Dror Bar-Natan: Classes: 2017: MAT 1350:
Algebraic Knot Theory — Poly-Time Computations:

About This Class

```
Dear All,
   This email is to shamelessly market to you the
graduate class I will be teaching this semester,
"MAT 1350 Algebraic Knot Theory - Polynomial Time
Computations".
   Over the last couple of years it emerged that
there exists a class of very strong knot invariants
that can be computed in polynomial-time (better
than the reigning champs, which are all exponential
time) - meaning that in principle these invariants
can be computed even for very very large knots.
Furthermore, there is interesting and promising
topology behind this class of invariants, and some
novel algebra, especially around Lie algebras and
their universal enveloping algebras. There are
also many directions to explore. [Yet unfortunately,
at least for now, some of this material is *hard*.]
   My class will be an introduction to the topic,
by means of pushing it further. If we work hard
during the semester, we may, just may, be able to
push things on from sl(2) to sl(3), thus turning
the invariant much stronger (even if a bit less
computable). It will be truly wonderful if we
succeed. Though even if we fail, we will learn a
great deal about knots and tangles and virtual
knots and virtual tangles and expansions and Lie
algebras and Lie bialgebras and about sophisticated
computations using Mathematica.
   The prerequisites are mathematical maturity and
no fear of computers, total comfort with linear
algebra: vector spaces, duals, quotients, tensor
products, etc., and some appreciation of Lie algebras.
   Sincerely, Dror Bar-Natan.
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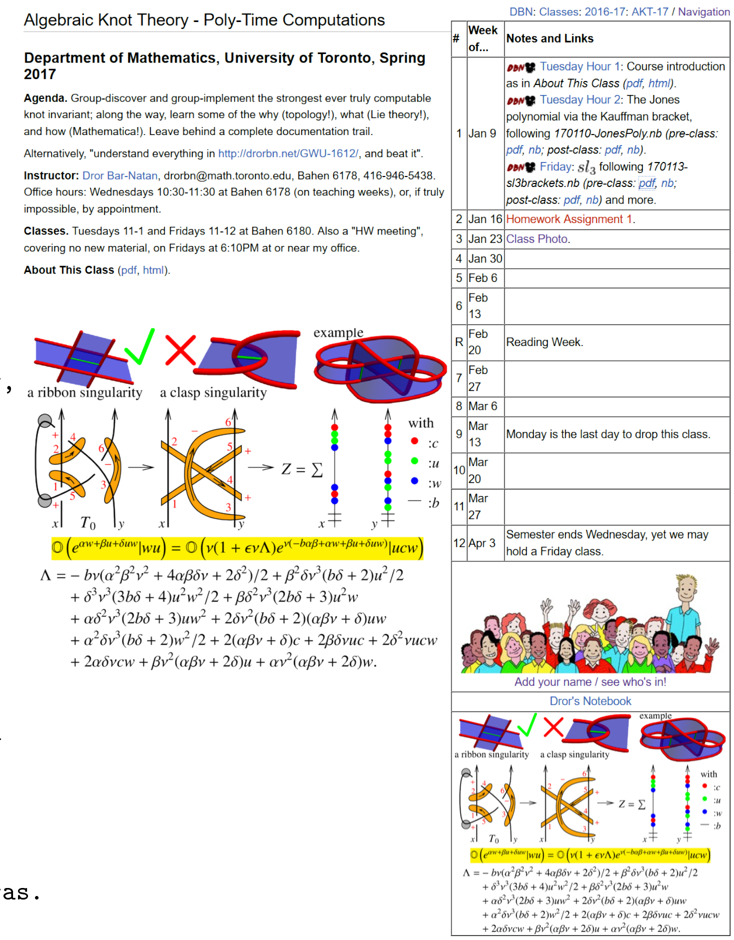
Shameless Marketing.

Web page @ http://drorbn.net/AKT-17:



DBN: Classes: 2016-17: AKT-17 / Navigation

Algebraic Knot Theory - Poly-Time Computations

Department of Mathematics, University of Toronto, Spring 2017

Agenda. Group-discover and group-implement the strongest ever truly computable knot invariant; along the way, learn some of the why (topology!), what (Lie theory!), and how (Mathematica!). Leave behind a complete documentation trail.

Alternatively, "understand everything in http://drorbn.net/GWU-1612/, and beat it".

Instructor: Dror Bar-Natan, drorbn@math.toronto.edu, Bahen 6178, 416-946-5438. Office hours: Wednesdays 10:30-11:30 at Bahen 6178 (on teaching weeks), or, if truly impossible, by appointment.

Classes. Tuesdays 11-1 and Fridays 11-12 at Bahen 6180. Also a "HW meeting", covering no new material, on Fridays at 6:10PM at or near my office.

About This Class (pdf, html).

