Pensieve header: The full Γ-Calculus, derived from pensieve://Projects/MetaCalculi/.

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Derived from Cheat Sheet Meta-Calculi, in http://drorbn.net/AcademicPensieve/Projects/MetaCalculi/

σ-calculus. $\sigma_1 * \sigma_2 = \sigma_1 \cup \sigma_2$, $\quad m_c^{ab}(\sigma) = (\sigma \setminus \{a,b\}) \cup (c \to \sigma_a \sigma_b) \,/\, (T_a, T_b \to T_c)$, $\quad \mathrm{tr}_c(\sigma)$

Gassner calculus / Γ-calculus. Preserves $C_1 :=$ [col sum = 1] ($\Leftrightarrow$OC) and ✓ C_2

• Except ui

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[\mu:=1-\beta,\ T_a,T_b\to T_c]{m_c^{ab}} \begin{array}{c|cc} \mu\omega & c & S \\ \hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu \\ S & \phi+\alpha\psi/\mu & \Xi+\psi\theta/\mu \end{array} \qquad \begin{array}{c|cc} \omega & c & S \\ \hline c & \alpha & \theta \\ S & \psi & \Xi \end{array} \xrightarrow[\mu:=1-\alpha]{\mathrm{tr}_c} \begin{array}{c|c} \mu\omega & S \\ \hline S & \Xi+\psi \end{array}$$

$$\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array} \xrightarrow[\mu:=T_a-1,\ \nu:=\alpha-\sigma_a]{\Delta_{bc}^a} \left(\begin{array}{c|ccc} \omega & b & c & S \\ \hline b & (\sigma_a-\alpha T_a-\nu T_c)/\mu & (T_b-1)T_c\nu/\mu & (T_b-1)T_c\theta/\mu \\ c & (T_c-1)\nu/\mu & (\alpha-\sigma_a T_a-\nu T_c)/\mu & (T_c-1)\theta/\mu \\ S & \phi & \phi & \Xi \end{array} \right)_{T_a \mapsto T_b T_c}$$

$$\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array} \xrightarrow{S^a} \left(\begin{array}{c|cc} \alpha\omega/\sigma_a & a & S \\ \hline a & 1/\alpha & \theta/\alpha \\ S & -\phi/\alpha & (\alpha\Xi-\phi\theta)/\alpha \end{array} \right)_{T_a \to T_a^{-1}}$$

Satisfies: ✓ $R_{12}^{\pm} /\!\!/ S^{1 \text{ or } 2} = R_{12}^{\mp}$.
✓ $S^a /\!\!/ S^a = I$.
✓ Assuming C_2, $\eta^a /\!\!/ \epsilon_a = \Delta_{bc}^a /\!\!/$

The map (tangle $T \mapsto$ matrix A) is anti-multiplicative. The MVA mod units: $L \mapsto$

```
<< KnotTheory`
```

Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.

