Pensieve header: Γ-Calculus.

```
<< KnotTheory`
```

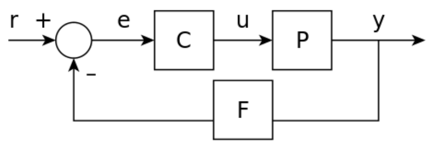
```
Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

Linear Control Theory

From the Wikipedia entry “Control Theory” (edited):

Closed-loop transfer function [edit]

This is called a single-input-single-output (*SISO*) control system; *MIMO* (i.e., Multi-Input-Multi-Output) systems, with more than one input/output, are common. In such cases variables are represented through vectors instead of simple scalar values. For some distributed parameter systems the vectors may be infinite-dimensional (typically functions).



Solving for $Y(s)$ in terms of $R(s)$ gives

$$Y(s)=\left(\frac{P(s)C(s)}{1+F(s)P(s)C(s)}\right)R(s)=H(s)R(s).$$

$$M=\begin{pmatrix}\alpha & \beta & \theta\\ \gamma & \delta & \epsilon\\ \phi & \psi & \Xi\end{pmatrix};\ \{y_1, y_2, y_3\} == M.\{x_1, x_2, x_3\}$$

$$\{y_1, y_2, y_3\} == \{\alpha x_1 + \beta x_2 + \theta x_3,\ \gamma x_1 + \delta x_2 + \epsilon x_3,\ \phi x_1 + \psi x_2 + \Xi x_3\}$$

```
? Thread
```

Thread[*f*[*args*]] "threads" *f* over any lists that appear in *args*.
Thread[*f*[*args*], *h*] threads *f* over any objects with head *h* that appear in *args*.
Thread[*f*[*args*], *h*, *n*] threads *f* over objects with head *h* that appear in the first *n* *args*. >>

```
Thread[f[{1, 2, 3}, {4, 5, 6}]]
```

{f[1, 4], f[2, 5], f[3, 6]}

$$\text{Thread}[\{y_1, y_2, y_3\} == M.\{x_1, x_2, x_3\}]$$

$$\{y_1 == \alpha x_1 + \beta x_2 + \theta x_3,\ y_2 == \gamma x_1 + \delta x_2 + \epsilon x_3,\ y_3 == \phi x_1 + \psi x_2 + \Xi x_3\}$$

$$\text{eqns} = \text{Thread}[\{y_1, y_2, y_3\} == M.\{x_1, x_2, x_3\}] \cup \{y_1 == x_2\}$$

$$\{y_1 == x_2,\ y_1 == \alpha x_1 + \beta x_2 + \theta x_3,\ y_2 == \gamma x_1 + \delta x_2 + \epsilon x_3,\ y_3 == \phi x_1 + \psi x_2 + \Xi x_3\}$$

$$\text{Solve}[\text{eqns}, \{y_2, y_3\}]$$

{}

$$\text{Solve}[\text{eqns}, \{x_2, y_1, y_2, y_3\}]$$

$$\left\{\left\{x_2 \to -\frac{\alpha x_1 + \theta x_3}{-1+\beta},\ y_1 \to -\frac{\alpha x_1 + \theta x_3}{-1+\beta},\ y_2 \to -\frac{1}{-1+\beta}(\gamma x_1 - \beta \gamma x_1 + \alpha \delta x_1 + \epsilon x_3 - \beta \epsilon x_3 + \delta \theta x_3),\right.\right.$$
$$\left.\left. y_3 \to -\frac{1}{-1+\beta}(\phi x_1 - \beta \phi x_1 + \alpha \psi x_1 + \Xi x_3 - \beta \Xi x_3 + \theta \psi x_3)\right\}\right\}$$

```
sol = Solve[eqns, {x2, y1, y2, y3}][[1]]
```

$$\left\{x_2 \to -\frac{\alpha x_1 + \Theta x_3}{-1+\beta},\ y_1 \to -\frac{\alpha x_1 + \Theta x_3}{-1+\beta},\ y_2 \to -\frac{1}{-1+\beta}(\gamma x_1 - \beta\gamma x_1 + \alpha\delta x_1 + \epsilon x_3 - \beta\epsilon x_3 + \delta\Theta x_3),\ y_3 \to -\frac{1}{-1+\beta}(\phi x_1 - \beta\phi x_1 + \alpha\psi x_1 + \Xi x_3 - \beta\Xi x_3 + \Theta\psi x_3)\right\}$$

```
sol = First[Solve[eqns, {x2, y1, y2, y3}]]
```

$$\left\{x_2 \to -\frac{\alpha x_1 + \Theta x_3}{-1+\beta},\ y_1 \to -\frac{\alpha x_1 + \Theta x_3}{-1+\beta},\ y_2 \to -\frac{1}{-1+\beta}(\gamma x_1 - \beta\gamma x_1 + \alpha\delta x_1 + \epsilon x_3 - \beta\epsilon x_3 + \delta\Theta x_3),\ y_3 \to -\frac{1}{-1+\beta}(\phi x_1 - \beta\phi x_1 + \alpha\psi x_1 + \Xi x_3 - \beta\Xi x_3 + \Theta\psi x_3)\right\}$$

```
y2 /. sol
```

$$-\frac{1}{-1+\beta}(\gamma x_1 - \beta\gamma x_1 + \alpha\delta x_1 + \epsilon x_3 - \beta\epsilon x_3 + \delta\Theta x_3)$$

```
Coefficient[y2 /. sol, x3]
```

$$-\frac{\epsilon - \beta\epsilon + \delta\Theta}{-1+\beta}$$

```
Coefficient[y2 /. sol, x3] // Expand
```

$$-\frac{\epsilon}{-1+\beta} + \frac{\beta\epsilon}{-1+\beta} - \frac{\delta\Theta}{-1+\beta}$$

```
Coefficient[y2 /. sol, x3] // Simplify
```

$$\frac{\epsilon - \beta\epsilon + \delta\Theta}{1-\beta}$$

```
Coefficient[y2 /. sol, x3] // Factor
```

$$\frac{-\epsilon + \beta\epsilon - \delta\Theta}{-1+\beta}$$

```
Coefficient[y2 /. sol, x3] // Apart
```

$$\epsilon - \frac{\delta\Theta}{-1+\beta}$$

