4,1}}\,e_1\left(f_1\,b_{2,3}+f_4\,b_{2,4}\right)}{1+h_1\,b_{2,1}}+ \\
& \Big(\mathbb{e}^{h_1(a_{1,1}+a_{1,2}+a_{2,1}+a_{2,2}+a_{3,1}+a_{3,2})+h_4(a_{4,1}+a_{4,2})}\,e_1\left(f_1\left(b_{3,3}+h_1\left(-b_{2,3}\,b_{3,1}+b_{2,1}\,b_{3,3}\right)\right)+f_4\left(b_{3,4}+h_1\left(-b_{2,4}\,b_{3,1}+b_{2,1}\,b_{3,4}\right)\right)\right)\Big)\Big/ \\
& \left(1+h_1\left(b_{2,1}+\mathbb{e}^{h_1(a_{1,2}+a_{2,2}+a_{3,2})+h_4 a_{4,2}}\,b_{3,1}+b_{3,2}\right)+h_1^2\left(-b_{2,2}\,b_{3,1}+b_{2,1}\,b_{3,2}\right)\right)-\frac{h_1\left(f_1\,b_{2,3}+f_4\,b_{2,4}\right)\left(e_1\,b_{1,1}+e_4\,b_{4,1}\right)}{1+h_1\,b_{2,1}}- \\
& \Big(h_1\left(-\frac{h_1\left(f_1\,b_{2,3}+f_4\,b_{2,4}\right)b_{3,1}}{1+h_1\,b_{2,1}}+f_1\,b_{3,3}+f_4\,b_{3,4}\right)\left(e_1\,b_{1,2}\left(1+h_1\,b_{2,1}\right)+\mathbb{e}^{h_1(a_{1,1}+a_{2,1}+a_{3,1})+h_4 a_{4,1}}\,e_1\,b_{2,2}-h_1\,b_{2,2}\right. \\
& \quad \left.\left(e_1\,b_{1,1}+e_4\,b_{4,1}\right)+\mathbb{e}^{h_1(a_{1,2}+a_{2,2}+a_{3,2})+h_4 a_{4,2}}\left(e_1\left(b_{1,1}+\mathbb{e}^{h_1(a_{1,1}+a_{2,1}+a_{3,1})+h_4 a_{4,1}}\,b_{2,1}\right)+e_4\,b_{4,1}\right)+e_4\left(1+h_1\,b_{2,1}\right)b_{4,2}\right)\Big)\Big/ \\
& \left(1+h_1\left(b_{2,1}+\mathbb{e}^{h_1(a_{1,2}+a_{2,2}+a_{3,2})+h_4 a_{4,2}}\,b_{3,1}+b_{3,2}\right)+h_1^2\left(-b_{2,2}\,b_{3,1}+b_{2,1}\,b_{3,2}\right)\right)+ \\
& \Big(\mathbb{e}^{h_1(a_{1,3}+a_{2,3}+a_{3,3})+h_4 a_{4,3}}\,f_1\Big(e_1\Big(b_{1,2}\left(1+h_1\,b_{2,1}\right)+b_{1,1}\left(\mathbb{e}^{h_1(a_{1,2}+a_{2,2}+a_{3,2})+h_4 a_{4,2}}-h_1\,b_{2,2}\right)+\mathbb{e}^{h_1(a_{1,1}+a_{2,1}+a_{3,1})+h_4 a_{4,1}} \\
& \quad \Big(b_{2,2}\left(1-\mathbb{e}^{h_1(a_{1,2}+a_{2,2}+a_{3,2})+h_4 a_{4,2}}\,h_1\,b_{3,1}\right)+\mathbb{e}^{h_1(a_{1,2}+a_{2,2}+a_{3,2})+h_4 a_{4,2}}\left(\mathbb{e}^{h_1(a_{1,2}+a_{2,2}+a_{3,2})+h_4 a_{4,2}}\,b_{3,1}+b_{3,2}\right)+ \\
& \quad \mathbb{e}^{h_1(a_{1,2}+a_{2,2}+a_{3,2})+h_4 a_{4,2}}\,b_{2,1}\left(1+h_1\,b_{3,2}\right)\Big)\Big)+e_4\left(\left(\mathbb{e}^{h_1(a_{1,2}+a_{2,2}+a_{3,2})+h_4 a_{4,2}}-h_1\,b_{2,2}\right)b_{4,1}+\left(1+h_1\,b_{2,1}\right)b_{4,2}\right)\Big)\Big)\Big/ \\
& \left(1+h_1\left(b_{2,1}+\mathbb{e}^{h_1(a_{1,2}+a_{2,2}+a_{3,2})+h_4 a_{4,2}}\,b_{3,1}+b_{3,2}\right)+h_1^2\left(-b_{2,2}\,b_{3,1}+b_{2,1}\,b_{3,2}\right)\right)+e_4\,f_1\,b_{4,3}+e_4\,f_4\,b_{4,4}\Big]
\end{aligned}
$$

```
(ζ // m1,2→1 // m1,3→1) ≡ (ζ // m2,3→2 // m1,2→1)
```

True



Reidemeister 3:

```
lhs = 𝔼[X⁺1,4 X⁺2,3 X⁺5,6] // m1,5→1 // m2,6→2 // m3,4→3
```

$$\mathbb{E}\left[1,\ h_2\,l_3+h_1\left(l_2+l_3\right),\ \frac{\left(-1+\mathbb{e}^{h_1}\right)e_1\left(f_2+f_3\right)}{h_1}+\frac{\mathbb{e}^{h_1}\left(-1+\mathbb{e}^{h_2}\right)e_2\,f_3}{h_2}\right]$$

```
rhs = 𝔼[X⁺1,2 X⁺4,3 X⁺5,6] // m1,4→1 // m2,5→2 // m3,6→3
```

$$\mathbb{E}\left[1,\ h_2\,l_3+h_1\left(l_2+l_3\right),\ \frac{\mathbb{e}^{h_1}\left(-1+\mathbb{e}^{h_2}\right)e_2\,f_3\,h_1+\left(-1+\mathbb{e}^{h_1}\right)e_1\left(f_2+f_3\right)h_2}{h_1\,h_2}\right]$$

```
lhs ≡ rhs
```

True

Homework.

1. Use the same methodology to verify $m_{a,b\to c}$ // $m_{d,e\to f}$ == $m_{d,e\to f}$ // $m_{a,b\to c}$.
2. Likewise, verify the two types of R2 moves.
3. Make sure that R1 gives no trouble.
4. Implement the “polished version” of the main theorem below, and verify that everything works.

The Main $\mathfrak{g}_0$ Theorem, Polished Version

Polished Version. With $\overline{e}=\frac{(e^h-1)}{h}e$, the $\mathfrak{g}_0$ invariant of any S-component tangle T can be written in the form $Z(T)=\mathbb{O}\left(\omega^{-1}\,\mathbb{e}^{L+\omega^{-1}Q}\ \middle|\ \prod_{i\in S}\overline{e}_i\,l_i\,f_i\right)$, where ω is a scalar (meaning, a polynomial in the variables $t_i=\mathbb{e}^{h_i}$), where $L=\sum a_{ij}\,h_i\,l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij} and where $Q=\sum b_{ij}\,\overline{e}_i\,f_j$ is a balanced quadratic in the variables $\overline{e}_i$ and f_j with scalar coefficients b_{ij}. Furthermore, after setting $t_i=t$ for all i, the invariant $Z(T)$ is poly-time computable.