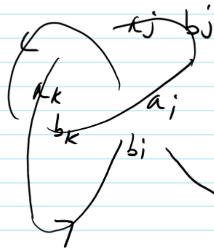
Pensieve header: The main ${\mathfrak g}_0$ theorem.

Reminders

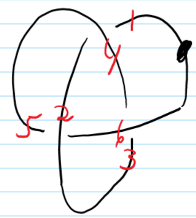
1. Fill out “Graduate Course Participation Form” (see http://drorbn.net/AcademicPensieve/Classes/17-1350-AKT/GCP.jpg, in time).
2. Make sure that you have Mathematica and that you play with these programs!
3. Change meeting time?



$R=\sum a_i\otimes b_i \in A\otimes A=\widehat{U(g)}\otimes\widehat{U(g)}$

s.t. $R^{12}R^{13}R^{23}=R^{23}R^{13}R^{12}$

$\to \sum_{i,j,k} b_i a_j b_k a_i b_j a_k \in U(g)$



Today: $g_0=\langle h,e,l,f\rangle / {h \text{ central} \atop [e,l]=-e}$ $[f,l]=f$ $[e,f]=h$

$r=h\otimes l+e\otimes f$ $R=\exp(r)$

Note $U(g_0)^{\otimes S}=U(\oplus_S g_0)=U\left(\langle h_i,e_i,l_i,f_i\rangle / {h_i \text{ central} \atop [e_i,l_j]=\delta_{ij}e_i \text{ etc.}}\right)$

Theorem. $R=e^{h\otimes l+e\otimes f}=\mathbb{O}\left(\exp\left(hl+\frac{e^h-1}{h}ef\right)\middle|\, e\otimes lf\right)$.

Representing g_0

```
Date: Sun, 23 May 1999 17:24:53 +0200 (MET DST)
From: Dylan Thurston <Dylan.Thurston@math.unige.ch>
To: Dror Bar-Natan <drorbn@math.huji.ac.il>
Subject: Oh, yeah...

Duflo had another possibly useful comment: he said that there is a simpler (non-trivial) algebra than
sl_2, given by the relations (more or less)

        [x,y] = z
        [x,t] = x
        [y,t] = -y
        [t,z] = [x,z] = [y,z] = 0

I hope I got that right; it should be the Lie algebra of matrices

        0 x z
        0 t y
        0 0 0

He says that one can write down the exponential explicitly, that commutators very quickly go to zero,
etc.  Apparently it plays the same role for reducible lie algebras that sl_2 plays for semi-simple ones.
It’s worth thinking about.

Best,
        Dylan
```

```
ρh = (0 0 -1
      0 0  0
      0 0  0); ρe = (0 0 0
                     0 0 1
                     0 0 0); ρl = (0 0 0
                                   0 1 0
                                   0 0 0); ρf = (0 1 0
                                                 0 0 0
                                                 0 0 0);
B[x_?MatrixQ, y_?MatrixQ] := x.y - y.x;
```

```
MatrixForm /@ {B[ρe, ρl], B[ρf, ρl], B[ρe, ρf], B[ρh, ρe], B[ρh, ρl], B[ρh, ρf]}
```

```
{(0 0  0
  0 0 -1
  0 0  0), (0 1 0
            0 0 0
            0 0 0), (0 0 -1
                     0 0  0
                     0 0  0), (0 0 0
                               0 0 0
                               0 0 0), (0 0 0
                                        0 0 0
                                        0 0 0), (0 0 0
                                                 0 0 0
                                                 0 0 0)}
```

Implementing $\mathfrak{g}_0$

```
PBWRule = {e → 1, l → 2, f → 3};
B[U@e, U@l] = -U@e;
B[U@f, U@l] = U@f;
B[U@e, U@f] = h U[];
```

```
$TD = 3; ℏ /: ℏ^d_. /; d > $TD := 0;
```

```
x_ ≤ y_ := OrderedQ[{x, y} /. PBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. PBWRule];
Simp[ℰ_] := Collect[ℰ, _U, Expand];
```

```
U_{i_}[ℰ_] := ℰ /. {h → h_i, t → t_i, u_U :→ Replace[u, x_ :→ x_i, 1]};
B[U[(x_)_{i_}], U[(y_)_{i_}]] := B[U[x_i], U[y_i]] = U_i[B[U@x, U@y]];
B[U[(x_)_{i_}], U[(y_)_{j_}]] /; i =!= j := 0;
B[x_, x_] = 0;
B[U[y_], U[x_]] := B[U[y], U[x]] = Simp[-B[U[x], U[y]]];
B[x_, y_] := x ** y - y ** x;
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
0 ** _ = _ ** 0 = 0;
x_ ** U[] := x; U[] ** x_ := x;
(a_ * x_U) ** (b_ * y_U) := If[a b === 0, 0, Simp[a b (x ** y)]];
(a_ * x_U) ** y_ := Simp[a (x ** y)]; x_ ** (a_ * y_U) := Simp[a (x ** y)];
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
```

```
U[xx___, x_] ** U[y_, yy___] := If[x ≤ y, U[xx, x, y, yy], U@xx ** (U@y ** U@x + B[U@x, U@y]) ** U@yy];
```

```
UU[l___, x_^n_, r___] := UU[l, Sequence @@ Table[x, {n}], r];
UU[l___, 1, r___] := UU[l, r];
UU[] = U[];
UU[l_, r___] := U[l] ** UU[r];
```