Γ-Calculus

```
ΓSimp = Factor; SetAttributes[ΓCollect, Listable];
ΓCollect[Γ[ω_, σ_, λ_]] := ΓCollect[ΓSimp][Γ[ω, σ, λ]];
ΓCollect[simp_][Γ[ω_, σ_, λ_]] := Γ[simp[ω], simp[σ],
   Collect[λ, h_, Collect[#, t_, simp] &]];
dL[Γ[_, _, λ_]] := Union[Cases[λ, (h | t)a_ :> a, Infinity]];
Γ[ω1_, _, _][ω] := ω1;
Γ[ω_, σ_, λ_][Σ] := (∂h# σ) & /@ dL[Γ[ω, σ, λ]];
Γ[ω_, σ_, λ_][A] := Module[{S = dL[Γ[ω, σ, λ]]}, Outer[ΓSimp[(∂t#1h#2 λ)] &, S, S]];
ΓForm[Γ[ω_, σ_, λ_]] := Module[{S, M},
   S = dL[Γ[ω, σ, λ]];
   M = Γ[ω, σ, λ][A] // Transpose;
   PrependTo[M, s# & /@ S];
   M = Join[
     {Prepend[s# & /@ S, ω]},
     Transpose[M],
     {Prepend[Γ[ω, σ, λ][Σ], "Γ"]}
    ];
   MatrixForm[M]
  ];
ΓForm[else_] := else /. Γ[ω_, σ_, λ_] :> ΓForm[Γ[ω, σ, λ]];
Format[Γ[ω_, σ_, λ_], StandardForm] := ΓForm[Γ[ω, σ, λ]];
```

```
Γ /: Γ[ω1_, σ1_, μ1_] == Γ[ω2_, σ2_, μ2_] := Module[
  {S},
  S = dL[Γ[ω1, σ1, μ1]] ⋃ dL[Γ[ω2, σ2, μ2]];
  (ω1 == ω2) && (And @@ ((∂h# σ1 == ∂h# σ2) & /@ S)) && (
    And @@ Flatten[Outer[
       (∂t#1,h#2 μ1 == ∂t#1,h#2 μ2) &,
       S, S
      ]]
   )
 ]
```

```
Γ /: Γ[ω1_, σ1_, λ1_] Γ[ω2_, σ2_, λ2_] := Γ[ω1 * ω2, σ1 + σ2, λ1 + λ2];
dm_{i_ j_ → k_}[Γ[ω_, σ_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
   ( α  β  θ )   ( ∂_{t_i,h_i} λ  ∂_{t_i,h_j} λ  ∂_{t_i} λ )
   ( γ  δ  ϵ ) = ( ∂_{t_j,h_i} λ  ∂_{t_j,h_j} λ  ∂_{t_j} λ ) /. (t | h)_{i|j} → 0;
   ( ϕ  ψ  Ξ )   (   ∂_{h_i} λ      ∂_{h_j} λ        λ     )
   ΓCollect[Γ[(μ = 1 - β) ω,
      h_k (∂_{h_i} σ) (∂_{h_j} σ) + (σ /. h_{i|j} → 0),
      {t_k, 1}.( γ + α δ / μ   ϵ + δ θ / μ ).{h_k, 1}
               ( ϕ + α ψ / μ   Ξ + θ ψ / μ )
     ]] /. {T_i → T_k, T_j → T_k, b_i → b_k, b_j → b_k} // ΓCollect
  ];
dm[a_, b_, c_][Γ[ω_, σ_, λ_]] := dm_{ab→c}[Γ[ω, σ, λ]];
dη[a_][γ_Γ] := γ /. {(h | t)_a → 0, T_a → 1};
```

```
tr[a_][Γ[ω_, σ_, λ_]] := Module[{α, θ, ψ, Ξ},
   ( α  θ ) = ( ∂_{t_a,h_a} λ  ∂_{t_a} λ ) /. (t | h)_a → 0;
   ( ψ  Ξ )   (   ∂_{h_a} λ        λ     )
   Γ[ω (1 - α), σ /. h_a → 0, Ξ + ψ * θ / (1 - α)] // ΓCollect];
```

```
FullStitch[γ1_Γ, γ2_Γ] := Module[{S1, S2, S, γ, τ},
   S = (S1 = dL[γ1]) ⋃ (S2 = dL[γ2]);
   γ = γ1 (Times @@ (Γ /@ ϵ /@ Complement[S, S1]));
   γ *= (γ2 /. {h_{a_} :→ h_{τ[a]}, t_{a_} :→ t_{τ[a]}, T_{a_} :→ T_{τ[a]}})
      (Times @@ (Γ /@ ϵ /@ τ /@ Complement[S, S2]));
   Do[
    γ = γ // dm[s, τ[s], s],
    {s, S}
   ];
   γ
  ];
Γ /: γ1_Γ ** γ2_Γ := Module[{S1, S2, S, γ1p, γ2p},
   S = (S1 = dL[γ1]) ⋃ (S2 = dL[γ2]);
