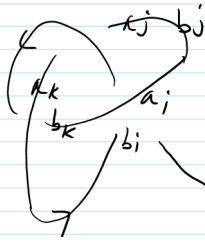


Pensieve header: Implementing  $U(\mathfrak{g}_0)$ .

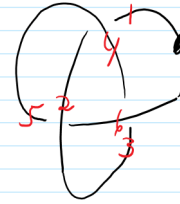
## Reminders



$$R = \sum a_i \otimes b_j \in A \otimes A = U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

$$\text{s.t. } R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}$$

$$\sum_{i,j,k} b_j a_i b_k a_i b_j a_k \in U(\mathfrak{g})$$



PBW:  $\mathfrak{g} = \langle x_1, \dots, x_k \rangle \Rightarrow \{x_1^{a_1} x_2^{a_2} \dots x_k^{a_k} : a_i \in \mathbb{Z}_{\geq 0}\}$  is a basis of  $U(\mathfrak{g})$ .

Today:  $\mathfrak{g}_0 = \langle h, e, l, f \rangle / [e, l] = e \quad [l, f] = f \quad [f, e] = h$

$$r = h \otimes l + f \otimes e \quad R = \exp(r)$$

Note  $U(\mathfrak{g}_0)^{\otimes S} = U(\bigoplus_S \mathfrak{g}_0) = U(\langle h_i, e_{ij}, l_i, f_i \rangle / [e_{ij}, l_j] = f_{ij}, e_i \text{ etc.})$   $h_i$  central

## Global Mathematica Initialization

```
FileNameJoin[{$BaseDirectory, "Kernel", "init.m"}]
```

C:\ProgramData\Mathematica\Kernel\init.m

In the above file, I have (among other things):

```
If[$FrontEnd != Null,
  SetOptions[$FrontEnd,
    InputAliases -> {"Zhe" -> "Ж", "zhe" -> "ж", "<<" -> "<", "<=" -> "<=", ">>" -> ">", ">=" -> ">"}
  ]
]
```

Ж <

Implementing  $\mathfrak{g}_0$ 

```
PBWRule = {e -> 1, l -> 2, f -> 3};
B[U@1, U@e] = - (B[U@e, U@1] = U@e);
B[U@f, U@1] = - (B[U@1, U@f] = U@f);
B[U@e, U@f] = - (B[U@f, U@e] = h U[]);
```

```
B[U@e, U@f]
```

```
-h U[]
```

```
U_i[_e_] := e /. {h -> h_i, t -> t_i, u_U -> Replace[u, x_ -> x_i, 1]};
B[x_, x_] = 0;
B[U[(x_)_i], U[(y_)_i]] := B[U[x_i], U[y_i]] = U_i[B[U@x, U@y]];
B[U[(x_)_i], U[(y_)_j]] /. i != j := 0;
B[x_, y_] := x**y - y**x;
```

$B[U@e_1, U@f_2]$

0

$B[U@e_2, U@f_2]$

$-h_2 U[]$

```
x_ ≤ y_ := OrderedQ[{x, y} /. PBWRule];
x_ < y_ := ! OrderedQ[{y, x} /. PBWRule];
```

$z = (a + b + c)^5$   
 $(a + b + c)^5$

$z // \text{Expand}$

$a^5 + 5 a^4 b + 10 a^3 b^2 + 10 a^2 b^3 + 5 a b^4 + b^5 + 5 a^4 c + 20 a^3 b c + 30 a^2 b^2 c + 20 a b^3 c +$   
 $5 b^4 c + 10 a^3 c^2 + 30 a^2 b c^2 + 30 a b^2 c^2 + 10 b^3 c^2 + 10 a^2 c^3 + 20 a b c^3 + 10 b^2 c^3 + 5 a c^4 + 5 b c^4 + c^5$

$z // \text{Simplify}$

$(a + b + c)^5$

$a^5 + 5 a^4 b + 10 a^3 b^2 + 10 a^2 b^3 + 5 a b^4 + b^5 + 5 a^4 c + 20 a^3 b c + 30 a^2 b^2 c + 20 a b^3 c + 5 b^4 c +$   
 $10 a^3 c^2 + 30 a^2 b c^2 + 30 a b^2 c^2 + 10 b^3 c^2 + 10 a^2 c^3 + 20 a b c^3 + 10 b^2 c^3 + 5 a c^4 + 5 b c^4 + c^5 // \text{Simplify}$