$$\mathbb{O}\left(e^{\alpha w+\beta u+\delta uw}|wu\right) = \mathbb{O}\left(\nu(1+\epsilon\nu\Lambda)e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|ucw\right)$$

$$\begin{aligned}\Lambda = &-b\nu(\alpha^2\beta^2\nu^2+4\alpha\beta\delta\nu+2\delta^2)/2+\beta^2\delta\nu^3(b\delta+2)u^2/2\\ &+\delta^3\nu^3(3b\delta+4)u^2w^2/2+\beta\delta^2\nu^3(2b\delta+3)u^2w\\ &+\alpha\delta^2\nu^3(2b\delta+3)uw^2+2\delta\nu^2(b\delta+2)(\alpha\beta\nu+\delta)uw\\ &+\alpha^2\delta\nu^3(b\delta+2)w^2/2+2(\alpha\beta\nu+\delta)c+2\beta\delta\nu uc+2\delta^2\nu ucw\\ &+2\alpha\delta\nu cw+\beta\nu^2(\alpha\beta\nu+2\delta)u+\alpha\nu^2(\alpha\beta\nu+2\delta)w.\end{aligned}$$

#	Week of...	Notes and Links
1	Jan 9	Tuesday Hour 1: Course introduction as in *About This Class (pdf, html)*. Tuesday Hour 2: The Jones polynomial via the Kauffman bracket, following *170110-JonesPoly.nb (pre-class: pdf, nb; post-class: pdf, nb)*. Friday: sl_3 following *170113-sl3brackets.nb (pre-class: pdf, nb; post-class: pdf, nb)* and more.
2	Jan 16	Homework Assignment 1.
3	Jan 23	Class Photo.
4	Jan 30	
5	Feb 6	
6	Feb 13	
R	Feb 20	Reading Week.
7	Feb 27	
8	Mar 6	
9	Mar 13	Monday is the last day to drop this class.
10	Mar 20	
11	Mar 27	
12	Apr 3	Semester ends Wednesday, yet we may hold a Friday class.

Add your name / see who's in!

Dror's Notebook

Classes. Tuesdays 11-1 and Fridays 11-12 at Bahen 6180. There will also be a "HW meeting", covering no new material, on Fridays at 6:10PM at or near my office.

Instructor. Dror Bar-Natan, drorbn@math.toronto.edu, http://www.math.toronto.edu/~drorbn/, Bahen 6178, 416-946-5438. Office hours: See website, or by appointment, or by opportunity.

Agenda. Group-discover and group-implement the strongest ever truly computable knot invariant. Along the way, learn some of the why (topology!), what (Lie theory!), and how (Mathematica!). Leave behind a complete documentation trail.

Alternatively, "understand everything in http://drorbn.net/dbnvp/GWU-1612.php (video there, handout attached), and beat it".

Plan. Yes, I mean it. I plan to do deep and real and significant research as a part of this class; the sketch I already know, and the details and the implementation we will work out. We will split our time at a roughly 1:1 ratio between background material (Tuesdays and later Fridays), and shortest and most direct&brute path to discovery (Fridays and later Tuesdays).

Warning. By the nature of these plans, this will be a hard class, and likely messy, and possibly we will fail to achieve the main goal. Yet even then we will learn a great deal.

Optimistic tentative "Background Material" Plan. Course introduction (hour 1). Knots, Reidemeister moves and the Jones polynomial (h2). Tangles and a faster Jones program (h3). The Alexander polynomial as a determinant and using Γ-calculus (h4-5). Seifert surfaces and genus, ribbon knots and "algebraic knot theory" (h6-7). Finite type invariants and expansions (h8-9). The relationship with metrized Lie algebras and PBW (h10-11). The variants v, w, bv, and rv, and their expansions (h12-14). Lie bialgebras and solvable approximation (h15-16). Odds and ends (h17-18).

Optimistic tentative "Direct&Brute Path" Plan. The Lie algebras sl_2, sl_3, $\mathfrak{g}_0$ and $\mathfrak{g}_1$, universal enveloping algebras and low degree computations (hours 1-2). Algebras, Yang-Baxter elements, and invariants (h3). Ordering symbols and commutation relations for $\mathfrak{g}_0$ (h4-5). The $\mathfrak{g}_0$ invariant (h6). The Λόγος and $\mathfrak{g}_1$ computations (h7-8). Morse knots and the $\mathfrak{g}_1$ invariant (h9). $\mathfrak{g}_0$ and $\mathfrak{g}_1$ as approximations of sl_2, approximating sl_3 (h10). The sl_3^0 invariant (h10-11). The sl_3^1 invariant, fame, and glory (h12-18).

Prerequisites. Mathematical maturity and no fear of computers. Total comfort with linear algebra: vector spaces, duals, quotients, tensor products, etc. Some appreciation of Lie algebras.

Mathematica. Almost everything we say, we will immediately implement using Mathematica[1]. If you are seriously taking this class, you **must** have a working copy on your computer. If you don't already have one, "Mathematica Student Edition" (web-search that) is CDN$150. Save the receipt! I will make every effort to refund this expense for students officially taking this class.