```
Table[Coefficient[yi /. sol, xj] // Apart, {i, {2, 3}}, {j, {1, 3}}] // MatrixForm
```

$$\begin{pmatrix} \gamma - \frac{\alpha\delta}{-1+\beta} & \epsilon - \frac{\delta\Theta}{-1+\beta} \\ \phi - \frac{\alpha\psi}{-1+\beta} & \Xi - \frac{\Theta\psi}{-1+\beta} \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta & \Theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \to \begin{pmatrix} \gamma - \frac{\alpha\delta}{-1+\beta} & \epsilon - \frac{\delta\Theta}{-1+\beta} \\ \phi - \frac{\alpha\psi}{-1+\beta} & \Xi - \frac{\Theta\psi}{-1+\beta} \end{pmatrix}$$

Γ0-Calculus

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \Theta_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \Theta_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \Theta_3 \\ \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix}$$

$$\{\{\alpha_{11}, \alpha_{12}, \alpha_{13}, \Theta_1\}, \{\alpha_{21}, \alpha_{22}, \alpha_{23}, \Theta_2\}, \{\alpha_{31}, \alpha_{32}, \alpha_{33}, \Theta_3\}, \{\phi_1, \phi_2, \phi_3, \Xi\}\}$$

$$Q = \text{Expand}\left[\{t_1, t_2, t_3, t_S\}.\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \Theta_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \Theta_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \Theta_3 \\ \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix}.\{h_1, h_2, h_3, h_S\}\right]$$

$$\Xi h_S t_S + h_1 t_1 \alpha_{11} + h_2 t_1 \alpha_{12} + h_3 t_1 \alpha_{13} + h_1 t_2 \alpha_{21} + h_2 t_2 \alpha_{22} + h_3 t_2 \alpha_{23} + h_1 t_3 \alpha_{31} + h_2 t_3 \alpha_{32} + h_3 t_3 \alpha_{33} + h_S t_1 \Theta_1 + h_S t_2 \Theta_2 + h_S t_3 \Theta_3 + h_1 t_S \phi_1 + h_2 t_S \phi_2 + h_3 t_S \phi_3$$

$\partial_{t_2} Q$

$h_1\, \alpha_{21} + h_2\, \alpha_{22} + h_3\, \alpha_{23} + h_S\, \Theta_2$

$\partial_{h_3} \partial_{t_2} Q$

α_{23}

$$\zeta = \Gamma 0\left[\{t_1, t_2, t_3, t_S\}.\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \Theta_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \Theta_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \Theta_3 \\ \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix}.\{h_1, h_2, h_3, h_S\}\right]$$

$$\Gamma 0\left[h_S\,(\Xi\, t_S + t_1\,\Theta_1 + t_2\,\Theta_2 + t_3\,\Theta_3) + h_1\,(t_1\,\alpha_{11} + t_2\,\alpha_{21} + t_3\,\alpha_{31} + t_S\,\phi_1) + h_2\,(t_1\,\alpha_{12} + t_2\,\alpha_{22} + t_3\,\alpha_{32} + t_S\,\phi_2) + h_3\,(t_1\,\alpha_{13} + t_2\,\alpha_{23} + t_3\,\alpha_{33} + t_S\,\phi_3)\right]$$

```
Γ0 /: Collect[Γ0[λ_]] := Γ0[Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ0[λ_]] := Module[{S, M},
   S = Union@Cases[Γ0[λ], (h | t)_a_ :→ a, ∞];
   M = Outer[Factor[∂_{h_#1 t_#2} λ] &, S, S];
   M = Prepend[M, t_# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h_# & /@ S, "Γ0"]];
   M // MatrixForm];
```

ζ

$$\begin{pmatrix} \Gamma 0 & h_1 & h_2 & h_3 & h_S \\ t_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \Theta_1 \\ t_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \Theta_2 \\ t_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \Theta_3 \\ t_S & \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix}$$

$m_{a\_b\_\to c\_}[\Gamma 0[\lambda\_]] := \text{Module}\Big[\{\alpha, \beta, \gamma, \delta, \theta, \epsilon, \phi, \psi, \Xi\},$

$$\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} = \begin{pmatrix} \partial_{t_a,h_a}\lambda & \partial_{t_a,h_b}\lambda & \partial_{t_a}\lambda \\ \partial_{t_b,h_a}\lambda & \partial_{t_b,h_b}\lambda & \partial_{t_b}\lambda \\ \partial_{h_a}\lambda & \partial_{h_b}\lambda & \lambda \end{pmatrix} \;/.\; (t\,|\,h)_{a|b} \to 0;$$

$$\Gamma 0\left[\{t_c, 1\}.\begin{pmatrix} \gamma - \frac{\alpha\,\delta}{-1+\beta} & \epsilon - \frac{\delta\,\theta}{-1+\beta} \\ \phi - \frac{\alpha\,\psi}{-1+\beta} & \Xi - \frac{\theta\,\psi}{-1+\beta} \end{pmatrix}.\{h_c, 1\}\right] \text{ // Collect}\Big];$$

ζ // $m_{1,2\to 5}$

$$\begin{pmatrix} \Gamma 0 & h_3 & h_5 & h_S \\ t_3 & \frac{-\alpha_{13}\,\alpha_{32}-\alpha_{33}+\alpha_{12}\,\alpha_{33}}{-1+\alpha_{12}} & \frac{-\alpha_{31}+\alpha_{12}\,\alpha_{31}-\alpha_{11}\,\alpha_{32}}{-1+\alpha_{12}} & \frac{-\alpha_{32}\,\Theta_1-\Theta_3+\alpha_{12}\,\Theta_3}{-1+\alpha_{12}} \\ t_5 & \frac{-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}}{-1+\alpha_{12}} & \frac{-\alpha_{21}+\alpha_{12}\,\alpha_{21}-\alpha_{11}\,\alpha_{22}}{-1+\alpha_{12}} & \frac{-\alpha_{22}\,\Theta_1-\Theta_2+\alpha_{12}\,\Theta_2}{-1+\alpha_{12}} \\ t_S & \frac{-\alpha_{13}\,\phi_2-\phi_3+\alpha_{12}\,\phi_3}{-1+\alpha_{12}} & \frac{-\phi_1+\alpha_{12}\,\phi_1-\alpha_{11}\,\phi_2}{-1+\alpha_{12}} & \frac{-\Xi+\Xi\,\alpha_{12}-\Theta_1\,\phi_2}{-1+\alpha_{12}} \end{pmatrix}$$

(ζ // $m_{12\to 1}$ // $m_{13\to 1}$)

$$\begin{pmatrix} \Gamma 0 & h_1 & h_S \\ t_1 & \frac{\alpha_{31}-\alpha_{12}\,\alpha_{31}-\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{23}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{11}\,\alpha_{32}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{21}\,\alpha_{33}-\alpha_{12}\,\alpha_{21}\,\alpha_{33}+\alpha_{11}\,\alpha_{22}\,\alpha_{33}}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} & \frac{\alpha_{32}\,\Theta_1-\alpha_{23}\,\alpha_{32}\,\Theta_1+\alpha_{22}\,\alpha_{33}\,\Theta_1+\alpha_{13}\,\alpha_{32}\,\Theta_2+\alpha_{33}\,\Theta_2-\alpha_{12}\,\alpha_{33}\,\Theta_2+\Theta_3-c}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ t_S & \frac{\phi_1-\alpha_{12}\,\phi_1-\alpha_{13}\,\alpha_{22}\,\phi_1-\alpha_{23}\,\phi_1+\alpha_{12}\,\alpha_{23}\,\phi_1+\alpha_{11}\,\phi_2+\alpha_{13}\,\alpha_{21}\,\phi_2-\alpha_{11}\,\alpha_{23}\,\phi_2+\alpha_{21}\,\phi_3-\alpha_{12}\,\alpha_{21}\,\phi_3+\alpha_{11}\,\alpha_{22}\,\phi_3}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} & \frac{\Xi-\Xi\,\alpha_{12}-\Xi\,\alpha_{13}\,\alpha_{22}-\Xi\,\alpha_{23}+\Xi\,\alpha_{12}\,\alpha_{23}+\Theta_1\,\phi_2-\alpha_{23}\,\Theta_1\,\phi_2+\alpha_{13}\,\Theta_2}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \end{pmatrix}$$

(ζ // $m_{12\to 1}$ // $m_{13\to 1}$) == (ζ // $m_{23\to 2}$ // $m_{12\to 1}$)

True

$\Gamma 0$ /: $\Gamma 0[\lambda 1\_]\ \Gamma 0[\lambda 2\_] := \Gamma 0[\lambda 1 + \lambda 2]$;

The YB Elements for Γ0-Calculus