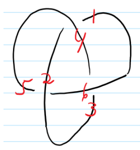
```
UProducts[{}, 0] = {UU[]};
UProducts[{}, n_Integer] /; n > 0 = {};
UProducts[{x_, xs___}, n_Integer] :=
  Sort@Flatten@Table[UU[x^k] ** u, {k, 0, n}, {u, UProducts[{xs}, n - k]}];
UProducts[xs_List, k_Integer, n_Integer] := UProducts[Flatten@Table[x_j, {x, xs}, {j, k}], n];
UProducts[any__, {n_}] := Flatten@Table[UProducts[any, k], {k, 0, n}];
```

```
r_{i_,j_} := Simp[ℏ (h_i UU[l_j] + UU[e_i, f_j])]
```

```
UExp[u_] := Module[{s, t, k},
   s = t = U[]; k = 0;
   While[k < 20 ∧ 0 =!= (t = t ** u), s += t / (++k) !];
   Simp[s]];
R_{i_,j_} := UExp[r_{i,j}];
```

```
m[i_, j_, k_][ℰ_] := Simp[ℰ /. {
    u_U :> UU @@ Join[DeleteCases[u, x_{i|j}], U @@ Cases[u, x_i :> x_k], U @@ Cases[u, x_j :> x_k]],
    h_{i|j} → h_k}]
```

The Invariant of the Trefoil




```
; $TD = 2; R_{4,1} ** R_{2,5} ** R_{6,3}
```

$$U[] + \hbar h_4 U[l_1] + \hbar h_6 U[l_3] + \hbar h_2 U[l_5] + \left(\hbar + \frac{\hbar^2 h_2}{2}\right) U[e_2, f_5] + \left(\hbar + \frac{\hbar^2 h_4}{2}\right) U[e_4, f_1] + \left(\hbar + \frac{\hbar^2 h_6}{2}\right) U[e_6, f_3] +$$
$$\frac{1}{2}\hbar^2 h_4^2 U[l_1, l_1] + \hbar^2 h_4 h_6 U[l_1, l_3] + \hbar^2 h_2 h_4 U[l_1, l_5] + \frac{1}{2}\hbar^2 h_6^2 U[l_3, l_3] + \hbar^2 h_2 h_6 U[l_3, l_5] + \frac{1}{2}\hbar^2 h_2^2 U[l_5, l_5] +$$
$$\hbar^2 h_4 U[e_2, l_1, f_5] + \hbar^2 h_6 U[e_2, l_3, f_5] + \hbar^2 h_2 U[e_2, l_5, f_5] + \hbar^2 h_4 U[e_4, l_1, f_1] + \hbar^2 h_6 U[e_4, l_3, f_1] +$$
$$\hbar^2 h_2 U[e_4, l_5, f_1] + \hbar^2 h_4 U[e_6, l_1, f_3] + \hbar^2 h_6 U[e_6, l_3, f_3] + \hbar^2 h_2 U[e_6, l_5, f_3] + \frac{1}{2}\hbar^2 U[e_2, e_2, f_5, f_5] +$$
$$\hbar^2 U[e_2, e_4, f_1, f_5] + \hbar^2 U[e_2, e_6, f_3, f_5] + \frac{1}{2}\hbar^2 U[e_4, e_4, f_1, f_1] + \hbar^2 U[e_4, e_6, f_1, f_3] + \frac{1}{2}\hbar^2 U[e_6, e_6, f_3, f_3]$$

```
Timing[$TD = 3; R_{4,1} ** R_{2,5} ** R_{6,3} // m[1, 2, 1] // m[1, 3, 1] // m[1, 4, 1] // m[1, 5, 1] // m[1, 6, 1]]
```

$$\Big\{0.609375, \left(1 - 2\hbar h_1 + \hbar^2 h_1^2 + \frac{2}{3}\hbar^3 h_1^3\right) U[] +$$
$$\left(3\hbar h_1 - 6\hbar^2 h_1^2 + 3\hbar^3 h_1^3\right) U[l_1] + \left(3\hbar - \frac{3\hbar^2 h_1}{2} - \frac{3}{2}\hbar^3 h_1^2\right) U[e_1, f_1] + \left(\frac{9}{2}\hbar^2 h_1^2 - 9\hbar^3 h_1^3\right) U[l_1, l_1] +$$
$$\left(9\hbar^2 h_1 - \frac{9}{2}\hbar^3 h_1^2\right) U[e_1, l_1, f_1] + \frac{9}{2}\hbar^3 h_1^3 U[l_1, l_1, l_1] + \left(\frac{9\hbar^2}{2} + \frac{9\hbar^3 h_1}{2}\right) U[e_1, e_1, f_1, f_1] +$$
$$\frac{27}{2}\hbar^3 h_1^2 U[e_1, l_1, l_1, f_1] + \frac{27}{2}\hbar^3 h_1 U[e_1, e_1, l_1, f_1, f_1] + \frac{9}{2}\hbar^3 U[e_1, e_1, e_1, f_1, f_1, f_1]\Big\}$$

```
Timing[$TD = 5; R4,1 ** R2,5 ** R6,3 // m[1, 2, 1] // m[1, 3, 1] // m[1, 4, 1] // m[1, 5, 1] // m[1, 6, 1]]
```