Class Photo. To help me learn your names, I will take a class photo on Wednesday of the third week of classes. I will post the picture on the class' web site and you will be encouraged to identify yourself on the Class Photo page (`http://drorbn.net/?title=AKT-17/Class_Photo`) of the class' wiki.

Videos and Wiki. We will videotape all classes and the course's web site will be centered around these videos. I have set up a system (see below) that allows anyone signed-up to index and annotate these videos on a wiki, and allows for the inclusion and linking of other pages and further material to this wiki.

Anyone signed-up can, is welcome and is encouraged to edit and add to the class' web site. In particular, students can post video annotations, notes, comments, pictures, solution to open problems, whatever. Some rules, though —

- This wiki is a part of my (Dror's) academic web page. All postings on it must be class-related (or related to one of the other projects I'm involved with).
- I (Dror) will allow myself to exercise editorial control, when necessary.
- The titles of all pages related to this class should contain and preferably begin with the string "`AKT-17`", just like the title the classes' main page.
- For most `AKT-17` pages, it is a good idea to put a line containing only the string `{{AKT-17/Navigation}}` at the top of the page. This template inserts the class' "navigation panel" at the top right of the page.
- To edit the navigation panel itself, click on the word "Navigation" on the upper right of the panel. Use caution! Such edits affect many other pages!
- Neatness matters! Material that is posted in an appealing manner will be read more, and thus will be more useful.
- Some further editing help is available at `http://drorbn.net/?title=Help:Contents`.

Wiki Sign-Up. Email me with you full name, email address and preferred userid if you need an account on the class wiki.

Homework assignments will usually be jointly written, usually on the Friday HW meetings, usually they will be assigned on Mondays, and usually be due on the following Monday. There will be about 11 assignments; your HW mark will be the average of your best 6 assignments. Late assignments will be marked down by 1% per day.

Student Presentations. During class I will occasionally suggest topics for student presentations. Typically these will involve reading some research papers and lecturing to class about them a week or two after the end of formal classes.

Final Exam. If people will so prefer, there will also be a Final Examination.

Good deeds. You will be able to earn "good deed" points throughout the term. You may earn up to 90 good deed points for removing one of the major roadblocks we will encounter, up to 80 good deed points for writing a book-quality open-source and copyleft exposition of a significant and deep portion of this class. More easily, for lively participation in and markup of the class wiki, you may receive up to about 40 good deed points.

Semi-Final grade. The higher of 70% HW and 30% Presentation Mark or Final Examination, or 20% HW and 80% Presentation Mark or Final Examination.

Final Grade. If you earn $0 \leq \gamma \leq 90$ good deed points during the term, and your semi-final grade is σ, your final grade will be $\gamma + (100-\gamma)\sigma/100$. This can be 100 even with $\gamma = 0$, yet with $\gamma = 90$, it will be 95 even with $\sigma = 50$.

Dror's Open Notebook for this class is at `http://drorbn.net/AcademicPensieve/Classes/17-1350-AKT/`. Use at your own risk.

last modified January 9, 2017, 9:28pm

[1] **Q.** Can I use language X instead of Mathemtica? **A.** Theoretically, you could and it would be a wonderful contribution if you did. In practice you'd be taking something very difficult and turning it into nearly impossible. So the honest answer is "No".

How to use this site

January-18-12
8:59 PM

Click to read my copyleft policy

Click to go to the main WKO paper page

download the video file for offline viewing

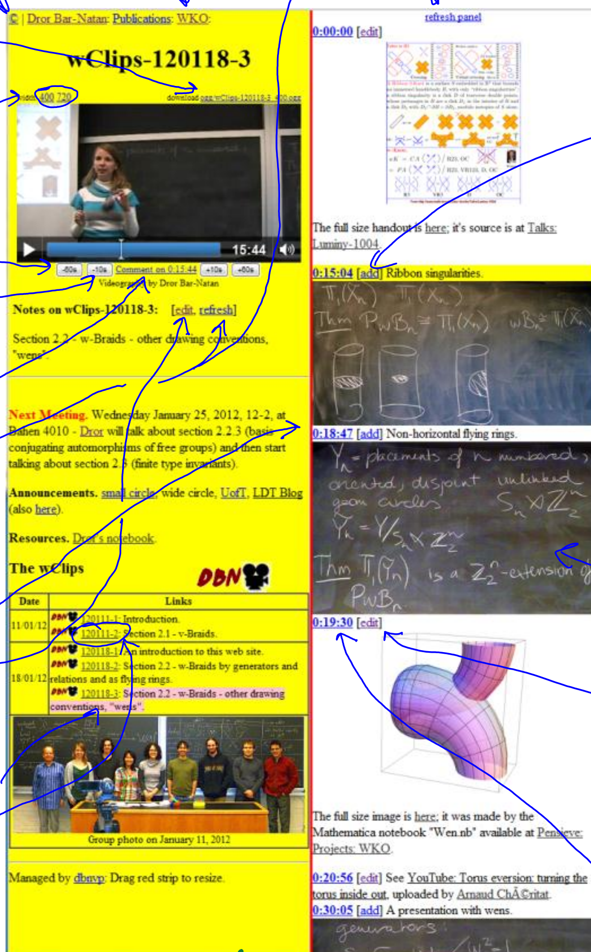
Change the resolution of this video

click to jump 60 second back or 10

Click to add a wiki comment on minute 0:15:44 of this video. Then hit "refresh" and your new comment will appear on the right panel, about here

Edit the title of this specific video; it will be pinked-out in the table of content.