$(a + b + c)^5$

$a^5 + 5 a^4 b + 10 a^3 b^2 + 10 a^2 b^3 + 5 a b^4 + b^5 + 5 a^4 c + 20 a^3 b c + 30 a^2 b^2 c + 20 a b^3 c + 5 b^4 c +$   
 $10 a^3 c^2 + 30 a^2 b c^2 + 30 a b^2 c^2 + 10 b^3 c^2 + 10 a^2 c^3 + 20 a b c^3 + 10 b^2 c^3 + 5 a c^4 + 5 b c^4 + c^5 // \text{Factor}$

$(a + b + c)^5$

$\text{Collect}[z, c]$

$a^5 + 5 a^4 b + 10 a^3 b^2 + 10 a^2 b^3 + 5 a b^4 + b^5 + (5 a^4 + 20 a^3 b + 30 a^2 b^2 + 20 a b^3 + 5 b^4) c +$   
 $(10 a^3 + 30 a^2 b + 30 a b^2 + 10 b^3) c^2 + (10 a^2 + 20 a b + 10 b^2) c^3 + (5 a + 5 b) c^4 + c^5$

$\text{Collect}[z, c, \text{Simplify}]$

$(a + b)^5 + 5 (a + b)^4 c + 10 (a + b)^3 c^2 + 10 (a + b)^2 c^3 + 5 (a + b) c^4 + c^5$

```
Simp[ε_] := Collect[ε, _U, Expand];
```

? NonCommutativeMultiply

$a ** b ** c$  is a general associative, but non-commutative, form of multiplication. >>

$1 ** 3 = 7$

 Set: Tag NonCommutativeMultiply in 1 \*\* 3 is Protected.

7

$(\#^2) \&[7]$

49

$(x \mapsto x^2) [5]$

25

$\lambda[a_ + b_] := \lambda[a] + \lambda[b]$

$\lambda[x + y]$  $\lambda[x] + \lambda[y]$  $f /@ \{1, 2, 3, 4\}$  $\{f[1], f[2], f[3], f[4]\}$  $(x \mapsto x^3) /@ \{1, 2, 3, 4\}$  $\{1, 8, 27, 64\}$  $\mu[z\_Plus] := \mu /@ z$  $\mu[x + y]$  $\mu[x] + \mu[y]$ 

```

Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
0 ** _ = _ ** 0 = 0;
x_ ** U[] := x; U[] ** x_ := x;
(a_ ** x_U) ** (b_ ** y_U) := If[ab === 0, 0, Simp[ab (x ** y)]];
(a_ ** x_U) ** y_ := Simp[a (x ** y)]; x_ ** (a_ ** y_U) := Simp[a (x ** y)];
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;

```

 $(U@x1 + U@y1) ** U@z1$ ... \$IterationLimit: Iteration limit of 4096 exceeded.... \$IterationLimit: Iteration limit of 4096 exceeded. $\text{Hold}[U[x1] ** U[z1]] + \text{Hold}[U[y1] ** U[z1]]$ 

```

U[xx___, x_] ** U[y_, yy___] := If[x ≤ y, U[xx, x, y, yy], U@xx ** (U@y ** U@x + B[U@x, U@y]) ** U@yy];

```

 $U[e_1, l_1, f_1] ** U[e_1, l_1, f_1]$  $U[e_1, e_1, f_1, f_1] + h_1 U[e_1, l_1, l_1, f_1] - 2 U[e_1, e_1, l_1, f_1, f_1] + U[e_1, e_1, l_1, l_1, f_1, f_1]$ 

```

UU[L___, x^n_, r___] := UU[L, Sequence@@Table[x, {n}], r];
UU[L___, 1, r___] := UU[L, r];
UU[] = U[];
UU[L_, r___] := U[L] ** UU[r];