   γ1p = γ1 (Times @@ (Γ /@ ϵ /@ Complement[S, S1]));
   γ2p = γ2 (Times @@ (Γ /@ ϵ /@ Complement[S, S2]));
   Γ[
    γ1p[ω] * γ2p[ω],
    (γ1p[Σ] γ2p[Σ]).(h_# & /@ S),
    (t_# & /@ S).(γ2p[A].γ1p[A]).(h_# & /@ S)
   ]
  ];
```

```
Γ /: Γ[ω_, σ_, λ_]^-1 := Module[{S = dL[Γ[ω, σ, λ]]},
   Γ[
    ω^-1, Collect[σ, h_, (1/#) &],
    (t_# & /@ S).Inverse[Outer[ΓSimp[(∂_{t_#1 h_#2} λ)] &, S, S]].(h_# & /@ S)
   ]
  ];
```

```
dA[a_][Γ[ω_, σ_, λ_]] := Module[
   {α, θ, ϕ, Ξ, σa},
   ( α  θ )   ( ∂_{t_a,h_a} λ   ∂_{t_a} λ )
   ( ϕ  Ξ ) = ( ∂_{h_a} λ       λ         ) /. (h | t)_a → 0;
   σa = ∂_{h_a} σ;
   ΓCollect[Γ[
     α ω / σa,
     ((σ /. h_a → 0) + h_a / σa),
     {t_a, 1}.( 1    θ         ).{h_a, 1} / α
              ( -ϕ   α Ξ - ϕ θ )
    ]]
  ];
dS[a_][γ_Γ] := ΓCollect[dA[a][γ] /. {T_a → 1/T_a, b_a → -b_a}];
```

```
qΔ[a_, x_, y_][Γ[ω_, σ_, λ_]] := Module[
   {α, θ, ϕ, Ξ, σa, Ta, M},
   ( α  θ )   ( ∂_{t_a,h_a} λ   ∂_{t_a} λ )
   ( ϕ  Ξ ) = ( ∂_{h_a} λ       λ         ) /. (h | t)_a → 0 /. T_a → Ta;
   σa = ∂_{h_a} σ /. T_a → Ta;
   M = ( (-σa+α Ta+(-α+σa) T_y)/(-1+Ta)   ((-1+T_x) T_y (α-σa))/(-1+Ta)   (θ (-1+T_x) T_y)/(-1+Ta) )
       ( ((-1+T_y) (α-σa))/(-1+Ta)        (-α+σa Ta+(α-σa) T_y)/(-1+Ta)   (θ (-1+T_y))/(-1+Ta)     )
       ( ϕ                                ϕ                               Ξ                        );
   ΓCollect[Γ[
     ω /. T_a → T_x T_y,
     ((σ /. h_a → 0) + (h_x + h_y) σa) /. T_a | Ta → T_x T_y,
     {t_x, t_y, 1}.M.{h_x, h_y, 1} /. Ta → T_x T_y
    ]]
  ];
```

```
Mirror[γ_Γ] := Module[{γ1},
   γ1 = γ // (dS @@ dL[γ]);
   γ1[[3]] = γ1[[3]] /. {t_a_ :→ h_a, h_a_ :→ t_a};
   γ1];
```

```
tσ[rules___Rule][γ_Γ] := ΓCollect[
   γ /. {t_u_ :> t_(u /. {rules}), T_u_ :> T_(u /. {rules}), b_u_ :> b_(u /. {rules})}
  ];
hσ[rules___Rule][γ_Γ] := ΓCollect[γ /. h_x_ :> h_(x /. {rules})];
```

```
SetAttributes[Γ, Listable];
Γ[p_Times | p_NonCommutativeMultiply] := Γ /@ p;
Γ[ϵ[a_]] := Γ[1, h_a, h_a t_a];
Γ[Xp[a_, b_]] := Γ[1, h_a + h_b T_a, {t_a, t_b}.(1  1 - T_a
                                                 0  T_a    ).{h_a, h_b}];
Γ[Xm[a_, b_]] := Γ[Xp[a, b]] /. T_a → 1/T_a;
```

```
MVA[Γ, L_Link] := Module[{Hs, ω, σ, μ, A},
  {ω, σ, μ} = List @@ Z[Γ, L];
  Hs = Rest[h_# & /@ (First /@ Skeleton[L])];
  A = Outer[Coefficient[μ, #1 * #2] &, Hs, Hs /. h_a_ :> t_a];
  Factor[ω Det[A - IdentityMatrix@Length@Hs] / (1 - T_Skeleton[L][[1,1]])]
 ]
```