Click to jump to the "section 2.1" video.



Left panel Right panel

drag red line left or right to change panel sizes.

click "add" to add a wiki comment on minute 15:04

This blackboard shot was taken at minute 15:04 of the talk. This more or less means that the discussion of its content ended at that minute.

Click the blackboard shot to enlarge it (opens in another tab)

click "edit" to edit the wiki comment for minute 19:30

click the 19:30 link to jump to minute 19:30 of the video.

Wiki: To add/modify wiki comments you must have an account on Dror's wiki. Email drorbn@math.toronto.edu to ask for one, if you don't already have one.

Dror Bar-Natan: Talks: GWU-1612:

On Elves and Invariants

Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen.

ωεβ:=http://drorbn.net/GWU-1612/

Work in Progress! Fluid! Help Needed!

Abstract. Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

Three steps to the computation of ρ_1:

1. Preparation. Given K, results

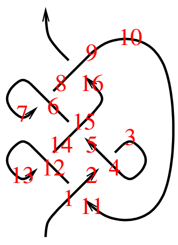
$\langle$*long word* $\|$ *simple formulas*$\rangle$.

2. Rewrite rules. Make the *word* simpler and the *formulas* more complicated, until the *word* "*elf*" is reached.

3. Readout. The invariant ρ_1 is read from the last *formulas*.

$$\text{Knot } K \;\xrightarrow{\text{preparation}}\; \langle elf\ldots elf \,\|\, \omega_0; L_0; Q_0; P_0\rangle \;\xrightarrow{\text{rewrite rules}}\; \langle elf \,\|\, \omega; -; -; P\rangle \;\xrightarrow{\text{readout}}\; \rho_1(K) = \rho_1(\omega, P)$$

Preparation. Draw K using a 0-framed 0-rotation planar diagram D where all crossings are pointing up. Walk along D labeling features by $1, \ldots, m$ in order: over-passes, under-passes, and right-heading cups and caps ("±-cuaps"). If x is a xing, let i_x and j_x be the labels on its over/under strands, and let s_x be 0 if it right-handed and -1 otherwise. If c is a cuap, let i_c be its label and s_c be its sign. Set



$$(L; Q; P) = \sum_{x:\,(i,j,s)} (-)^s \left(l_j;\ t^s e_i f_j;\ (-t)^s e_i l_{(1+s)i-sj} f_j + l_i l_j + \frac{t^{2s} e_i^2 f_j^2}{4} \right) + \sum_{c:\,(i,s)} (0;\ 0;\ s \cdot l_i)\,.$$

This done, output $\langle e_1 l_1 f_1 e_2 l_2 f_2 \cdots e_m l_m f_m \,\|\, 1; L; Q; P\rangle$.

In *formulas*, L is always $\mathbb{Z}$-linear in $\{l_i\}$, Q is an R-linear combination of $\{e_i f_j\}$ where $R := \mathbb{Q}[t^{\pm 1}]$, and P is an R-linear combination of $\{1,\ l_i,\ l_i l_j,\ e_i f_j,\ e_i l_j f_k,\ e_i e_j f_k f_l\}$. (The key to computability!)

Rewrite Rules. Manipulate $\langle$*word* $\|$ *formulas*$\rangle$ expressions using the rewrite rules below, until you come to the form $\langle e_1 l_1 f_1 \,\|\, \omega; -; -; P\rangle$. Output (ω, P).

Rule 1, Deletions. If a letter appears in *word* but not in *formulas*, you can delete it.