```
? ReplacePart
```

ReplacePart[*expr*, *i* → *new*] yields an expression in which the i^{th} part of *expr* is replaced by *new*.
ReplacePart[*expr*, {i_1 → new_1, i_2 → new_2, …}] replaces parts at positions i_n by new_n.
ReplacePart[*expr*, {*i*, *j*, …} → *new*] replaces the part at position {*i*, *j*, …}.
ReplacePart[*expr*, {{i_1, j_1, …} → new_1, …}] replaces parts at positions {i_n, j_n, …} by new_n.
ReplacePart[*expr*, {{i_1, j_1, …}, …} → *new*] replaces all parts at positions {i_n, j_n, …} by *new*.
ReplacePart[*i* → *new*] represents an operator form of ReplacePart that can be applied to an expression. ≫

```
ReplacePart[f[a, b, c], 2 → d]
```

```
f[a, d, c]
```

```
R_{n_,i_,j_} := ReplacePart[IdentityMatrix[n], {
    {i, i} → α, {i, j} → β,
    {j, i} → γ, {j, j} → δ
  }];
```

$R_{2,1,2}$

$\{\{\alpha, \beta\}, \{\gamma, \delta\}\}$

$R_{3,1,3}$

$\{\{\alpha, 0, \beta\}, \{0, 1, 0\}, \{\gamma, 0, \delta\}\}$

MatrixForm /@ $\{R_{2,1,2}, R_{3,1,3}\}$

$$\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \begin{pmatrix}\alpha & 0 & \beta\\ 0 & 1 & 0\\ \gamma & 0 & \delta\end{pmatrix}\right\}$$

$R_{3,1,2}.R_{3,1,3}.R_{3,2,3}$

$$\{\{\alpha^2, \alpha\beta+\alpha\beta\gamma, \beta^2+\alpha\beta\delta\}, \{\alpha\gamma, \beta\gamma^2+\alpha\delta, \beta\delta+\beta\gamma\delta\}, \{\gamma, \gamma\delta, \delta^2\}\}$$

$R_{3,1,2}.R_{3,1,3}.R_{3,2,3}$ // MatrixForm

$$\begin{pmatrix}\alpha^2 & \alpha\beta+\alpha\beta\gamma & \beta^2+\alpha\beta\delta\\ \alpha\gamma & \beta\gamma^2+\alpha\delta & \beta\delta+\beta\gamma\delta\\ \gamma & \gamma\delta & \delta^2\end{pmatrix}$$

$R_{3,2,3}.R_{3,1,3}.R_{3,1,2}$ // MatrixForm

$$\begin{pmatrix}\alpha^2 & \alpha\beta & \beta\\ \alpha\gamma+\alpha\beta\gamma & \beta^2\gamma+\alpha\delta & \beta\delta\\ \gamma^2+\alpha\gamma\delta & \gamma\delta+\beta\gamma\delta & \delta^2\end{pmatrix}$$

```
Flatten[{{1, 2, 3}, {4, 5}}]
```

```
{1, 2, 3, 4, 5}
```

Flatten[$R_{3,1,2}.R_{3,1,3}.R_{3,2,3}$]

$$\{\alpha^2, \alpha\beta+\alpha\beta\gamma, \beta^2+\alpha\beta\delta, \alpha\gamma, \beta\gamma^2+\alpha\delta, \beta\delta+\beta\gamma\delta, \gamma, \gamma\delta, \delta^2\}$$

eqns = Thread[Flatten[$R_{3,1,2}.R_{3,1,3}.R_{3,2,3}$] == Flatten[$R_{3,2,3}.R_{3,1,3}.R_{3,1,2}$]]

$$\{\text{True}, \alpha\beta+\alpha\beta\gamma == \alpha\beta, \beta^2+\alpha\beta\delta == \beta, \alpha\gamma == \alpha\gamma+\alpha\beta\gamma, \beta\gamma^2+\alpha\delta == \beta^2\gamma+\alpha\delta, \beta\delta+\beta\gamma\delta == \beta\delta, \gamma == \gamma^2+\alpha\gamma\delta, \gamma\delta == \gamma\delta+\beta\gamma\delta, \text{True}\}$$

```
Solve[eqns, {α, β, γ, δ}]
```

Solve: Equations may not give solutions for all "solve" variables.

{{β → 0, γ → 0}, {β → 0, γ → 1 - α δ}, {β → 1 - α δ, γ → 0}, {α → 0, β → 1, γ → 1, δ → 0}}

```
MatrixForm /@ (R2,1,2 /. Solve[eqns, {α, β, γ, δ}])
```

Solve: Equations may not give solutions for all "solve" variables.

$$\left\{\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}, \begin{pmatrix}\alpha & 0\\ 1-\alpha\delta & \delta\end{pmatrix}, \begin{pmatrix}\alpha & 1-\alpha\delta\\ 0 & \delta\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\}$$