$$
\begin{aligned}
\{&11.625,\ \left(1 - 2\hbar h_1 + \hbar^2 h_1^2 + \tfrac{2}{3}\hbar^3 h_1^3 - \tfrac{5}{12}\hbar^4 h_1^4 - \tfrac{23}{30}\hbar^5 h_1^5\right) U[\,] + \\
&\left(3\hbar h_1 - 6\hbar^2 h_1^2 + 3\hbar^3 h_1^3 + 2\hbar^4 h_1^4 - \tfrac{5}{4}\hbar^5 h_1^5\right) U[l_1] + \left(3\hbar - \tfrac{3\hbar^2 h_1}{2} - \tfrac{3}{2}\hbar^3 h_1^2 + \tfrac{7}{8}\hbar^4 h_1^3 + \tfrac{61}{40}\hbar^5 h_1^4\right) U[e_1, f_1] + \\
&\left(\tfrac{9}{2}\hbar^2 h_1^2 - 9\hbar^3 h_1^3 + \tfrac{9}{2}\hbar^4 h_1^4 + 3\hbar^5 h_1^5\right) U[l_1, l_1] + \left(9\hbar^2 h_1 - \tfrac{9}{2}\hbar^3 h_1^2 - \tfrac{9}{2}\hbar^4 h_1^3 + \tfrac{21}{8}\hbar^5 h_1^4\right) U[e_1, l_1, f_1] + \\
&\left(\tfrac{9}{2}\hbar^3 h_1^3 - 9\hbar^4 h_1^4 + \tfrac{9}{2}\hbar^5 h_1^5\right) U[l_1, l_1, l_1] + \left(\tfrac{9\hbar^2}{2} + \tfrac{9\hbar^3 h_1}{2} + \tfrac{9}{8}\hbar^4 h_1^2 - \tfrac{3}{8}\hbar^5 h_1^3\right) U[e_1, e_1, f_1, f_1] + \\
&\left(\tfrac{27}{2}\hbar^3 h_1^2 - \tfrac{27}{4}\hbar^4 h_1^3 - \tfrac{27}{4}\hbar^5 h_1^4\right) U[e_1, l_1, l_1, f_1] + \left(\tfrac{27}{8}\hbar^4 h_1^4 - \tfrac{27}{4}\hbar^5 h_1^5\right) U[l_1, l_1, l_1, l_1] + \\
&\left(\tfrac{27\hbar^3 h_1}{2} + \tfrac{27}{2}\hbar^4 h_1^2 + \tfrac{27}{8}\hbar^5 h_1^3\right) U[e_1, e_1, l_1, f_1, f_1] + \left(\tfrac{27}{2}\hbar^4 h_1^3 - \tfrac{27}{4}\hbar^5 h_1^4\right) U[e_1, l_1, l_1, l_1, f_1] + \\
&\tfrac{81}{40}\hbar^5 h_1^5 U[l_1, l_1, l_1, l_1, l_1] + \left(\tfrac{9\hbar^3}{2} + \tfrac{45\hbar^4 h_1}{4} + \tfrac{117}{8}\hbar^5 h_1^2\right) U[e_1, e_1, e_1, f_1, f_1, f_1] + \\
&\left(\tfrac{81}{4}\hbar^4 h_1^2 + \tfrac{81}{4}\hbar^5 h_1^3\right) U[e_1, e_1, l_1, l_1, f_1, f_1] + \tfrac{81}{8}\hbar^5 h_1^4 U[e_1, l_1, l_1, l_1, l_1, f_1] + \\
&\left(\tfrac{27\hbar^4 h_1}{2} + \tfrac{135}{4}\hbar^5 h_1^2\right) U[e_1, e_1, e_1, l_1, f_1, f_1, f_1] + \tfrac{81}{4}\hbar^5 h_1^3 U[e_1, e_1, l_1, l_1, l_1, f_1, f_1] + \\
&\left(\tfrac{27\hbar^4}{8} + \tfrac{27\hbar^5 h_1}{2}\right) U[e_1, e_1, e_1, e_1, f_1, f_1, f_1, f_1] + \tfrac{81}{4}\hbar^5 h_1^2 U[e_1, e_1, e_1, l_1, l_1, f_1, f_1, f_1] + \\
&\tfrac{81}{8}\hbar^5 h_1 U[e_1, e_1, e_1, e_1, l_1, f_1, f_1, f_1, f_1] + \tfrac{81}{40}\hbar^5 U[e_1, e_1, e_1, e_1, e_1, f_1, f_1, f_1, f_1, f_1]\}
\end{aligned}
$$

Ordering Symbols

```
𝕆[poly_, specs___] := Module[{vs, us, z},
  vs = Join @@ (First /@ {specs});
  us = Join @@ ({specs} /. (l_ → s_) ⧴ (l /. x_i_ ⧴ x_s));
  Simp@Total[CoefficientRules[Normal@Series[poly, {ℏ, 0, $TD}], vs] /. (p_ → c_) ⧴ c UU @@ (us^p)]
 ]
```

Theorem. $R = e^{h\otimes l + e\otimes f} = \mathbb{O}\left(\exp\left(hl + \frac{e^h-1}{h}\, ef\right) \middle| \; e\otimes lf\right)$.