Rule 2, Merges. In *word*, you can replace *adjacent* $v_i v_j$ with v_k (for $v \in \{e, l, f\}$) while making the same changes in *formulas* (provided k creates no naming clashes). E.g.,

$$\langle \ldots e_i e_j \ldots \,\|\, Z\rangle \to \langle \ldots e_k \ldots \,\|\, Z|_{e_i, e_j \to e_k}\rangle.$$

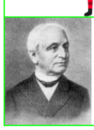
Rule 3, *le* Sorts. Provided k introduces no clashes, given $\langle \ldots l_j e_i \ldots \,\|\, \omega; L; Q; P\rangle$, decompose $L = \lambda l_j + L'$, $Q = \alpha e_i + Q'$, write $P = P(e_i, l_j)$ (with messy coefficients), set $q = \mathbb{e}^\gamma \beta e_k + \gamma l_k$, and output

$$\left\langle \ldots e_k l_k \ldots \,\|\, \omega; L|_{l_j \to l_k}; t^\lambda \alpha e_k + Q'; \mathbb{e}^{-q} P(\partial_\beta, \partial_\gamma) \mathbb{e}^q |_{\beta \to \alpha/\omega,\, \gamma \to \lambda \log t} \right\rangle.$$

Rule 4, *fl* Sorts. Provided k introduces no clashes, given $\langle \ldots f_i l_j \ldots \,\|\, \omega; L; Q; P\rangle$, decompose $L = \lambda l_j + L'$, $Q = \alpha f_i + Q'$, write $P = P(f_i, l_j)$ (with messy coefficients), set $q = \mathbb{e}^\gamma \beta f_k + \gamma l_k$, and output

$$\left\langle \ldots l_k f_k \ldots \,\|\, \omega; L|_{l_j \to l_k}; t^\lambda \alpha f_k + Q'; \mathbb{e}^{-q} P(\partial_\beta, \partial_\gamma) \mathbb{e}^q |_{\beta \to \alpha/\omega,\, \gamma \to \lambda \log t} \right\rangle.$$

Happy Birthday, Scott!



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

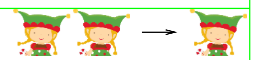
Rule 5, *fe* Sorts. Provided k introduces no clashes, given $\langle \ldots f_i e_j \ldots \,\|\, \omega; L; Q; P\rangle$, decompose $Q = Q_{fe} f_i e_j + Q_f f_i + Q_e e_j + Q'$ write $P = P(f_i, e_j)$ (with messy coefficients), set $\mu = 1 + (t-1)\delta$ and $q = ((1-t)\alpha\beta + \beta e_k + \alpha f_k + \delta e_k f_k)/\mu$, and output

$$\left\langle \ldots e_k f_k \ldots \left\| \begin{matrix} \mu\omega; L;\ \mu\omega q + \mu Q'; \\ \omega^4 \Lambda_k + \mathbb{e}^{-q} P(\partial_\alpha, \partial_\beta)(\mathbb{e}^q) \end{matrix} \right. \right\rangle \Bigg|_{\substack{\alpha \to Q_f/\omega,\, \beta \to Q_e/\omega, \\ \delta \to Q_{fe}/\omega}},$$

where Λ_k is the Λόγος, "a principle of order and knowledge":

$$\begin{aligned} \Lambda_k = \frac{t+1}{4} \Big(& -\delta(\mu+1)\left(\beta^2 e_k^2 + \alpha^2 f_k^2\right) - \delta^3(3\mu+1) e_k^2 f_k^2 \\ & - 2\left(\beta e_k + \alpha f_k\right)\left(\alpha\beta + 2\delta\mu + \delta^2(2\mu+1) e_k f_k + 2\delta\mu^2 l_k\right) \\ & - 4(\alpha\beta + \delta\mu)\left(\delta(\mu+1) e_k f_k + \mu^2 l_k\right) - 4\delta^2\mu^2 e_k f_k l_k \\ & + (t-1)\left(2(\alpha\beta + \delta\mu)^2 - \alpha^2\beta^2\right) \Big). \end{aligned}$$

elf merges, m_k^{ij}, are defined as compositions



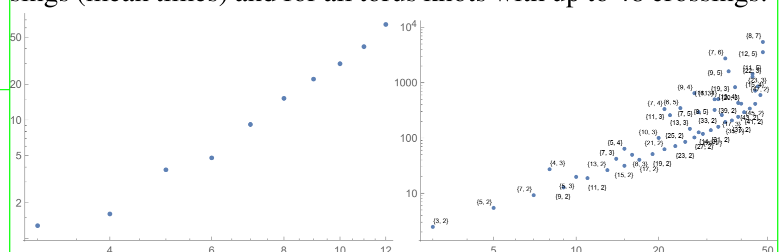
$$e_i l_i \overline{f_i e_j} l_j f_j \xrightarrow{S_x^{f_i e_j}} e_i \overline{l_i e_x}\, \overline{f_x l_j} f_j \xrightarrow{S_x^{l_i e_x} /\!/ S_x^{f_x l_j}} \overline{e_i e_x}\, \overline{l_x l_x}\, \overline{f_x f_j} \xrightarrow{i,j,x \to k} e_k l_k f_k$$

Readout. Given $\langle elf \,\|\, \omega; -; -; P\rangle$, output

$$\rho_1(K) := \frac{t(P|_{e,l,f \to 0} - t\omega'\omega^3)}{(t-1)^2\omega^2}.$$

(ω is the Alexander polynomial, L and Q are not interesting).