```

 $UU[e_1, l_1, f_1, e_1, l_1, f_1]$  $U[e_1, e_1, f_1, f_1] + h_1 U[e_1, l_1, l_1, f_1] - 2 U[e_1, e_1, l_1, f_1, f_1] + U[e_1, e_1, l_1, l_1, f_1, f_1]$  $UU[e_1, l_1, f_1, e_1^4, l_1, f_1]$ 

$$-12 h_1 U[e_1, e_1, e_1, e_1, l_1, f_1] + 4 U[e_1, e_1, e_1, e_1, e_1, f_1, f_1] +$$

$$4 h_1 U[e_1, e_1, e_1, e_1, l_1, l_1, f_1] - 5 U[e_1, e_1, e_1, e_1, e_1, l_1, f_1, f_1] + U[e_1, e_1, e_1, e_1, e_1, l_1, l_1, f_1, f_1]$$

## Testing g0

```

UProducts[{}, 0] = {UU[]};
UProducts[{}, n_Integer] /; n > 0 = {};
UProducts[{x_, xs___}, n_Integer] :=
  Sort@Flatten@Table[UU[x^k] ** u, {k, 0, n}, {u, UProducts[{xs}, n - k]}];
UProducts[xs_List, k_Integer, n_Integer] := UProducts[Flatten@Table[xj, {x, xs}, {j, k}], n];
UProducts[any___, {n_}] := Flatten@Table[UProducts[any, k], {k, 0, n}];

```

**UProducts**[{e<sub>1</sub>, l<sub>1</sub>, f<sub>1</sub>}, 4]

{U[e<sub>1</sub>, e<sub>1</sub>, e<sub>1</sub>, e<sub>1</sub>], U[e<sub>1</sub>, e<sub>1</sub>, e<sub>1</sub>, f<sub>1</sub>], U[e<sub>1</sub>, e<sub>1</sub>, e<sub>1</sub>, l<sub>1</sub>], U[e<sub>1</sub>, e<sub>1</sub>, f<sub>1</sub>, f<sub>1</sub>], U[e<sub>1</sub>, e<sub>1</sub>, l<sub>1</sub>, f<sub>1</sub>],  
 U[e<sub>1</sub>, e<sub>1</sub>, l<sub>1</sub>, l<sub>1</sub>], U[e<sub>1</sub>, f<sub>1</sub>, f<sub>1</sub>, f<sub>1</sub>], U[e<sub>1</sub>, l<sub>1</sub>, f<sub>1</sub>, f<sub>1</sub>], U[e<sub>1</sub>, l<sub>1</sub>, l<sub>1</sub>, f<sub>1</sub>], U[e<sub>1</sub>, l<sub>1</sub>, l<sub>1</sub>, l<sub>1</sub>],  
 U[f<sub>1</sub>, f<sub>1</sub>, f<sub>1</sub>, f<sub>1</sub>], U[l<sub>1</sub>, f<sub>1</sub>, f<sub>1</sub>, f<sub>1</sub>], U[l<sub>1</sub>, l<sub>1</sub>, f<sub>1</sub>, f<sub>1</sub>], U[l<sub>1</sub>, l<sub>1</sub>, l<sub>1</sub>, f<sub>1</sub>], U[l<sub>1</sub>, l<sub>1</sub>, l<sub>1</sub>, l<sub>1</sub>]}

**B**[U@f<sub>1</sub>, U@e<sub>1</sub>]

h<sub>1</sub> U[]