Testing

```
Xp_a_,b_ := Xp[a, b]; Xm_a_,b_ := Xm[a, b];
```

```
{Xp_ab, Xm_ab} // Γ
```

$$\left\{\begin{pmatrix} 1 & s_a & s_b \\ s_a & 1 & 1-T_a \\ s_b & 0 & T_a \\ \Gamma & 1 & T_a \end{pmatrix}, \begin{pmatrix} 1 & s_a & s_b \\ s_a & 1 & \frac{-1+T_a}{T_a} \\ s_b & 0 & \frac{1}{T_a} \\ \Gamma & 1 & \frac{1}{T_a} \end{pmatrix}\right\}$$

Meta-Associativity

```
n = 4; γ0 = Γ[ω, Sum[h_a σ_a, {a, 1, n}], Sum[t_a h_b α_(10 a+b), {a, 1, n}, {b, 1, n}]]
```

$$\begin{pmatrix} \omega & s_1 & s_2 & s_3 & s_4 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ s_4 & \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \\ \Gamma & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{pmatrix}$$

```
γ0 // dm[1, 2, 1] // dm[1, 3, 1]
```

$$\begin{pmatrix} \omega\left(1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}\right) & s_1 \\ s_1 & \frac{\alpha_{31}-\alpha_{12}\,\alpha_{31}-\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{23}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{11}\,\alpha_{32}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{21}\,\alpha_{33}-\alpha_{12}\,\alpha_{21}\,\alpha_{33}+\alpha_{11}\,\cdots}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ s_4 & \frac{\alpha_{41}-\alpha_{12}\,\alpha_{41}-\alpha_{13}\,\alpha_{22}\,\alpha_{41}-\alpha_{23}\,\alpha_{41}+\alpha_{12}\,\alpha_{23}\,\alpha_{41}+\alpha_{11}\,\alpha_{42}+\alpha_{13}\,\alpha_{21}\,\alpha_{42}-\alpha_{11}\,\alpha_{23}\,\alpha_{42}+\alpha_{21}\,\alpha_{43}-\alpha_{12}\,\alpha_{21}\,\alpha_{43}+\alpha_{11}\,\cdots}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ \Gamma & \sigma_1\,\sigma_2\,\sigma_3 \end{pmatrix}$$

```
γ0 // dm[2, 3, 2] // dm[1, 2, 1]
```

$$\begin{pmatrix} \omega\left(1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}\right) & s_1 \\ s_1 & \frac{\alpha_{31}-\alpha_{12}\,\alpha_{31}-\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{23}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{11}\,\alpha_{32}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{21}\,\alpha_{33}-\alpha_{12}\,\alpha_{21}\,\alpha_{33}+\alpha_{11}\,\cdots}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ s_4 & \frac{\alpha_{41}-\alpha_{12}\,\alpha_{41}-\alpha_{13}\,\alpha_{22}\,\alpha_{41}-\alpha_{23}\,\alpha_{41}+\alpha_{12}\,\alpha_{23}\,\alpha_{41}+\alpha_{11}\,\alpha_{42}+\alpha_{13}\,\alpha_{21}\,\alpha_{42}-\alpha_{11}\,\alpha_{23}\,\alpha_{42}+\alpha_{21}\,\alpha_{43}-\alpha_{12}\,\alpha_{21}\,\alpha_{43}+\alpha_{11}\,\cdots}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ \Gamma & \sigma_1\,\sigma_2\,\sigma_3 \end{pmatrix}$$

```
(γ0 // dm[1, 2, 1] // dm[1, 3, 1]) == (γ0 // dm[2, 3, 2] // dm[1, 2, 1])
True
```