```
MatrixForm /@ (R2,1,2 /. Solve[eqns ⋃ {α == 0, δ == t}, {α, β, γ, δ}])
```

$$\left\{\begin{pmatrix}0 & 0\\ 1 & t\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & t\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & t\end{pmatrix}\right\}$$

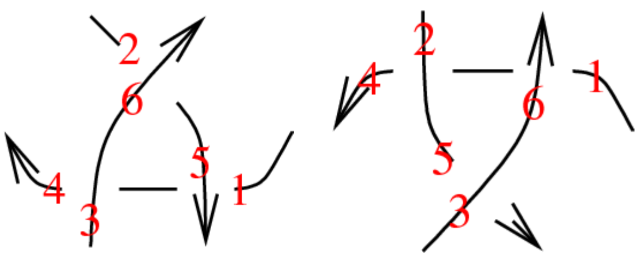
```
MatrixForm /@ (R2,1,2 /. Solve[eqns ⋃ {α == 1, δ == t}, {α, β, γ, δ}])
```

$$\left\{\begin{pmatrix}1 & 0\\ 1-t & t\end{pmatrix}, \begin{pmatrix}1 & 1-t\\ 0 & t\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix}\right\}$$

$$\begin{pmatrix}1 & 1-t\\ 0 & t\end{pmatrix} \text{ // Inverse}$$

$$\left\{\left\{1, \frac{-1+t}{t}\right\}, \left\{0, \frac{1}{t}\right\}\right\}$$

```
Γ0[Xp_{a_ b_} * more_.] := Γ0[{t_a, t_b}.(1  1-T; 0  T).{h_a, h_b}] Γ0[more];
Γ0[Xm_{a_ b_} * more_.] := (Γ0[Xp_{ab}] /. T → 1/T) Γ0[more];
Γ0[1] = Γ0[0];
```



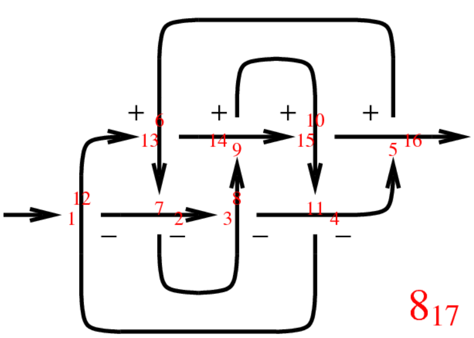

```
{Xm51 Xm62 Xp34 // Γ0 // m14→1 // m25→2 // m36→3,  Xp61 Xm24 Xm35 // Γ0 // m14→1 // m25→2 // m36→3}
```

$$\left\{\begin{pmatrix}\Gamma 0 & h_1 & h_2 & h_3\\ t_1 & 1 & 0 & 0\\ t_2 & \frac{-1+T}{T} & \frac{1}{T} & 0\\ t_3 & -\frac{-1+T}{T} & \frac{-1+T}{T} & 1\end{pmatrix}, \begin{pmatrix}\Gamma 0 & h_1 & h_2 & h_3\\ t_1 & 1 & 0 & 0\\ t_2 & \frac{-1+T}{T} & \frac{1}{T} & 0\\ t_3 & -\frac{-1+T}{T} & \frac{-1+T}{T} & 1\end{pmatrix}\right\}$$

Two types of R2!

```
{Xp12 Xm34 // Γ0 // m13→1 // m24→2, Xp12 Xm34 // Γ0 // m13→1 // m42→2}
```

$$\left\{\begin{pmatrix}\Gamma 0 & h_1 & h_2\\ t_1 & 1 & 0\\ t_2 & 0 & 1\end{pmatrix}, \begin{pmatrix}\Gamma 0 & h_1 & h_2\\ t_1 & 1 & 0\\ t_2 & 0 & 1\end{pmatrix}\right\}$$



8_{17}

```
z = Γ0[Xm12,1 Xm27 Xm83 Xm4,11 Xp16,5 Xp6,13 Xp14,9 Xp10,15];
Do[z = z // m1k→1, {k, 2, 16}]; z
```

Γ0	h_1
t_1	1