**UProducts**[{e, l, f}, 2, {3}]

{U[], U[e<sub>1</sub>], U[e<sub>2</sub>], U[f<sub>1</sub>], U[f<sub>2</sub>], U[l<sub>1</sub>], U[l<sub>2</sub>], U[e<sub>1</sub>, e<sub>1</sub>], U[e<sub>1</sub>, e<sub>2</sub>], U[e<sub>1</sub>, f<sub>1</sub>], U[e<sub>1</sub>, f<sub>2</sub>],  
 U[e<sub>1</sub>, l<sub>1</sub>], U[e<sub>1</sub>, l<sub>2</sub>], U[e<sub>2</sub>, e<sub>2</sub>], U[e<sub>2</sub>, f<sub>1</sub>], U[e<sub>2</sub>, f<sub>2</sub>], U[e<sub>2</sub>, l<sub>1</sub>], U[e<sub>2</sub>, l<sub>2</sub>], U[f<sub>1</sub>, f<sub>1</sub>], U[f<sub>1</sub>, f<sub>2</sub>],  
 U[f<sub>2</sub>, f<sub>2</sub>], U[l<sub>1</sub>, f<sub>1</sub>], U[l<sub>1</sub>, f<sub>2</sub>], U[l<sub>1</sub>, l<sub>1</sub>], U[l<sub>1</sub>, l<sub>2</sub>], U[l<sub>2</sub>, f<sub>1</sub>], U[l<sub>2</sub>, f<sub>2</sub>], U[l<sub>2</sub>, l<sub>2</sub>], U[e<sub>1</sub>, e<sub>1</sub>, e<sub>1</sub>],  
 U[e<sub>1</sub>, e<sub>1</sub>, e<sub>2</sub>], U[e<sub>1</sub>, e<sub>1</sub>, f<sub>1</sub>], U[e<sub>1</sub>, e<sub>1</sub>, f<sub>2</sub>], U[e<sub>1</sub>, e<sub>1</sub>, l<sub>1</sub>], U[e<sub>1</sub>, e<sub>1</sub>, l<sub>2</sub>], U[e<sub>1</sub>, e<sub>2</sub>, e<sub>2</sub>],  
 U[e<sub>1</sub>, e<sub>2</sub>, f<sub>1</sub>], U[e<sub>1</sub>, e<sub>2</sub>, f<sub>2</sub>], U[e<sub>1</sub>, e<sub>2</sub>, l<sub>1</sub>], U[e<sub>1</sub>, e<sub>2</sub>, l<sub>2</sub>], U[e<sub>1</sub>, f<sub>1</sub>, f<sub>1</sub>], U[e<sub>1</sub>, f<sub>1</sub>, f<sub>2</sub>], U[e<sub>1</sub>, f<sub>2</sub>, f<sub>2</sub>],  
 U[e<sub>1</sub>, l<sub>1</sub>, f<sub>1</sub>], U[e<sub>1</sub>, l<sub>1</sub>, f<sub>2</sub>], U[e<sub>1</sub>, l<sub>1</sub>, l<sub>1</sub>], U[e<sub>1</sub>, l<sub>1</sub>, l<sub>2</sub>], U[e<sub>1</sub>, l<sub>2</sub>, f<sub>1</sub>], U[e<sub>1</sub>, l<sub>2</sub>, f<sub>2</sub>], U[e<sub>1</sub>, l<sub>2</sub>, l<sub>2</sub>],  
 U[e<sub>2</sub>, e<sub>2</sub>, e<sub>2</sub>], U[e<sub>2</sub>, e<sub>2</sub>, f<sub>1</sub>], U[e<sub>2</sub>, e<sub>2</sub>, f<sub>2</sub>], U[e<sub>2</sub>, e<sub>2</sub>, l<sub>1</sub>], U[e<sub>2</sub>, e<sub>2</sub>, l<sub>2</sub>], U[e<sub>2</sub>, f<sub>1</sub>, f<sub>1</sub>], U[e<sub>2</sub>, f<sub>1</sub>, f<sub>2</sub>],  
 U[e<sub>2</sub>, f<sub>2</sub>, f<sub>2</sub>], U[e<sub>2</sub>, l<sub>1</sub>, f<sub>1</sub>], U[e<sub>2</sub>, l<sub>1</sub>, f<sub>2</sub>], U[e<sub>2</sub>, l<sub>1</sub>, l<sub>1</sub>], U[e<sub>2</sub>, l<sub>1</sub>, l<sub>2</sub>], U[e<sub>2</sub>, l<sub>2</sub>, f<sub>1</sub>], U[e<sub>2</sub>, l<sub>2</sub>, f<sub>2</sub>],  
 U[e<sub>2</sub>, l<sub>2</sub>, l<sub>2</sub>], U[f<sub>1</sub>, f<sub>1</sub>, f<sub>1</sub>], U[f<sub>1</sub>, f<sub>1</sub>, f<sub>2</sub>], U[f<sub>1</sub>, f<sub>2</sub>, f<sub>2</sub>], U[f<sub>2</sub>, f<sub>2</sub>, f<sub>2</sub>], U[l<sub>1</sub>, f<sub>1</sub>, f<sub>1</sub>], U[l<sub>1</sub>, f<sub>1</sub>, f<sub>2</sub>],  
 U[l<sub>1</sub>, f<sub>2</sub>, f<sub>2</sub>], U[l<sub>1</sub>, l<sub>1</sub>, f<sub>1</sub>], U[l<sub>1</sub>, l<sub>1</sub>, f<sub>2</sub>], U[l<sub>1</sub>, l<sub>1</sub>, l<sub>1</sub>], U[l<sub>1</sub>, l<sub>1</sub>, l<sub>2</sub>], U[l<sub>1</sub>, l<sub>2</sub>, f<sub>1</sub>], U[l<sub>1</sub>, l<sub>2</sub>, f<sub>2</sub>],  
 U[l<sub>1</sub>, l<sub>2</sub>, l<sub>2</sub>], U[l<sub>2</sub>, f<sub>1</sub>, f<sub>1</sub>], U[l<sub>2</sub>, f<sub>1</sub>, f<sub>2</sub>], U[l<sub>2</sub>, f<sub>2</sub>, f<sub>2</sub>], U[l<sub>2</sub>, l<sub>2</sub>, f<sub>1</sub>], U[l<sub>2</sub>, l<sub>2</sub>, f<sub>2</sub>], U[l<sub>2</sub>, l<sub>2</sub>, l<sub>2</sub>]}