Experimental Analysis (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Power. On the 250 knots with at most 10 crossings, the pair (ω, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 xings, always ρ_1 is symmetric under $t \leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1, always $\deg \rho_1^+ \leq 2g - 1$, where g is the 3-genus of K (equallity for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

Why Works? The Lie algebra $\mathfrak{g}_1$ (below) is a "solvable approximation of sl_2".

Theorem. The map (as defined below)

$$\langle w \,\|\, \omega; L; Q; P\rangle \mapsto \mathbb{O}\left(\omega^{-1} \mathbb{e}^{L \log t + \omega^{-1} Q} (1 + \epsilon\omega^{-4} P):\ w\right) \in \hat{\mathcal{U}}(\mathfrak{g}_1)$$

is well defined modulo the sorting rules. It maps the initial preparation to a product of "R-matrices" and "cuap values" satisfying the usual moves for Morse knots (R3, etc.). (And hence the result is a "quantum invariant", except computed very differently; no representation theory!).

1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle h, e', l, f\rangle$ over the ring $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with h central and with $[f, l] = f$, $[e', l] = -e'$, and $[e', f] = h - 2\epsilon l$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1 \supset \langle h, e', f, \epsilon h, \epsilon e', \epsilon l, \epsilon f\rangle \supset \langle h, \epsilon h, \epsilon e', \epsilon l, \epsilon f\rangle \supset 0$. Pragmatics: declare $\deg(h, e', l, f, \epsilon) = (1, 1, 0, 0, 1)$ and set $t := \mathbb{e}^h$ and $e := (t-1)e'/h$.

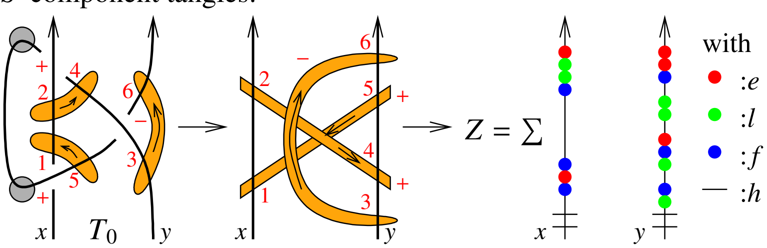
How did it arise? $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^-/\mathfrak{h} =: sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+ = \langle l, f\rangle/[f, l] = f$ is a Lie bialgebra with $\delta\colon \mathfrak{b}^+ \to \mathfrak{b}^+ \otimes \mathfrak{b}^+$ by $\delta\colon (l, f) \mapsto (0, l \wedge f)$. Going back, $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle h', e', l, f\rangle/\cdots$. **Idea.** Replace $\delta \to \epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$. At $k = 1$, get $[f, l] = f$, $[f, h'] = -\epsilon f$, $[l, e'] = e'$, $[h', e'] = -\epsilon e'$, $[h', l] = 0$, and $[e', f] = h' - \epsilon l$. Now note that $h' + \epsilon l$ is central, so switch to $h := h' + \epsilon l$. This is $\mathfrak{g}_1$.

Ordering Symbols. $\mathbb{O}(poly \mid specs)$ plants the variables of *poly* in $\hat{\mathcal{S}}(\oplus_i \mathfrak{g})$ along $\hat{\mathcal{U}}(\mathfrak{g})$ according to *specs*. E.g.,

$$\mathbb{O}\left(e_1 \mathbb{e}^{e_3} l_1^3 l_2 f_3^9 \mid f_3 l_1 e_1 e_3 l_2\right) = f^9 l^3 e \mathbb{e}^e l \in \hat{\mathcal{U}}(\mathfrak{g}).$$

This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})$ using commutative polynomials / power series. In $\mathfrak{g}_1$, no need to specify h / t.

Algebras and Invariants. Given any unital algebra A (even better if A is Hopf; typically, $A \sim \hat{\mathcal{U}}(\mathfrak{g})$), appropriate orange $R \in A \otimes A$, and appropriate cuaps $\in A$, get an $A^{\otimes S}$-valued invariant of pure S-component tangles:



What we didn't say (more, including videos, in ωεβ/Talks).

- ρ_1 is "line" in the coloured Jones polynomial; related to Melvin-Morton-Rozansky.
- ρ_1 extends to "rotational virtual tangles" and is a projection of the universal finite type invariant of such.
- ρ_1 seems to have a better chance than anything else we know to detect a counterexample to slice=ribbon.
- ρ_1 leads to many questions and a very long to-do list. Years of work, many papers ahead. Have fun!

Demo Programs. ωεβ/Demo

Formatting (prints differ ☹)

```
CF[ℰ_] := Module[{vars = Union@Cases[ℰ, e_ | l_ | f_, ∞]},
   If[vars === {}, Factor[ℰ],
    Total[CoefficientRules[ℰ, vars] /.