```
z = Γ0[Xm12,1 Xm27 Xm83 Xm4,11 Xp16,5 Xp6,13 Xp14,9 Xp10,15];
Do[Print[z];
 z = z // m1k→1, {k, 2, 16}];
z
```

Γ0	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9	h_{10}	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}	h_{16}
t_1	$\frac{1}{T}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_2	0	1	0	0	0	0	$\frac{-1+T}{T}$	0	0	0	0	0	0	0	0	0
t_3	0	0	$\frac{1}{T}$	0	0	0	0	0	0	0	0	0	0	0	0	0
t_4	0	0	0	1	0	0	0	0	0	0	$\frac{-1+T}{T}$	0	0	0	0	0
t_5	0	0	0	0	T	0	0	0	0	0	0	0	0	0	0	0
t_6	0	0	0	0	0	1	0	0	0	0	0	0	1 - T	0	0	0
t_7	0	0	0	0	0	0	$\frac{1}{T}$	0	0	0	0	0	0	0	0	0
t_8	0	0	$\frac{-1+T}{T}$	0	0	0	0	1	0	0	0	0	0	0	0	0
t_9	0	0	0	0	0	0	0	0	T	0	0	0	0	0	0	0
t_{10}	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1 - T	0
t_{11}	0	0	0	0	0	0	0	0	0	0	$\frac{1}{T}$	0	0	0	0	0
t_{12}	$\frac{-1+T}{T}$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
t_{13}	0	0	0	0	0	0	0	0	0	0	0	0	T	0	0	0
t_{14}	0	0	0	0	0	0	0	0	1 - T	0	0	0	0	1	0	0
t_{15}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	T	0
t_{16}	0	0	0	0	1 - T	0	0	0	0	0	0	0	0	0	0	1

Γ0	h_1	h_3	h_4	h_5	h_6	h_7	h_8	h_9	h_{10}	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}	h_{16}
t_1	$\frac{1}{T}$	0	0	0	0	$\frac{-1+T}{T}$	0	0	0	0	0	0	0	0	0
t_3	0	$\frac{1}{T}$	0	0	0	0	0	0	0	0	0	0	0	0	0
t_4	0	0	1	0	0	0	0	0	0	$\frac{-1+T}{T}$	0	0	0	0	0
t_5	0	0	0	T	0	0	0	0	0	0	0	0	0	0	0
t_6	0	0	0	0	1	0	0	0	0	0	0	1 - T	0	0	0
t_7	0	0	0	0	0	$\frac{1}{T}$	0	0	0	0	0	0	0	0	0
t_8	0	$\frac{-1+T}{T}$	0	0	0	0	1	0	0	0	0	0	0	0	0
t_9	0	0	0	0	0	0	0	T	0	0	0	0	0	0	0
t_{10}	0	0	0	0	0	0	0	0	1	0	0	0	0	1 - T	0
t_{11}	0	0	0	0	0	0	0	0	0	$\frac{1}{T}$	0	0	0	0	0
t_{12}	$\frac{-1+T}{T}$	0	0	0	0	0	0	0	0	0	1	0	0	0	0
t_{13}	0	0	0	0	0	0	0	0	0	0	0	T	0	0	0
t_{14}	0	0	0	0	0	0	0	1 - T	0	0	0	0	1	0	0
t_{15}	0	0	0	0	0	0	0	0	0	0	0	0	0	T	0
t_{16}	0	0	0	1 - T	0	0	0	0	0	0	0	0	0	0	1

Γ0	h_1	h_4	h_5	h_6	h_7	h_8	h_9	h_{10}	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}	h_{16}
t_1	$\frac{1}{T^2}$	0	0	0	$\frac{-1+T}{T^2}$	0	0	0	0	0	0	0	0	0
t_4	0	1	0	0	0	0	0	0	$\frac{-1+T}{T}$	0	0	0	0	0
t_5	0	0	T	0	0	0	0	0	0	0	0	0	0	0
t_6	0	0	0	1	0	0	0	0	0	0	1 - T	0	0	0
t_7	0	0	0	0	$\frac{1}{T}$	0	0	0	0	0	0	0	0	0
t_8	$\frac{-1+T}{T^2}$	0	0	0	$\frac{(-1+T)^2}{T^2}$	1	0	0	0	0	0	0	0	0
t_9	0	0	0	0	0	0	T	0	0	0	0	0	0	0
t_{10}	0	0	0	0	0	0	0	1	0	0	0	0	1 - T	0
t_{11}	0	0	0	0	0	0	0	0	$\frac{1}{T}$	0	0	0	0	0
t_{12}	$\frac{-1+T}{T}$	0	0	0	0	0	0	0	0	1	0	0	0	0
t_{13}	0	0	0	0	0	0	0	0	0	0	T	0	0	0
t_{14}	0	0	0	0	0	0	1 - T	0	0	0	0	1	0	0
t_{15}	0	0	0	0	0	0	0	0	0	0	0	0	T	0
t_{16}	0	0	1 - T	0	0	0	0	0	0	0	0	0	0	1

Γ0	h_1	h_5	h_6	h_7	h_8	h_9	h_{10}	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}	h_{16}
t_1	$\frac{1}{T^2}$	0	0	$\frac{-1+T}{T^2}$	0	0	0	$\frac{-1+T}{T}$	0	0	0	0	0
t_5	0	T	0	0	0	0	0	0	0	0	0	0	0
t_6	0	0	1	0	0	0	0	0	0	$1-T$	0	0	0
t_7	0	0	0	$\frac{1}{T}$	0	0	0	0	0	0	0	0	0
t_8	$\frac{-1+T}{T^2}$	0	0	$\frac{(-1+T)^2}{T^2}$	1	0	0	0	0	0	0	0	0
t_9	0	0	0	0	0	T	0	0	0	0	0	0	0
t_{10}	0	0	0	0	0	0	1	0	0	0	0	$1-T$	0
t_{11}	0	0	0	0	0	0	0	$\frac{1}{T}$	0	0	0	0	0
t_{12}	$\frac{-1+T}{T}$	0	0	0	0	0	0	0	1	0	0	0	0
t_{13}	0	0	0	0	0	0	0	0	0	T	0	0	0
t_{14}	0	0	0	0	0	$1-T$	0	0	0	0	1	0	0
t_{15}	0	0	0	0	0	0	0	0	0	0	0	T	0
t_{16}	0	$1-T$	0	0	0	0	0	0	0	0	0	0	1

Γ0	h_1	h_6	h_7	h_8	h_9	h_{10}	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}	h_{16}
t_1	$\frac{1}{T}$	0	$\frac{-1+T}{T}$	0	0	0	$-1+T$	0	0	0	0	0
t_6	0	1	0	0	0	0	0	0	$1-T$	0	0	0
t_7	0	0	$\frac{1}{T}$	0	0	0	0	0	0	0	0	0
t_8	$\frac{-1+T}{T^2}$	0	$\frac{(-1+T)^2}{T^2}$	1	0	0	0	0	0	0	0	0
t_9	0	0	0	0	T	0	0	0	0	0	0	0
t_{10}	0	0	0	0	0	1	0	0	0	0	$1-T$	0
t_{11}	0	0	0	0	0	0	$\frac{1}{T}$	0	0	0	0	0
t_{12}	$\frac{-1+T}{T}$	0	0	0	0	0	0	1	0	0	0	0
t_{13}	0	0	0	0	0	0	0	0	T	0	0	0
t_{14}	0	0	0	0	$1-T$	0	0	0	0	1	0	0
t_{15}	0	0	0	0	0	0	0	0	0	0	T	0
t_{16}	$-\frac{-1+T}{T^2}$	0	$-\frac{(-1+T)^2}{T^2}$	0	0	0	$-\frac{(-1+T)^2}{T}$	0	0	0	0	1

Γ0	h_1	h_7	h_8	h_9	h_{10}	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}	h_{16}
t_1	$\frac{1}{T}$	$\frac{-1+T}{T}$	0	0	0	$-1+T$	0	$1-T$	0	0	0
t_7	0	$\frac{1}{T}$	0	0	0	0	0	0	0	0	0
t_8	$\frac{-1+T}{T^2}$	$\frac{(-1+T)^2}{T^2}$	1	0	0	0	0	0	0	0	0