**bas** = **UProducts** [{e, l, f}, 2, {3}]

{U[], U[e<sub>1</sub>], U[e<sub>2</sub>], U[f<sub>1</sub>], U[f<sub>2</sub>], U[l<sub>1</sub>], U[l<sub>2</sub>], U[e<sub>1</sub>, e<sub>1</sub>], U[e<sub>1</sub>, e<sub>2</sub>], U[e<sub>1</sub>, f<sub>1</sub>], U[e<sub>1</sub>, f<sub>2</sub>],  
 U[e<sub>1</sub>, l<sub>1</sub>], U[e<sub>1</sub>, l<sub>2</sub>], U[e<sub>2</sub>, e<sub>2</sub>], U[e<sub>2</sub>, f<sub>1</sub>], U[e<sub>2</sub>, f<sub>2</sub>], U[e<sub>2</sub>, l<sub>1</sub>], U[e<sub>2</sub>, l<sub>2</sub>], U[f<sub>1</sub>, f<sub>1</sub>], U[f<sub>1</sub>, f<sub>2</sub>],  
 U[f<sub>2</sub>, f<sub>2</sub>], U[l<sub>1</sub>, f<sub>1</sub>], U[l<sub>1</sub>, f<sub>2</sub>], U[l<sub>1</sub>, l<sub>1</sub>], U[l<sub>1</sub>, l<sub>2</sub>], U[l<sub>2</sub>, f<sub>1</sub>], U[l<sub>2</sub>, f<sub>2</sub>], U[l<sub>2</sub>, l<sub>2</sub>], U[e<sub>1</sub>, e<sub>1</sub>, e<sub>1</sub>],  
 U[e<sub>1</sub>, e<sub>1</sub>, e<sub>2</sub>], U[e<sub>1</sub>, e<sub>1</sub>, f<sub>1</sub>], U[e<sub>1</sub>, e<sub>1</sub>, f<sub>2</sub>], U[e<sub>1</sub>, e<sub>1</sub>, l<sub>1</sub>], U[e<sub>1</sub>, e<sub>1</sub>, l<sub>2</sub>], U[e<sub>1</sub>, e<sub>2</sub>, e<sub>2</sub>],  
 U[e<sub>1</sub>, e<sub>2</sub>, f<sub>1</sub>], U[e<sub>1</sub>, e<sub>2</sub>, f<sub>2</sub>], U[e<sub>1</sub>, e<sub>2</sub>, l<sub>1</sub>], U[e<sub>1</sub>, e<sub>2</sub>, l<sub>2</sub>], U[e<sub>1</sub>, f<sub>1</sub>, f<sub>1</sub>], U[e<sub>1</sub>, f<sub>1</sub>, f<sub>2</sub>], U[e<sub>1</sub>, f<sub>2</sub>, f<sub>2</sub>],  
 U[e<sub>1</sub>, l<sub>1</sub>, f<sub>1</sub>], U[e<sub>1</sub>, l<sub>1</sub>, f<sub>2</sub>], U[e<sub>1</sub>, l<sub>1</sub>, l<sub>1</sub>], U[e<sub>1</sub>, l<sub>1</sub>, l<sub>2</sub>], U[e<sub>1</sub>, l<sub>2</sub>, f<sub>1</sub>], U[e<sub>1</sub>, l<sub>2</sub>, f<sub>2</sub>], U[e<sub>1</sub>, l<sub>2</sub>, l<sub>2</sub>],  