Brute Proof.

```
$TD = 8; 𝕆[Exp[ℏ h1 l2 + (e^(ℏ h1) - 1)/h1 e1 f2], {e1} → 1, {l2, f2} → 2] == R1,2
```

True

Debts.

1. How does the $\mathbb{O}$ program work?
2. Why is the above theorem true?
3. What do you do with it?

old above / new below

Theorem. $R = e^{h\otimes l + e\otimes f} = \mathbb{O}\left(\exp\left(hl + \frac{e^h-1}{h}\, ef\right) \middle| \; e\otimes lf\right)$.

Gentler Proof, Presented Brutely.

```
MatrixForm /@ {h ρl + e ρf, h ρl, (e^h - 1)/h e ρf}
```

$$
\left\{ \begin{pmatrix} 0 & e & 0 \\ 0 & h & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{e\,(-1+e^h)}{h} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}
$$

$$\text{MatrixForm /@ }\left\{\text{MatrixExp}[h\,\rho l + e\,\rho f],\ \text{MatrixExp}[h\,\rho l].\text{MatrixExp}\left[\frac{e^{h}-1}{h}\,e\,\rho f\right]\right\}$$

$$\left\{\begin{pmatrix} 1 & \frac{e\,(-1+e^{h})}{h} & 0 \\ 0 & e^{h} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \frac{e\,(-1+e^{h})}{h} & 0 \\ 0 & e^{h} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

The Invariant of the Trefoil, Again

$$\text{Timing}\Big[\$TD = 3;\ \mathbb{O}\Big[\text{Exp}\Big[\hbar\,h\,l_1 + \frac{e^{\hbar h}-1}{h}\,e_4\,f_1 + \hbar\,h\,l_5 + \frac{e^{\hbar h}-1}{h}\,e_2\,f_5 + \hbar\,h\,l_3 + \frac{e^{\hbar h}-1}{h}\,e_6\,f_3\Big],\ \{l_1, f_1, e_2, l_3, f_3, e_4, l_5, f_5, e_6\} \to 1\Big]\ /.\ h_1 \to h\Big]$$

$$\Big\{0.078125,\ \left(1 - 2\,h\,\hbar + h^2\,\hbar^2 + \frac{2\,h^3\,\hbar^3}{3}\right) U[\,] + \left(3\,h\,\hbar - 6\,h^2\,\hbar^2 + 3\,h^3\,\hbar^3\right) U[l_1] + \left(3\,\hbar - \frac{3\,h\,\hbar^2}{2} - \frac{3\,h^2\,\hbar^3}{2}\right) U[e_1, f_1] + \left(\frac{9\,h^2\,\hbar^2}{2} - 9\,h^3\,\hbar^3\right) U[l_1, l_1] + \left(9\,h\,\hbar^2 - \frac{9\,h^2\,\hbar^3}{2}\right) U[e_1, l_1, f_1] + \frac{9}{2}\,h^3\,\hbar^3\,U[l_1, l_1, l_1] + \left(\frac{9\,\hbar^2}{2} + \frac{9\,h\,\hbar^3}{2}\right) U[e_1, e_1, f_1, f_1] + \frac{27}{2}\,h^2\,\hbar^3\,U[e_1, l_1, l_1, f_1] + \frac{27}{2}\,h\,\hbar^3\,U[e_1, e_1, l_1, f_1, f_1] + \frac{9}{2}\,\hbar^3\,U[e_1, e_1, e_1, f_1, f_1, f_1]\Big\}$$

$$\text{Timing}\Big[\$TD = 7;\ \mathbb{O}\Big[\text{Exp}\Big[\hbar\,h\,l_1 + \frac{e^{\hbar h}-1}{h}\,e_4\,f_1 + \hbar\,h\,l_5 + \frac{e^{\hbar h}-1}{h}\,e_2\,f_5 + \hbar\,h\,l_3 + \frac{e^{\hbar h}-1}{h}\,e_6\,f_3\Big],\ \{l_1, f_1, e_2, l_3, f_3, e_4, l_5, f_5, e_6\} \to 1\Big]\ /.\ h_1 \to h\Big]$$