      (p_ → c_) :→ Factor[c] Times @@ (vars^p)] ]];
CF[ℰ_𝔼] := CF /@ ℰ;
```

Preparation

```
𝔼[i_, j_, s_] := 𝔼[1, (-1)^s l_j, (-t)^s e_i f_j,
   t^s e_i l_((1+s) i-s j) f_j + (-1)^s l_i l_j + (-t^2)^s e_i^2 f_j^2/4];
𝔼[i_, s_] := 𝔼[1, 0, 0, s l_i];
𝔼 /: 𝔼[1, L1_, Q1_, P1_] 𝔼[1, L2_, Q2_, P2_] :=
  𝔼[1, L1 + L2, Q1 + Q2, P1 + P2];
```

Preparing the Trefoil

```
z1 = (𝔼[1, 11, 0] 𝔼[4, 2, -1] 𝔼[15, 5, 0]
   𝔼[6, 8, -1] 𝔼[9, 16, 0] 𝔼[12, 14, -1] 𝔼[3, -1] 𝔼[7, +1]
   𝔼[10, -1] 𝔼[13, +1])
```

$$\mathbb{E}\Big[1,\ -l_2 + l_5 - l_8 + l_{11} - l_{14} + l_{16},$$
$$-\frac{e_4 f_2}{t} + e_{15} f_5 - \frac{e_6 f_8}{t} + e_1 f_{11} - \frac{e_{12} f_{14}}{t} + e_9 f_{16},$$
$$-\frac{e_4^2 f_2^2}{4t^2} + \frac{1}{4} e_{15}^2 f_5^2 - \frac{e_6^2 f_8^2}{4t^2} + \frac{1}{4} e_1^2 f_{11}^2 - \frac{e_{12}^2 f_{14}^2}{4t^2} + \frac{1}{4} e_9^2 f_{16}^2 + e_1 f_{11} l_1 +$$
$$\frac{e_4 f_2 l_2}{t} - l_3 - l_2 l_4 + l_7 + \frac{e_6 f_8 l_8}{t} - l_6 l_8 + e_9 f_{16} l_9 - l_{10} +$$
$$l_1 l_{11} + l_{13} + \frac{e_{12} f_{14} l_{14}}{t} - l_{12} l_{14} + e_{15} f_5 l_{15} + l_5 l_{15} + l_9 l_{16}\Big]$$

Differential Polynomials (Implementing $P(\partial_\alpha, \partial_\beta)(f)$)

```
DP_(x_→D_α_, y_→D_β_)[P_][f_] :=
 Total[CoefficientRules[P, {x, y}] /.
   ({m_, n_} → c_) :→ c D[f, {α, m}, {β, n}]]
```

***le* and *fl* Sorts**

```
S_(l_j_ (x:e|f)_i_ →k_)[𝔼[ω_, L_, Q_, P_]] :=
 With[{λ = ∂_(l_j) L, α = ∂_(x_i) Q, q = 𝕖^γ β x_k + γ l_k}, CF[
   𝔼[ω, L /. l_j → l_k, t^λ α x_k + (Q /. x_i → 0),
    𝕖^-q DP_(l_j→D_γ, x_i→D_β)[P][𝕖^q] /. {β → α / ω, γ → λ Log[t]}] ]];
```

The Λόγος

```
Λ[k_] := ((t - 1) (2 (α β + δ μ)^2 - α^2 β^2) - 4 e_k l_k f_k δ^2 μ^2 -
    δ (1 + μ) (f_k^2 α^2 + e_k^2 β^2) - e_k^2 f_k^2 δ^3 (1 + 3 μ) -
    2 (α β + 2 δ μ + e_k f_k δ^2 (1 + 2 μ) + 2 l_k δ μ^2) (f_k α + e_k β) -
    4 (l_k μ^2 + e_k f_k δ (1 + μ)) (α β + δ μ)) (1 + t) / 4;
```

***fe* Sorts**

```
S_(f_i_ e_j_ →k_)[𝔼[ω_, L_, Q_, P_]] :=
 With[{q = ((1 - t) α β + β e_k + α f_k + δ e_k f_k) / μ}, CF[
   𝔼[μ ω, L, μ ω q + μ (Q /. f_i | e_j → 0),
     μ^4 𝕖^-q DP_(f_i→D_α, e_j→D_β)[P][𝕖^q] + ω^4 Λ[k]] /. μ → 1 + (t - 1) δ /.
   {α → ω^-1 (∂_(f_i) Q /. e_j → 0), β → ω^-1 (∂_(e_j) Q /. f_i → 0),
    δ → ω^-1 ∂_(f_i,e_j) Q}]];
```

Elf Merges

```
m_(i_,j_→k_)[Z_𝔼] := Module[{x, z},
  CF[(Z // S_(f_i e_j→x) // S_(l_i e_x→x) // S_(f_x l_j→x)) /. z_i|j|x → z_k]]
```

Rewriting the Trefoil (by merging 16 elves)

```
(Do[z1 = z1 // m_(1,k→1), {k, 2, 16}]; z1)
```

$$\mathbb{E}\Big[\frac{1-t+t^2}{t},\ 0,\ 0,\ \frac{(-1+t)\left(1-t+t^2\right)^2\left(1-t+2t^2\right)}{t^3} - \frac{2(1+t)\left(1-t+t^2\right)^3 e_1 f_1}{t^4} - \frac{2(-1+t)(1+t)\left(1-t+t^2\right)^3 l_1}{t^4}\Big]$$

Readout

```
ρ1[𝔼[ω_, _, _, P_]] := CF[t ((P /. e_ | l_ | f_ → 0) - t ω^3 (∂_t ω)) / ((t - 1)^2 ω^2)]
```

$\rho_1(3_1)$

```
ρ1[z1] // Expand
```

$$\frac{1}{t} + t$$

References.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* University of North Carolina PhD thesis, ωεβ/Ov.