t_9	0	0	0	T	0	0	0	0	0	0	0
t_{10}	0	0	0	0	1	0	0	0	0	$1-T$	0
t_{11}	0	0	0	0	0	$\frac{1}{T}$	0	0	0	0	0
t_{12}	$\frac{-1+T}{T}$	0	0	0	0	0	1	0	0	0	0
t_{13}	0	0	0	0	0	0	0	T	0	0	0
t_{14}	0	0	0	$1-T$	0	0	0	0	1	0	0
t_{15}	0	0	0	0	0	0	0	0	0	T	0
t_{16}	$-\frac{-1+T}{T^2}$	$-\frac{(-1+T)^2}{T^2}$	0	0	0	$-\frac{(-1+T)^2}{T}$	0	0	0	0	1

Γ0	h_1	h_8	h_9	h_{10}	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}	h_{16}
t_1	$\frac{1}{T}$	0	0	0	$-1+T$	0	$1-T$	0	0	0
t_8	$\frac{-1+T}{T}$	1	0	0	$\frac{(-1+T)^3}{T}$	0	$-\frac{(-1+T)^3}{T}$	0	0	0
t_9	0	0	T	0	0	0	0	0	0	0
t_{10}	0	0	0	1	0	0	0	0	$1-T$	0
t_{11}	0	0	0	0	$\frac{1}{T}$	0	0	0	0	0
t_{12}	$\frac{-1+T}{T}$	0	0	0	0	1	0	0	0	0
t_{13}	0	0	0	0	0	0	T	0	0	0
t_{14}	0	0	$1-T$	0	0	0	0	1	0	0
t_{15}	0	0	0	0	0	0	0	0	T	0
t_{16}	$-\frac{-1+T}{T}$	0	0	0	$-(-1+T)^2$	0	$\frac{(-1+T)^3}{T}$	0	0	1

$\Gamma 0$	h_1	h_9	h_{10}	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}	h_{16}
t_1	1	0	0	$\frac{(-1+T)\,(1-T+T^2)}{T}$	0	$-\frac{(-1+T)\,(1-T+T^2)}{T}$	0	0	0
t_9	0	T	0	0	0	0	0	0	0
t_{10}	0	0	1	0	0	0	0	$1-T$	0
t_{11}	0	0	0	$\frac{1}{T}$	0	0	0	0	0
t_{12}	$\frac{-1+T}{T}$	0	0	0	1	0	0	0	0
t_{13}	0	0	0	0	0	T	0	0	0
t_{14}	0	$1-T$	0	0	0	0	1	0	0
t_{15}	0	0	0	0	0	0	0	T	0
t_{16}	$-\frac{-1+T}{T}$	0	0	$-(-1+T)^2$	0	$\frac{(-1+T)^3}{T}$	0	0	1

$\Gamma 0$	h_1	h_{10}	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}	h_{16}
t_1	T	0	$(-1+T)\,(1-T+T^2)$	0	$-(-1+T)\,(1-T+T^2)$	0	0	0
t_{10}	0	1	0	0	0	0	$1-T$	0
t_{11}	0	0	$\frac{1}{T}$	0	0	0	0	0
t_{12}	$\frac{-1+T}{T}$	0	0	1	0	0	0	0
t_{13}	0	0	0	0	T	0	0	0
t_{14}	$1-T$	0	$-\frac{(-1+T)^2\,(1-T+T^2)}{T}$	0	$\frac{(-1+T)^2\,(1-T+T^2)}{T}$	1	0	0
t_{15}	0	0	0	0	0	0	T	0
t_{16}	$-\frac{-1+T}{T}$	0	$-(-1+T)^2$	0	$\frac{(-1+T)^3}{T}$	0	0	1

$\Gamma 0$	h_1	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}	h_{16}
t_1	T	$(-1+T)\,(1-T+T^2)$	0	$-(-1+T)\,(1-T+T^2)$	0	$1-T$	0
t_{11}	0	$\frac{1}{T}$	0	0	0	0	0
t_{12}	$\frac{-1+T}{T}$	0	1	0	0	0	0
t_{13}	0	0	0	T	0	0	0
t_{14}	$1-T$	$-\frac{(-1+T)^2\,(1-T+T^2)}{T}$	0	$\frac{(-1+T)^2\,(1-T+T^2)}{T}$	1	0	0
t_{15}	0	0	0	0	0	T	0
t_{16}	$-\frac{-1+T}{T}$	$-(-1+T)^2$	0	$\frac{(-1+T)^3}{T}$	0	0	1

$\Gamma 0$	h_1	h_{12}	h_{13}	h_{14}	h_{15}	h_{16}
t_1	$-\frac{1}{-2+2T-2T^2+T^3}$	0	$\frac{(-1+T)\,(1-T+T^2)}{T\,(-2+2T-2T^2+T^3)}$	0	$\frac{-1+T}{T\,(-2+2T-2T^2+T^3)}$	0
t_{12}	$\frac{-1+T}{T}$	1	0	0	0	0
t_{13}	0	0	T	0	0	0
t_{14}	$\frac{-1+T}{-2+2T-2T^2+T^3}$	0	$-\frac{(-1+T)^2\,(1-T+T^2)}{T\,(-2+2T-2T^2+T^3)}$	1	$-\frac{(-1+T)^3\,(1-T+T^2)}{T\,(-2+2T-2T^2+T^3)}$	0
t_{15}	0	0	0	0	T	0
t_{16}	$\frac{(-1+T)\,(2-2T+T^2)}{T\,(-2+2T-2T^2+T^3)}$	0	$-\frac{(-1+T)^3\,(2-T+T^2)}{T\,(-2+2T-2T^2+T^3)}$	0	$-\frac{(-1+T)^3}{-2+2T-2T^2+T^3}$	1

$\Gamma 0$	h_1	h_{13}	h_{14}	h_{15}	h_{16}
t_1	$\frac{(-2+T)\,(-1+2T-T^2+T^3)}{T\,(-2+2T-2T^2+T^3)}$	$\frac{(-1+T)\,(1-T+T^2)}{T\,(-2+2T-2T^2+T^3)}$	0	$\frac{-1+T}{T\,(-2+2T-2T^2+T^3)}$	0
t_{13}	0	T	0	0	0
t_{14}	$\frac{-1+T}{-2+2T-2T^2+T^3}$	$-\frac{(-1+T)^2\,(1-T+T^2)}{T\,(-2+2T-2T^2+T^3)}$	1	$-\frac{(-1+T)^3\,(1-T+T^2)}{T\,(-2+2T-2T^2+T^3)}$	0
t_{15}	0	0	0	T	0
t_{16}	$\frac{(-1+T)\,(2-2T+T^2)}{T\,(-2+2T-2T^2+T^3)}$	$-\frac{(-1+T)^3\,(2-T+T^2)}{T\,(-2+2T-2T^2+T^3)}$	0	$-\frac{(-1+T)^3}{-2+2T-2T^2+T^3}$	1

$\Gamma 0$	h_1	h_{14}	h_{15}	h_{16}
t_1	$\frac{(-2+T)\,T\,(-1+2T-T^2+T^3)}{1-4T+4T^2-3T^3+T^4}$	0	$\frac{(-1+T)\,T}{1-4T+4T^2-3T^3+T^4}$	0
t_{14}	$-\frac{(-1+T)\,(1-3T+3T^2-3T^3+T^4)}{T\,(1-4T+4T^2-3T^3+T^4)}$	1	$-\frac{(-1+T)^4\,(1-T+T^2)}{T\,(1-4T+4T^2-3T^3+T^4)}$	0
t_{15}	0	0	T	0
t_{16}	$-\frac{(-1+T)\,(-1+4T-8T^2+7T^3-4T^4+T^5)}{T\,(1-4T+4T^2-3T^3+T^4)}$	0	$-\frac{(-1+T)^3\,(1-T+T^2)}{T\,(1-4T+4T^2-3T^3+T^4)}$	1

$\Gamma 0$	h_1	h_{15}	h_{16}
t_1	$\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{T\,(1-4T+4T^2-3T^3+T^4)}$	$-\frac{(-1+T)\,(-1+4T-8T^2+7T^3-4T^4+T^5)}{T\,(1-4T+4T^2-3T^3+T^4)}$	0
t_{15}	0	T	0
t_{16}	$-\frac{(-1+T)\,(-1+4T-8T^2+7T^3-4T^4+T^5)}{T\,(1-4T+4T^2-3T^3+T^4)}$	$-\frac{(-1+T)^3\,(1-T+T^2)}{T\,(1-4T+4T^2-3T^3+T^4)}$	1

$\Gamma 0$	h_1	h_{16}
t_1	T	0
t_{16}	$1-T$	1

$$\begin{pmatrix} \Gamma 0 & h_1 \\ t_1 & 1 \end{pmatrix}$$

Γ-Calculus