 U[e<sub>2</sub>, e<sub>2</sub>, e<sub>2</sub>], U[e<sub>2</sub>, e<sub>2</sub>, f<sub>1</sub>], U[e<sub>2</sub>, e<sub>2</sub>, f<sub>2</sub>], U[e<sub>2</sub>, e<sub>2</sub>, l<sub>1</sub>], U[e<sub>2</sub>, e<sub>2</sub>, l<sub>2</sub>], U[e<sub>2</sub>, f<sub>1</sub>, f<sub>1</sub>], U[e<sub>2</sub>, f<sub>1</sub>, f<sub>2</sub>],  
 U[e<sub>2</sub>, f<sub>2</sub>, f<sub>2</sub>], U[e<sub>2</sub>, l<sub>1</sub>, f<sub>1</sub>], U[e<sub>2</sub>, l<sub>1</sub>, f<sub>2</sub>], U[e<sub>2</sub>, l<sub>1</sub>, l<sub>1</sub>], U[e<sub>2</sub>, l<sub>1</sub>, l<sub>2</sub>], U[e<sub>2</sub>, l<sub>2</sub>, f<sub>1</sub>], U[e<sub>2</sub>, l<sub>2</sub>, f<sub>2</sub>],  
 U[e<sub>2</sub>, l<sub>2</sub>, l<sub>2</sub>], U[f<sub>1</sub>, f<sub>1</sub>, f<sub>1</sub>], U[f<sub>1</sub>, f<sub>1</sub>, f<sub>2</sub>], U[f<sub>1</sub>, f<sub>2</sub>, f<sub>2</sub>], U[f<sub>2</sub>, f<sub>2</sub>, f<sub>2</sub>], U[l<sub>1</sub>, f<sub>1</sub>, f<sub>1</sub>], U[l<sub>1</sub>, f<sub>1</sub>, f<sub>2</sub>],  
 U[l<sub>1</sub>, f<sub>2</sub>, f<sub>2</sub>], U[l<sub>1</sub>, l<sub>1</sub>, f<sub>1</sub>], U[l<sub>1</sub>, l<sub>1</sub>, f<sub>2</sub>], U[l<sub>1</sub>, l<sub>1</sub>, l<sub>1</sub>], U[l<sub>1</sub>, l<sub>1</sub>, l<sub>2</sub>], U[l<sub>1</sub>, l<sub>2</sub>, f<sub>1</sub>], U[l<sub>1</sub>, l<sub>2</sub>, f<sub>2</sub>],  
 U[l<sub>1</sub>, l<sub>2</sub>, l<sub>2</sub>], U[l<sub>2</sub>, f<sub>1</sub>, f<sub>1</sub>], U[l<sub>2</sub>, f<sub>1</sub>, f<sub>2</sub>], U[l<sub>2</sub>, f<sub>2</sub>, f<sub>2</sub>], U[l<sub>2</sub>, l<sub>2</sub>, f<sub>1</sub>], U[l<sub>2</sub>, l<sub>2</sub>, f<sub>2</sub>], U[l<sub>2</sub>, l<sub>2</sub>, l<sub>2</sub>]}