$$\Big\{11.2969,\ \left(1-2h\hbar+h^2\hbar^2+\frac{2h^3\hbar^3}{3}-\frac{5h^4\hbar^4}{12}-\frac{23h^5\hbar^5}{30}+\frac{151h^6\hbar^6}{360}+\frac{871h^7\hbar^7}{1260}\right)U[\,]+$$
$$\left(3h\hbar-6h^2\hbar^2+3h^3\hbar^3+2h^4\hbar^4-\frac{5h^5\hbar^5}{4}-\frac{23h^6\hbar^6}{10}+\frac{151h^7\hbar^7}{120}\right)U[l_1]+$$
$$\left(3\hbar-\frac{3h\hbar^2}{2}-\frac{3h^2\hbar^3}{2}+\frac{7h^3\hbar^4}{8}+\frac{61h^4\hbar^5}{40}-\frac{67h^5\hbar^6}{80}-\frac{2323h^6\hbar^7}{1680}\right)U[e_1,f_1]+$$
$$\left(\frac{9h^2\hbar^2}{2}-9h^3\hbar^3+\frac{9h^4\hbar^4}{2}+3h^5\hbar^5-\frac{15h^6\hbar^6}{8}-\frac{69h^7\hbar^7}{20}\right)U[l_1,l_1]+$$
$$\left(9h\hbar^2-\frac{9h^2\hbar^3}{2}-\frac{9h^3\hbar^4}{2}+\frac{21h^4\hbar^5}{8}+\frac{183h^5\hbar^6}{40}-\frac{201h^6\hbar^7}{80}\right)U[e_1,l_1,f_1]+$$
$$\left(\frac{9h^3\hbar^3}{2}-9h^4\hbar^4+\frac{9h^5\hbar^5}{2}+3h^6\hbar^6-\frac{15h^7\hbar^7}{8}\right)U[l_1,l_1,l_1]+$$
$$\left(\frac{9\hbar^2}{2}+\frac{9h\hbar^3}{2}+\frac{9h^2\hbar^4}{8}-\frac{3h^3\hbar^5}{8}+\frac{111h^4\hbar^6}{80}+\frac{129h^5\hbar^7}{80}\right)U[e_1,e_1,f_1,f_1]+$$
$$\left(\frac{27h^2\hbar^3}{2}-\frac{27h^3\hbar^4}{4}-\frac{27h^4\hbar^5}{4}+\frac{63h^5\hbar^6}{16}+\frac{549h^6\hbar^7}{80}\right)U[e_1,l_1,l_1,f_1]+$$
$$\left(\frac{27h^4\hbar^4}{8}-\frac{27h^5\hbar^5}{4}+\frac{27h^6\hbar^6}{8}+\frac{9h^7\hbar^7}{4}\right)U[l_1,l_1,l_1,l_1]+$$
$$\left(\frac{27h\hbar^3}{2}+\frac{27h^2\hbar^4}{2}+\frac{27h^3\hbar^5}{8}-\frac{9h^4\hbar^6}{8}+\frac{333h^5\hbar^7}{80}\right)U[e_1,e_1,l_1,f_1,f_1]+$$
$$\left(\frac{27h^3\hbar^4}{2}-\frac{27h^4\hbar^5}{4}-\frac{27h^5\hbar^6}{4}+\frac{63h^6\hbar^7}{16}\right)U[e_1,l_1,l_1,l_1,f_1]+$$
$$\left(\frac{81h^5\hbar^5}{40}-\frac{81h^6\hbar^6}{20}+\frac{81h^7\hbar^7}{40}\right)U[l_1,l_1,l_1,l_1,l_1]+$$
$$\left(\frac{9\hbar^3}{2}+\frac{45h\hbar^4}{4}+\frac{117h^2\hbar^5}{8}+\frac{105h^3\hbar^6}{8}+\frac{849h^4\hbar^7}{80}\right)U[e_1,e_1,e_1,f_1,f_1,f_1]+$$
$$\left(\frac{81h^2\hbar^4}{4}+\frac{81h^3\hbar^5}{4}+\frac{81h^4\hbar^6}{16}-\frac{27h^5\hbar^7}{16}\right)U[e_1,e_1,l_1,l_1,f_1,f_1]+$$
$$\left(\frac{81h^4\hbar^5}{8}-\frac{81h^5\hbar^6}{16}-\frac{81h^6\hbar^7}{16}\right)U[e_1,l_1,l_1,l_1,l_1,f_1]+\left(\frac{81h^6\hbar^6}{80}-\frac{81h^7\hbar^7}{40}\right)U[l_1,l_1,l_1,l_1,l_1,l_1]+$$
$$\left(\frac{27h\hbar^4}{2}+\frac{135h^2\hbar^5}{4}+\frac{351h^3\hbar^6}{8}+\frac{315h^4\hbar^7}{8}\right)U[e_1,e_1,e_1,l_1,f_1,f_1,f_1]+$$
$$\left(\frac{81h^3\hbar^5}{4}+\frac{81h^4\hbar^6}{4}+\frac{81h^5\hbar^7}{16}\right)U[e_1,e_1,l_1,l_1,l_1,f_1,f_1]+\left(\frac{243h^5\hbar^6}{40}-\frac{243h^6\hbar^7}{80}\right)U[e_1,l_1,l_1,l_1,l_1,l_1,f_1]+$$
$$\frac{243}{560}h^7\hbar^7U[l_1,l_1,l_1,l_1,l_1,l_1,l_1]+\left(\frac{27\hbar^4}{8}+\frac{27h\hbar^5}{2}+\frac{459h^2\hbar^6}{16}+\frac{171h^3\hbar^7}{4}\right)U[e_1,e_1,e_1,e_1,f_1,f_1,f_1,f_1]+$$
$$\left(\frac{81h^2\hbar^5}{4}+\frac{405h^3\hbar^6}{8}+\frac{1053h^4\hbar^7}{16}\right)U[e_1,e_1,e_1,l_1,l_1,f_1,f_1,f_1]+$$
$$\left(\frac{243h^4\hbar^6}{16}+\frac{243h^5\hbar^7}{16}\right)U[e_1,e_1,l_1,l_1,l_1,l_1,f_1,f_1]+\frac{243}{80}h^6\hbar^7U[e_1,l_1,l_1,l_1,l_1,l_1,l_1,f_1]+$$
$$\left(\frac{81h\hbar^5}{8}+\frac{81h^2\hbar^6}{2}+\frac{1377h^3\hbar^7}{16}\right)U[e_1,e_1,e_1,e_1,l_1,f_1,f_1,f_1,f_1]+$$
$$\left(\frac{81h^3\hbar^6}{4}+\frac{405h^4\hbar^7}{8}\right)U[e_1,e_1,e_1,l_1,l_1,l_1,f_1,f_1,f_1]+\frac{729}{80}h^5\hbar^7U[e_1,e_1,l_1,l_1,l_1,l_1,l_1,f_1,f_1]+$$
$$\left(\frac{81\hbar^5}{40}+\frac{891h\hbar^6}{80}+\frac{162h^2\hbar^7}{5}\right)U[e_1,e_1,e_1,e_1,e_1,f_1,f_1,f_1,f_1,f_1]+$$
$$\left(\frac{243h^2\hbar^6}{16}+\frac{243h^3\hbar^7}{4}\right)U[e_1,e_1,e_1,e_1,l_1,l_1,f_1,f_1,f_1,f_1]+\frac{243}{16}h^4\hbar^7U[e_1,e_1,e_1,l_1,l_1,l_1,l_1,f_1,f_1,f_1]+$$
$$\left(\frac{243h\hbar^6}{40}+\frac{2673h^2\hbar^7}{80}\right)U[e_1,e_1,e_1,e_1,e_1,l_1,f_1,f_1,f_1,f_1,f_1]+$$
$$\frac{243}{16}h^3\hbar^7U[e_1,e_1,e_1,e_1,l_1,l_1,l_1,f_1,f_1,f_1,f_1]+$$
$$\left(\frac{81\hbar^6}{80}+\frac{567h\hbar^7}{80}\right)U[e_1,e_1,e_1,e_1,e_1,e_1,f_1,f_1,f_1,f_1,f_1,f_1]+$$
$$\frac{729}{80}h^2\hbar^7U[e_1,e_1,e_1,e_1,e_1,l_1,l_1,f_1,f_1,f_1,f_1,f_1]+$$
$$\frac{243}{80}h\hbar^7U[e_1,e_1,e_1,e_1,e_1,e_1,l_1,f_1,f_1,f_1,f_1,f_1,f_1]+$$
$$\frac{243}{560}\hbar^7U[e_1,e_1,e_1,e_1,e_1,e_1,e_1,f_1,f_1,f_1,f_1,f_1,f_1,f_1]\Big\}$$