```
Γ /: Collect[Γ[ω_, λ_]] := Γ[Simplify[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)_a_ :> a, ∞];
   M = Outer[Factor[∂_{h_#1 t_#2} λ] &, S, S];
   M = Prepend[M, t_# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h_# & /@ S, ω]];
   M // MatrixForm];
```

```
ξ = Γ[ω, {t1, t2, t3, tS}.(α11 α12 α13 θ1; α21 α22 α23 θ2; α31 α32 α33 θ3; ϕ1 ϕ2 ϕ3 Ξ).{h1, h2, h3, hS}]
```

$$\xi = \Gamma\left[\omega,\ \{t_1, t_2, t_3, t_S\}.\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3 \\ \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix}.\{h_1, h_2, h_3, h_S\}\right]$$

$$\begin{pmatrix} \omega & h_1 & h_2 & h_3 & h_S \\ t_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1 \\ t_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2 \\ t_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3 \\ t_S & \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix}$$

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
m_{a_ b_ → c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
   (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂_{t_a,h_a} λ  ∂_{t_a,h_b} λ  ∂_{t_a} λ;
                            ∂_{t_b,h_a} λ  ∂_{t_b,h_b} λ  ∂_{t_b} λ;
                            ∂_{h_a} λ      ∂_{h_b} λ      λ) /. (t | h)_{a|b} → 0;
   Γ[(μ = 1 - β) ω, {t_c, 1}.(γ + α δ / μ   ϵ + δ θ / μ; ϕ + α ψ / μ   Ξ + ψ θ / μ).{h_c, 1}] // Collect];
Γ[Xp_{a_ b_} * more_.] := Γ[1, {t_a, t_b}.(1  1 - T; 0  T).{h_a, h_b}] Γ[more];
Γ[Xm_{a_ b_} * more_.] := (Γ[Xp_{ab}] /. T → 1 / T) Γ[more];
Γ[1] = Γ[1, 0];
```

```
(ξ // m_{12→1} // m_{13→1})
```

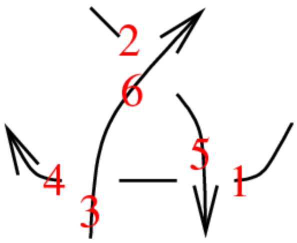
$$\begin{pmatrix} \omega\left(1 - \alpha_{13}\alpha_{22} + \alpha_{12}\left(-1+\alpha_{23}\right) - \alpha_{23}\right) & h_1 & \cdots \\ t_1 & \frac{\alpha_{31}-\alpha_{12}\alpha_{31}-\alpha_{13}\alpha_{22}\alpha_{31}-\alpha_{23}\alpha_{31}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{11}\alpha_{32}+\alpha_{13}\alpha_{21}\alpha_{32}-\alpha_{11}\alpha_{23}\alpha_{32}+\alpha_{21}\alpha_{33}-\alpha_{12}\alpha_{21}\alpha_{33}+\alpha_{11}\alpha_{22}\alpha_{33}}{1-\alpha_{12}-\alpha_{13}\alpha_{22}-\alpha_{23}+\alpha_{12}\alpha_{23}} & \alpha_{32}\theta_1-\alpha_{23}\alpha_{32}\cdots \\ t_S & \frac{\phi_1-\alpha_{12}\phi_1-\alpha_{13}\alpha_{22}\phi_1-\alpha_{23}\phi_1+\alpha_{12}\alpha_{23}\phi_1+\alpha_{11}\phi_2+\alpha_{13}\alpha_{21}\phi_2-\alpha_{11}\alpha_{23}\phi_2+\alpha_{21}\phi_3-\alpha_{12}\alpha_{21}\phi_3+\alpha_{11}\alpha_{22}\phi_3}{1-\alpha_{12}-\alpha_{13}\alpha_{22}-\alpha_{23}+\alpha_{12}\alpha_{23}} & \Xi-\Xi\,\alpha_{12}-\cdots \end{pmatrix}$$

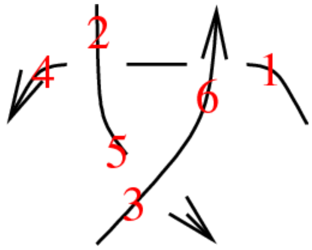
```
(ξ // m_{23→2} // m_{12→1})
```

$$\begin{pmatrix} \omega\left(1 - \alpha_{13}\alpha_{22} + \alpha_{12}\left(-1+\alpha_{23}\right) - \alpha_{23}\right) & h_1 & \cdots \\ t_1 & \frac{\alpha_{31}-\alpha_{12}\alpha_{31}-\alpha_{13}\alpha_{22}\alpha_{31}-\alpha_{23}\alpha_{31}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{11}\alpha_{32}+\alpha_{13}\alpha_{21}\alpha_{32}-\alpha_{11}\alpha_{23}\alpha_{32}+\alpha_{21}\alpha_{33}-\alpha_{12}\alpha_{21}\alpha_{33}+\alpha_{11}\alpha_{22}\alpha_{33}}{1-\alpha_{12}-\alpha_{13}\alpha_{22}-\alpha_{23}+\alpha_{12}\alpha_{23}} & \alpha_{32}\theta_1-\alpha_{23}\alpha_{32}\cdots \\ t_S & \frac{\phi_1-\alpha_{12}\phi_1-\alpha_{13}\alpha_{22}\phi_1-\alpha_{23}\phi_1+\alpha_{12}\alpha_{23}\phi_1+\alpha_{11}\phi_2+\alpha_{13}\alpha_{21}\phi_2-\alpha_{11}\alpha_{23}\phi_2+\alpha_{21}\phi_3-\alpha_{12}\alpha_{21}\phi_3+\alpha_{11}\alpha_{22}\phi_3}{1-\alpha_{12}-\alpha_{13}\alpha_{22}-\alpha_{23}+\alpha_{12}\alpha_{23}} & \Xi-\Xi\,\alpha_{12}-\cdots \end{pmatrix}$$

```
(ξ // m_{12→1} // m_{13→1}) == (ξ // m_{23→2} // m_{12→1})
```