**bas** = **UProducts** [{e, l, f}, 2, {3}];

**Table**[**B**[x, y] + **B**[y, x], {x, **bas**}, {y, **bas**}] // **Flatten** // **Union**

{0}

**bas** = **UProducts** [{e, l, f}, 2, {2}];

**Table**[

{x, y, z} = xyz;

**Simp**[**B**[**B**[x, y], z] + **B**[**B**[y, z], x] + **B**[**B**[z, x], y]],

{xyz, **Subsets**[**bas**, {3}]}]

// **Flatten** // **Union**

**bas** = **UProducts** [{e, l, f}, 2, {2}];

**Table**[

{x, y, z} = xyz;

**Simp**[x\*\* (y\*\* z) - (x\*\* y)\*\* z],

{xyz, **Subsets**[**bas**, {3}]}]

// **Flatten** // **Union**

{0}

## Testing CYBE

$$r_{i,j} := h_i \mathbb{U}[l_j] + \mathbb{U}[f_i, e_j]$$

$r_{1,2}$

$$h_1 \mathbb{U}[l_2] + \mathbb{U}[e_2, f_1]$$

$B[r_{1,2}, r_{1,3}]$

0

$B[r_{1,3}, r_{2,3}]$

$$h_2 \mathbb{U}[e_3, f_1] - h_1 \mathbb{U}[e_3, f_2]$$

$B[r_{1,2}, r_{2,3}]$

$$-h_2 \mathbb{U}[e_3, f_1] + h_1 \mathbb{U}[e_3, f_2]$$

$B[r_{1,2}, r_{1,3}] + B[r_{1,3}, r_{2,3}] + B[r_{1,2}, r_{2,3}]$

0

## Testing YBE

```

UExp[n_Integer, u_] := Module[{t},
  t = U[];
  Simp[t + Sum[ $\frac{t = t ** u}{k!}$ , {k, n}]]
];
R_{i,j}[n_] := UExp[n, r_{i,j}];

```

$UExp[5, U@e_1]$

$$U[] + U[e_1] + \frac{1}{2} U[e_1, e_1] + \frac{1}{6} U[e_1, e_1, e_1] + \frac{1}{24} U[e_1, e_1, e_1, e_1] + \frac{1}{120} U[e_1, e_1, e_1, e_1, e_1]$$

$R_{1,2}[4]$

$$\begin{aligned}
&U[] + h_1 U[l_2] + \left(1 - \frac{h_1}{2} + \frac{h_1^2}{6} - \frac{h_1^3}{24}\right) U[e_2, f_1] + \frac{1}{2} h_1^2 U[l_2, l_2] + \left(h_1 - \frac{h_1^2}{2} + \frac{h_1^3}{6}\right) U[e_2, l_2, f_1] + \\
&\frac{1}{6} h_1^3 U[l_2, l_2, l_2] + \left(\frac{1}{2} - \frac{h_1}{2} + \frac{7h_1^2}{24}\right) U[e_2, e_2, f_1, f_1] + \left(\frac{h_1^2}{2} - \frac{h_1^3}{4}\right) U[e_2, l_2, l_2, f_1] + \frac{1}{24} h_1^4 U[l_2, l_2, l_2, l_2] + \\
&\left(\frac{h_1}{2} - \frac{h_1^2}{2}\right) U[e_2, e_2, l_2, f_1, f_1] + \frac{1}{6} h_1^3 U[e_2, l_2, l_2, l_2, f_1] + \left(\frac{1}{6} - \frac{h_1}{4}\right) U[e_2, e_2, e_2, f_1, f_1, f_1] + \\
&\frac{1}{4} h_1^2 U[e_2, e_2, l_2, l_2, f_1, f_1] + \frac{1}{6} h_1 U[e_2, e_2, e_2, l_2, f_1, f_1, f_1] + \frac{1}{24} U[e_2, e_2, e_2, e_2, f_1, f_1, f_1, f_1]
\end{aligned}$$