Cyclicity of tr

```
n = 3; γ0 = Γ[ω, Sum_{a=0}^{n} h_a σ_a, Sum_{a=1}^{n} Sum_{b=1}^{n} t_a h_b α_{10 a+b}]
```

$$\begin{pmatrix} \omega & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \Gamma & \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$$

```
{γ0 // dm[1, 2, 1], γ0 // dm[1, 2, 1] // tr[1]}
```

$$\left\{\begin{pmatrix} -\omega\left(-1+\alpha_{12}\right) & s_1 & s_3 \\ s_1 & \frac{-\alpha_{21}+\alpha_{12}\,\alpha_{21}-\alpha_{11}\,\alpha_{22}}{-1+\alpha_{12}} & \frac{-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}}{-1+\alpha_{12}} \\ s_3 & \frac{-\alpha_{31}+\alpha_{12}\,\alpha_{31}-\alpha_{11}\,\alpha_{32}}{-1+\alpha_{12}} & \frac{-\alpha_{13}\,\alpha_{32}-\alpha_{33}+\alpha_{12}\,\alpha_{33}}{-1+\alpha_{12}} \\ \Gamma & \sigma_1\,\sigma_2 & \sigma_3 \end{pmatrix}, \begin{pmatrix} \omega\left(1-\alpha_{12}-\alpha_{21}+\alpha_{12}\,\alpha_{21}-\alpha_{11}\,\alpha_{22}\right) \\ s_3 & \alpha_{13}\,\alpha_{22}\,\alpha_{31}+\alpha_{23}\,\alpha_{31}-\alpha_{1}\cdots \\ \Gamma \end{pmatrix}\right.$$

```
{γ0 // dm[2, 1, 1], γ0 // dm[2, 1, 1] // tr[1]}
```

$$\left\{\begin{pmatrix} -\omega\left(-1+\alpha_{21}\right) & s_1 & s_3 \\ s_1 & \frac{-\alpha_{12}+\alpha_{12}\,\alpha_{21}-\alpha_{11}\,\alpha_{22}}{-1+\alpha_{21}} & \frac{-\alpha_{13}+\alpha_{13}\,\alpha_{21}-\alpha_{11}\,\alpha_{23}}{-1+\alpha_{21}} \\ s_3 & \frac{-\alpha_{22}\,\alpha_{31}-\alpha_{32}+\alpha_{21}\,\alpha_{32}}{-1+\alpha_{21}} & \frac{-\alpha_{23}\,\alpha_{31}-\alpha_{33}+\alpha_{21}\,\alpha_{33}}{-1+\alpha_{21}} \\ \Gamma & \sigma_1\,\sigma_2 & \sigma_3 \end{pmatrix}, \begin{pmatrix} \omega\left(1-\alpha_{12}-\alpha_{21}+\alpha_{12}\,\alpha_{21}-\alpha_{11}\,\alpha_{22}\right) \\ s_3 & \alpha_{13}\,\alpha_{22}\,\alpha_{31}+\alpha_{23}\,\alpha_{31}-\alpha_{1}\cdots \\ \Gamma \end{pmatrix}\right.$$

```
(γ0 // dm[1, 2, 1] // tr[1]) == (γ0 // dm[2, 1, 1] // tr[1])
True
```

Testing the MVA

```
SXForm[L_] := SXForm[
   Skeleton[L],
   Times @@ PD[L] /.
    X[i_, j_, k_, l_] :→ If[PositiveQ[X[i, j, k, l]], Xp[l, i], Xm[j, i]]
  ];
Z[L_] := Z[Identity, L];
Z[χ_, L_] := Module[{s, z},
   {s, z} = List @@ SXForm[L];
   z = χ[z];
   Do[z = z // dm[s⟦c, 1⟧, s⟦c, k⟧ , s⟦c, 1⟧], {c, Length[s]}, {k, 2, Length[s⟦c⟧]}];
   z
  ];
```

```
Z[Γ, Link["L6a4"]]
```