The Big $\mathfrak{g}_0$ Lemma.

1. $\mathbb{O}\left(e^{\gamma l+\beta e} \mid le\right)=\mathbb{O}\left(e^{\gamma l+e^{\gamma}\beta e} \mid el\right)$.
2. $\mathbb{O}\left(e^{\gamma l+\beta f} \mid fl\right)=\mathbb{O}\left(e^{\gamma l+e^{\gamma}\beta f} \mid lf\right)$.
3. $\mathbb{O}\left(e^{\beta e+\alpha f+\delta ef} \mid fe\right)=\mathbb{O}\left(\nu e^{\nu(-\alpha\beta h+\beta e+\alpha f+\delta ef)} \mid ef\right)$, with $\nu=(1+h\delta)^{-1}$.

Brute Proof.

$\$TD = 3;\ \mathbb{O}\left[e^{\hbar\gamma l_1+\beta\hbar e_1}, \{l_1, e_1\}\to 1\right]$

$$U[\,]+\left(\beta\hbar+\beta\gamma\hbar^2+\frac{1}{2}\beta\gamma^2\hbar^3\right)U[e_1]+\gamma\hbar\,U[l_1]+\left(\frac{\beta^2\hbar^2}{2}+\beta^2\gamma\hbar^3\right)U[e_1,e_1]+\left(\beta\gamma\hbar^2+\beta\gamma^2\hbar^3\right)U[e_1,l_1]+\frac{1}{2}\gamma^2\hbar^2\,U[l_1,l_1]+\frac{1}{6}\beta^3\hbar^3\,U[e_1,e_1,e_1]+\frac{1}{2}\beta^2\gamma\hbar^3\,U[e_1,e_1,l_1]+\frac{1}{2}\beta\gamma^2\hbar^3\,U[e_1,l_1,l_1]+\frac{1}{6}\gamma^3\hbar^3\,U[l_1,l_1,l_1]$$