True





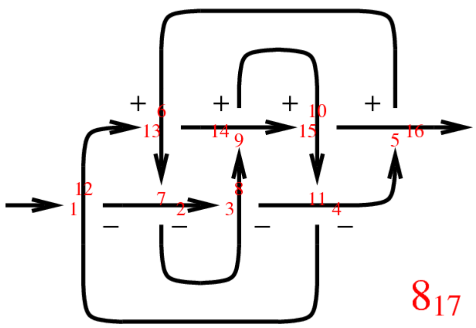
$\{Xm_{51}\, Xm_{62}\, Xp_{34}$ // Γ // $m_{14\to 1}$ // $m_{25\to 2}$ // $m_{36\to 3}$, $Xp_{61}\, Xm_{24}\, Xm_{35}$ // Γ // $m_{14\to 1}$ // $m_{25\to 2}$ // $m_{36\to 3}\}$

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & 1 & 0 & 0 \\ t_2 & \frac{-1+T}{T} & \frac{1}{T} & 0 \\ t_3 & -\frac{-1+T}{T} & \frac{-1+T}{T} & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & 1 & 0 & 0 \\ t_2 & \frac{-1+T}{T} & \frac{1}{T} & 0 \\ t_3 & -\frac{-1+T}{T} & \frac{-1+T}{T} & 1 \end{pmatrix}\right\}$$

Two types of R2.

$\{Xp_{12}\, Xm_{34}$ // Γ // $m_{13\to 1}$ // $m_{24\to 2}$, $Xp_{12}\, Xm_{34}$ // Γ // $m_{13\to 1}$ // $m_{42\to 2}\}$

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & 0 \\ t_2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & 0 \\ t_2 & 0 & 1 \end{pmatrix}\right\}$$



z = Γ[$Xm_{12,1}\, Xm_{27}\, Xm_{83}\, Xm_{4,11}\, Xp_{16,5}\, Xp_{6,13}\, Xp_{14,9}\, Xp_{10,15}$];
Do[z = z // $m_{1k\to 1}$, {k, 2, 16}]; z

$$\begin{pmatrix} 11 - \frac{1}{T^3} + \frac{4}{T^2} - \frac{8}{T} - 8T + 4T^2 - T^3 & h_1 \\ t_1 & 1 \end{pmatrix}$$

Alexander[Knot[8, 17]][t]

KnotTheory: Loading precomputed data in PD4Knots`.

$$11 - \frac{1}{t^3} + \frac{4}{t^2} - \frac{8}{t} - 8t + 4t^2 - t^3$$

Dror Bar-Natan: Classes: 2017: MAT 1350 AKT:
Handout for January 31, 2017:

Γ-Calculus

Derived from Cheat Sheet Meta-Calculi, in http://drorbn.net/AcademicPensieve/Projects/MetaCalculi/.

σ-calculus. $\sigma_1 * \sigma_2 = \sigma_1 \cup \sigma_2$, $m^{ab}_c(\sigma) = (\sigma\backslash\{a,b\}) \cup (c \to \sigma_a\sigma_b) \,/\, (T_a, T_b \to T_c)$, $\mathrm{tr}_c(\sigma) = \sigma\backslash c$, $R^{\pm}_{ab} \mapsto (a \to 1,\ b \to T_a^{\pm 1})$

Gassner calculus / Γ-calculus. Preserves $C_1 \coloneqq [\text{col sum} = 1]$ (⇔OC) and ✓ $C_2 \coloneqq [\forall a, b,\ (T_a - 1) \mid (A_{ab} - \delta_{ab}\sigma_b)]$

• Except under tr_c, at $T_* = 1$, $\omega = 1$ and $A = I$.

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[\substack{\mu:=1-\beta \\ T_a,T_b \to T_c}]{m^{ab}_c} \begin{array}{c|cc} \mu\omega & c & S \\ \hline c & \gamma + \alpha\delta/\mu & \epsilon + \delta\theta/\mu \\ S & \phi + \alpha\psi/\mu & \Xi + \psi\theta/\mu \end{array} \qquad \begin{array}{c|cc} \omega & c & S \\ \hline c & \alpha & \theta \\ S & \psi & \Xi \end{array} \xrightarrow[\mu:=1-\alpha]{\mathrm{tr}_c} \begin{array}{c|c} \mu\omega & S \\ \hline S & \Xi + \psi\theta/\mu \end{array} \qquad R^{\pm}_{ab} \underset{\Gamma}{=} \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1 - T_a^{\pm 1} \\ b & 0 & T_a^{\pm 1} \end{array}$$

$$\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array} \xrightarrow[\substack{\mu:=T_a-1 \\ \nu:=\alpha-\sigma_a}]{\Delta^a_{bc}} \left(\begin{array}{c|ccc} \omega & b & c & S \\ \hline b & (\sigma_a - \alpha T_a - \nu T_c)/\mu & (T_b - 1)T_c\nu/\mu & (T_b - 1)T_c\theta/\mu \\ c & (T_c - 1)\nu/\mu & (\alpha - \sigma_a T_a - \nu T_c)/\mu & (T_c - 1)\theta/\mu \\ S & \phi & \phi & \Xi \end{array} \right)_{T_a \mapsto T_b T_c}$$

Satisfies: ✓ $R^+_{13} /\!\!/ \Delta^1_{12} = R^+_{23} \# R^+_{13}$.
✓ $R^-_{13} /\!\!/ \Delta^1_{12} = R^-_{13} \# R^-_{23}$.
✓ $\Delta^a_{a_1a_2} /\!\!/ \Delta^b_{b_1b_2} /\!\!/ m^{a_1b_1}_{c_1} /\!\!/ m^{a_2b_2}_{c_2} = m^{ab}_c /\!\!/ \Delta^c_{c_1c_2}$.

$$\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array} \xrightarrow{S^a} \left(\begin{array}{c|cc} \alpha\omega/\sigma_a & a & S \\ \hline a & 1/\alpha & \theta/\alpha \\ S & -\phi/\alpha & (\alpha\Xi - \phi\theta)/\alpha \end{array} \right)_{T_a \to T_a^{-1}}$$

Satisfies: ✓ $R^{\pm}_{12} /\!\!/ S^{1 \text{ or } 2} = R^{\mp}_{12}$. ✓ $m^{ab}_c /\!\!/ S^c = S^a /\!\!/ S^b /\!\!/ m^{ba}_c$.
✓ $S^a /\!\!/ S^a = I$. ✓ $\Delta^a_{bc} /\!\!/ S^b /\!\!/ S^c = S^a /\!\!/ \Delta^a_{cb}$.
✓ Assuming C_2, $\eta^a /\!\!/ \epsilon_a = \Delta^a_{bc} /\!\!/ S^c /\!\!/ m^{bc}_a$ (also 3 variants).

The map (tangle $T \mapsto$ matrix A) is anti-multiplicative. The MVA mod units: $L \mapsto (\omega, A) \mapsto \omega \det{}'(A - I)/(1 - T')$ ✓