With[{n = 2}, Simp[R\_{1,2}[n] \*\* R\_{1,3}[n] \*\* R\_{2,3}[n] - R\_{2,3}[n] \*\* R\_{1,3}[n] \*\* R\_{1,2}[n]]] // Short

$$\left(-\frac{1}{2} h_1^2 h_2 + \frac{1}{4} h_1^2 h_2^2\right) U[<<1>>] + <<131>> + <<1>>$$

```

ToDegree[n_][e_] :=
  Simp[e /. {h_i -> hbar h_i, u_U -> hbar^Count[u,f] u}] /. a_. x_U -> Normal[Series[a, {hbar, 0, n}]] * x /. hbar -> 1

```

```
With[{n = 2}, Simp[R1,2[n] ** R1,3[n] ** R2,3[n] - R2,3[n] ** R1,3[n] ** R1,2[n]] // ToDegree[n + 2]]
- 1/2 h12 h2 U[e3, l2, f1] + 1/2 h13 U[e3, l2, f2] + 1/2 h12 h2 U[e3, l3, f1] - 1/2 h13 U[e3, l3, f2] - h1 h2 U[e2, e3, f1, f1] +
h12 U[e2, e3, f1, f2] + h1 h2 U[e3, e3, f1, f1] - h12 U[e3, e3, f1, f2] + 1/2 h12 h2 U[e3, l2, l2, f1] -
1/2 h13 U[e3, l2, l2, f2] - 1/2 h12 h2 U[e3, l3, l3, f1] + 1/2 h13 U[e3, l3, l3, f2] + h1 h2 U[e2, e3, l2, f1, f1] -
h12 U[e2, e3, l2, f1, f2] - h1 h2 U[e3, e3, l3, f1, f1] + h12 U[e3, e3, l3, f1, f2] + 1/2 h2 U[e2, e2, e3, f1, f1, f1] -
1/2 h1 U[e2, e2, e3, f1, f1, f2] - 1/2 h2 U[e3, e3, e3, f1, f1, f1] + 1/2 h1 U[e3, e3, e3, f1, f1, f2]
```

```
With[{n = 3}, Simp[R1,2[n] ** R1,3[n] ** R2,3[n] - R2,3[n] ** R1,3[n] ** R1,2[n]] // ToDegree[n]]
```

```
0
```

## The “Internal Multiplication” and Meta-Associativity

```
m[i_, j_, k_][e_] := Simp[e /. {
  u_ U => UU @@ Join[DeleteCases[u, x_i|j], U @@ Cases[u, x_i => xk], U @@ Cases[u, x_j => xk]],
  hi|j -> hk
}]
```

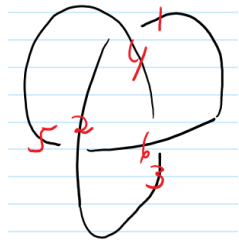
```
UU[e1, l4, f2]
```

```
UU[e1, l4, f2] // m[1, 2, 3]
```

```
UU[e1, l4, f2] // m[2, 1, 3]
```

```
Union@Table[
  (u // m[1, 2, 1] // m[1, 3, 1]) - (u // m[2, 3, 2] // m[1, 2, 1]),
  {u, UProducts[{e, l, f}, 4, {3}]}
]
```

## The Invariant of the Trefoil



```
With[{n = 2}, R4,1[n] ** R2,5[n] ** R6,3[n] // ToDegree[n]]
```

```
With[{n = 2},
```

```
R4,1[n] ** R2,5[n] ** R6,3[n] // ToDegree[n] // m[1, 2, 1] // m[1, 3, 1] // m[1, 4, 1] // m[1, 5, 1] //
m[1, 6, 1]
```

```
With[{n = 3},
```

```
R4,1[n] ** R2,5[n] ** R6,3[n] // ToDegree[n] // m[1, 2, 1] // m[1, 3, 1] // m[1, 4, 1] // m[1, 5, 1] //
m[1, 6, 1]
```

## Ordering Symbols

```
o[n_, poly_, specs___] := Module[{vs, us},  
  vs = Join@@(First /@ {specs});  
  us = Join@@({specs} /. (l_ -> s_) -> (l /. x_i_ -> x_s));  
  Total[CoefficientRules[Normal@Series[poly, {h, 0, n}], vs] /. (p_ -> c_) -> c UU@@(us^p)]]
```

TimeUsed[]