$\$TD = 3;\ \mathbb{O}\left[e^{\hbar\gamma l_1+e^{\hbar\gamma}\beta\hbar e_1}, \{e_1, l_1\}\to 1\right]$

$$U[\,]+\left(\beta\hbar+\beta\gamma\hbar^2+\frac{1}{2}\beta\gamma^2\hbar^3\right)U[e_1]+\gamma\hbar\,U[l_1]+\left(\frac{\beta^2\hbar^2}{2}+\beta^2\gamma\hbar^3\right)U[e_1,e_1]+\left(\beta\gamma\hbar^2+\beta\gamma^2\hbar^3\right)U[e_1,l_1]+\frac{1}{2}\gamma^2\hbar^2\,U[l_1,l_1]+\frac{1}{6}\beta^3\hbar^3\,U[e_1,e_1,e_1]+\frac{1}{2}\beta^2\gamma\hbar^3\,U[e_1,e_1,l_1]+\frac{1}{2}\beta\gamma^2\hbar^3\,U[e_1,l_1,l_1]+\frac{1}{6}\gamma^3\hbar^3\,U[l_1,l_1,l_1]$$

$\$TD = 6;\ \mathbb{O}\left[e^{\hbar\gamma l_1+\beta\hbar e_1}, \{l_1, e_1\}\to 1\right] == \mathbb{O}\left[e^{\hbar\gamma l_1+e^{\hbar\gamma}\beta\hbar e_1}, \{e_1, l_1\}\to 1\right]$

True

$\$TD = 6;\ \mathbb{O}\left[e^{\hbar\gamma l_1+\beta\hbar f_1}, \{f_1, l_1\}\to 1\right] == \mathbb{O}\left[e^{\hbar\gamma l_1+e^{n\gamma}\beta\hbar f_1}, \{l_1, f_1\}\to 1\right]$

True

$\$TD = 3;\ \mathbb{O}\left[e^{\hbar(\beta e_1+\alpha f_1+\delta e_1 f_1)}, \{f_1, e_1\}\to 1\right]$

$$\left(1-\delta\hbar h_1-\alpha\beta\hbar^2h_1+\delta^2\hbar^2h_1^2+2\alpha\beta\delta\hbar^3h_1^2-\delta^3\hbar^3h_1^3\right)U[\,]+\left(\beta\hbar-2\beta\delta\hbar^2h_1-\alpha\beta^2\hbar^3h_1+3\beta\delta^2\hbar^3h_1^2\right)U[e_1]+\left(\alpha\hbar-2\alpha\delta\hbar^2h_1-\alpha^2\beta\hbar^3h_1+3\alpha\delta^2\hbar^3h_1^2\right)U[f_1]+\left(\frac{\beta^2\hbar^2}{2}-\frac{3}{2}\beta^2\delta\hbar^3h_1\right)U[e_1,e_1]+\left(\delta\hbar+\alpha\beta\hbar^2-2\delta^2\hbar^2h_1-4\alpha\beta\delta\hbar^3h_1+3\delta^3\hbar^3h_1^2\right)U[e_1,f_1]+\left(\frac{\alpha^2\hbar^2}{2}-\frac{3}{2}\alpha^2\delta\hbar^3h_1\right)U[f_1,f_1]+\frac{1}{6}\beta^3\hbar^3\,U[e_1,e_1,e_1]+\left(\beta\delta\hbar^2+\frac{1}{2}\alpha\beta^2\hbar^3-3\beta\delta^2\hbar^3h_1\right)U[e_1,e_1,f_1]+\left(\alpha\delta\hbar^2+\frac{1}{2}\alpha^2\beta\hbar^3-3\alpha\delta^2\hbar^3h_1\right)U[e_1,f_1,f_1]+\frac{1}{6}\alpha^3\hbar^3\,U[f_1,f_1,f_1]+\frac{1}{2}\beta^2\delta\hbar^3\,U[e_1,e_1,e_1,f_1]+\left(\frac{\delta^2\hbar^2}{2}+\alpha\beta\delta\hbar^3-\frac{3}{2}\delta^3\hbar^3h_1\right)U[e_1,e_1,f_1,f_1]+\frac{1}{2}\alpha^2\delta\hbar^3\,U[e_1,f_1,f_1,f_1]+\frac{1}{2}\beta\delta^2\hbar^3\,U[e_1,e_1,e_1,f_1,f_1]+\frac{1}{2}\alpha\delta^2\hbar^3\,U[e_1,e_1,f_1,f_1,f_1]+\frac{1}{6}\delta^3\hbar^3\,U[e_1,e_1,e_1,f_1,f_1,f_1]$$

$\$TD = 6;\ \text{With}\left[\left\{\nu = (1+\hbar h\delta)^{-1}\right\},\right.$
$\quad \mathbb{O}\left[e^{\hbar(\beta e_1+\alpha f_1+\delta e_1 f_1)}, \{f_1, e_1\}\to 1\right] == \mathbb{O}\left[\nu\, e^{\hbar\nu(-\hbar h\alpha\beta+\beta e_1+\alpha f_1+\delta e_1 f_1)}, \{e_1, f_1\}\to 1\right]\ /.\ h_1\to h$
$\left.\right]$