KnotTheory: Loading precomputed data in PD4Links`.

$$\begin{pmatrix} \frac{(1-T_1-T_5+T_1T_5-T_9+T_1T_9+T_5T_9)(T_1+T_5-T_1T_5+T_9-T_1T_9-T_5T_9+T_1T_5T_9)}{T_1T_5T_9} & s_1 \\ s_1 & \frac{T_9(1-2T_1+T_1^2-T_5+3T_1T_5-T_1^2T_5-T_9+2T_1T_9-T_1^2T_9+T_5T_9-2T_1T_5T_9+T_1^2T_5T_9)}{(1-T_1-T_5+T_1T_5-T_9+T_1T_9+T_5T_9)(T_1+T_5-T_1T_5+T_9-T_1T_9-T_5T_9+T_1T_5T_9)} \\ s_5 & \frac{T_1(-1+T_5)(1-T_1+T_1T_5)(-1+T_9)}{(1-T_1-T_5+T_1T_5-T_9+T_1T_9+T_5T_9)(T_1+T_5-T_1T_5+T_9-T_1T_9-T_5T_9+T_1T_5T_9)} \\ s_9 & \frac{(-1+T_5)T_5(-1+T_9)(1-2T_1-T_9+T_1T_9)}{(1-T_1-T_5+T_1T_5-T_9+T_1T_9+T_5T_9)(T_1+T_5-T_1T_5+T_9-T_1T_9-T_5T_9+T_1T_5T_9)} \\ \Gamma & 1 \end{pmatrix}$$

```
trZ[L_] := Module[{γ},
  γ = Z[Γ, L];
  Do[
   γ = γ // tr[k],
   {k, Most[γ // dL]}];
  γ[ω] / (T_Last[γ//dL] - 1)]
```

```
trZ[Link["L10a4"]]
```

$$\frac{(-1+T_1)(-1+T_5)(1-3T_5+3T_5^2-3T_5^3+T_5^4)}{T_5^2}$$

```
MultivariableAlexander[Link["L10a4"]][T]
```

KnotTheory: Loading precomputed data in MultivariableAlexander4Links`.

$$-\frac{(-1+T[1])(-1+T[2])(1-3T[2]+3T[2]^2-3T[2]^3+T[2]^4)}{\sqrt{T[1]}\,T[2]^{5/2}}$$

```
Factor[1/trZ[#] (MultivariableAlexander[#][T] /. T[i_] :→ T_Skeleton[#]⟦i,1⟧)] & /@
 AllLinks[{2, 7}]
```

$$\Big\{-T_1^2T_3,\ -T_1^{3/2}T_5^{3/2},\ -\sqrt{T_1}\,T_5^{3/2},\ -T_1^{3/2}\sqrt{T_5},\ -T_1^2T_7^2,\ -T_1^2T_7^2,\ \frac{\sqrt{T_1}\sqrt{T_9}}{\sqrt{T_5}},\ T_1^{3/2}T_5^{3/2}T_9^{3/2},\ \frac{\sqrt{T_1}}{\sqrt{T_5}\sqrt{T_9}},$$
$$-\frac{1}{\sqrt{T_1}\sqrt{T_5}},\ -T_1^{3/2}T_5^{7/2},\ -\frac{\sqrt{T_1}}{T_5^{3/2}},\ -\frac{\sqrt{T_1}}{T_5^{3/2}},\ -T_1T_7^2,\ -\frac{1}{T_7},\ \frac{T_1^{3/2}\sqrt{T_5}}{\sqrt{T_9}},\ -\sqrt{T_1}\,T_5^{5/2},\ -\frac{T_5^{3/2}}{\sqrt{T_1}}\Big\}$$