True

The Big g_0 Lemma. (again)

1. $\mathbb{O}\left(e^{\gamma l+\beta e} \mid le\right)=\mathbb{O}\left(e^{\gamma l+e^{\gamma}\beta e} \mid el\right)$.
2. $\mathbb{O}\left(e^{\gamma l+\beta f} \mid fl\right)=\mathbb{O}\left(e^{\gamma l+e^{\gamma}\beta f} \mid lf\right)$.
3. $\mathbb{O}\left(e^{\beta e+\alpha f+\delta ef} \mid fe\right)=\mathbb{O}\left(\nu e^{\nu(-\alpha\beta h+\beta e+\alpha f+\delta ef)} \mid ef\right)$, with $\nu=(1+h\delta)^{-1}$.

Gentler Proofs of 1 & 2 and of 3 at δ=0.

$\text{MatrixForm /@ }\{\text{MatrixExp}[\gamma\,\rho l].\text{MatrixExp}[\beta\,\rho e],\ \text{MatrixExp}\left[e^{\gamma}\beta\,\rho e\right].\text{MatrixExp}[\gamma\,\rho l]\}$

$$\left\{\begin{pmatrix}1&0&0\\0&e^{\gamma}&e^{\gamma}\beta\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0\\0&e^{\gamma}&e^{\gamma}\beta\\0&0&1\end{pmatrix}\right\}$$

```
MatrixForm /@ {MatrixExp[β ρf].MatrixExp[γ ρl], MatrixExp[γ ρl].MatrixExp[e^γ β ρf]}
```

$$\left\{\begin{pmatrix}1 & e^{\gamma}\beta & 0\\ 0 & e^{\gamma} & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & e^{\gamma}\beta & 0\\ 0 & e^{\gamma} & 0\\ 0 & 0 & 1\end{pmatrix}\right\}$$

```
MatrixForm /@ {MatrixExp[α ρf].MatrixExp[β ρe], MatrixExp[-α β ρh].MatrixExp[β ρe].MatrixExp[α ρf]}
```

$$\left\{\begin{pmatrix}1 & \alpha & \alpha\beta\\ 0 & 1 & \beta\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & \alpha & \alpha\beta\\ 0 & 1 & \beta\\ 0 & 0 & 1\end{pmatrix}\right\}$$

The Main $\mathfrak{g}_0$ Theorem.

Raw Version. The $\mathfrak{g}_0$ invariant of any S-component tangle T can be written in the form $Z(T) = \mathbb{O}\left(\omega e^{L+Q} \mid \otimes_{i\in S} e_i l_i f_i\right)$, where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i = e^{h_i}$), where $L = \sum a_{ij} h_i l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij} and where $Q = \sum b_{ij} e_i f_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij}. Furthermore, after setting $h_i = h$ and $t_i = t$ for all i, the invariant $Z(T)$ is poly-time computable.

Proof. Indeed,

0. $R = e^{h\otimes l+e\otimes f} = \mathbb{O}\left(\exp\left(\text{hl} + \frac{e^h-1}{h}\text{ef}\right) \mid e\otimes\text{lf}\right)$,
1. $\mathbb{O}\left(e^{\gamma\text{l}+\beta\text{e}} \mid \text{le}\right) = \mathbb{O}\left(e^{\gamma\text{l}+e^{\gamma}\beta\text{e}} \mid \text{el}\right)$,
2. $\mathbb{O}\left(e^{\gamma\text{l}+\beta\text{f}} \mid \text{fl}\right) = \mathbb{O}\left(e^{\gamma\text{l}+e^{\gamma}\beta\text{f}} \mid \text{lf}\right)$,
3. $\mathbb{O}\left(e^{\beta\text{e}+\alpha\text{f}+\delta\text{ef}} \mid \text{fe}\right) = \mathbb{O}\left(\nu e^{\nu(-\alpha\beta\text{h}+\beta\text{e}+\alpha\text{f}+\delta\text{ef})} \mid \text{ef}\right)$, with $\nu = (1+\text{h}\delta)^{-1}$,

and the rest is straight-forward.

Polished Version. With $\overline{e} = \frac{(e^h-1)}{h} e$, the $\mathfrak{g}_0$ invariant of any S-component tangle T can be written in the form $Z(T) = \mathbb{O}\left(\omega^{-1} e^{L+\omega^{-1}Q} \mid \otimes_{i\in S} \overline{e}_i l_i f_i\right)$, where ω is a scalar (meaning, a polynomial in the variables $t_i = e^{h_i}$), where $L = \sum a_{ij} h_i l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij} and where $Q = \sum b_{ij} \overline{e}_i f_j$ is a balanced quadratic in the variables $\overline{e}_i$ and f_j with scalar coefficients b_{ij}. Furthermore, after setting $t_i = t$ for all i, the invariant $Z(T)$ is poly-time computable.

Debts.

1. Implement (and verify!).
2. Really prove part 3 of the big $\mathfrak{g}_0$-lemma.