[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal $U(1)$-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

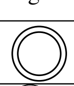
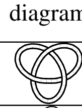
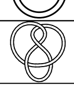
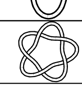
diagram	n_k^t / Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon; unknotting number / amphicheiral
	0_1^a / 0	1	0 / ✔; 0 / ✔
	3_1^a / t	$t-1$	1 / ✘; 1 / ✘
	4_1^a / 0	$3-t$	1 / ✘; 1 / ✔
	5_1^a / $2t^3+3t$	t^2-t+1	2 / ✘; 2 / ✘

diagram	n_k^t Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon unknotting number / amphicheiral	diagram	n_k^t Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon unknotting number / amphicheiral
	5_2^a	$2t-3$ $5t-4$	1 / ✘ 1 / ✘		6_1^a	$5-2t$ $t-4$	1 / ✔ 1 / ✘
	6_2^a	$-t^2+3t-3$ t^3-4t^2+4t-4	2 / ✘ 1 / ✘		6_3^a	t^2-3t+5 0	2 / ✘ 1 / ✔
	7_1^a	t^3-t^2+t-1 $3t^5+5t^3+6t$	3 / ✘ 3 / ✘		7_2^a	$3t-5$ $14t-16$	1 / ✘ 1 / ✘
	7_3^a	$2t^2-3t+3$ $-9t^3+8t^2-16t+12$	2 / ✘ 2 / ✘		7_4^a	$4t-7$ $32-24t$	1 / ✘ 2 / ✘
	7_5^a	$2t^2-4t+5$ $9t^3-16t^2+29t-28$	2 / ✘ 2 / ✘		7_6^a	$-t^2+5t-7$ $t^3-8t^2+19t-20$	2 / ✘ 1 / ✘
	7_7^a	t^2-5t+9 $8-3t$	2 / ✘ 1 / ✘		8_1^a	$7-3t$ $5t-16$	1 / ✘ 1 / ✘
	8_2^a	$-t^3+3t^2-3t+3$ $2t^5-8t^4+10t^3-12t^2+13t-12$	3 / ✘ 2 / ✘		8_3^a	$9-4t$ 0	1 / ✘ 2 / ✔
	8_4^a	$-2t^2+5t-5$ $3t^3-8t^2+6t-4$	2 / ✘ 2 / ✘		8_5^a	$-t^3+3t^2-4t+5$ $-2t^5+8t^4-13t^3+20t^2-22t+24$	3 / ✘ 2 / ✘
	8_6^a	$-2t^2+6t-7$ $5t^3-20t^2+28t-32$	2 / ✘ 2 / ✘		8_7^a	t^3-3t^2+5t-5 $-t^5+4t^4-10t^3+12t^2-13t+12$	3 / ✘ 1 / ✘
	8_8^a	$2t^2-6t+9$ $-t^3+4t^2-12t+16$	2 / ✔ 2 / ✘		8_9^a	$-t^3+3t^2-5t+7$ 0	3 / ✔ 1 / ✔
	8_{10}^a	t^3-3t^2+6t-7 $-t^5+4t^4-11t^3+16t^2-21t+20$	3 / ✘ 2 / ✘		8_{11}^a	$-2t^2+7t-9$ $5t^3-24t^2+39t-44$	2 / ✘ 1 / ✘
	8_{12}^a	$t^2-7t+13$ 0	2 / ✘ 2 / ✔		8_{13}^a	$2t^2-7t+11$ $-t^3+4t^2-14t+20$	2 / ✘ 1 / ✘
	8_{14}^a	$-2t^2+8t-11$ $5t^3-28t^2+57t-68$	2 / ✘ 1 / ✘		8_{15}^a	$3t^2-8t+11$ $21t^3-64t^2+120t-140$	2 / ✘ 2 / ✘
	8_{16}^a	t^3-4t^2+8t-9 $t^5-6t^4+17t^3-28t^2+35t-36$	3 / ✘ 2 / ✘		8_{17}^a	$-t^3+4t^2-8t+11$ 0	3 / ✘ 1 / ✔
	8_{18}^a	$-t^3+5t^2-10t+13$ 0	3 / ✘ 2 / ✔		8_{19}^n	t^3-t^2+1 $-3t^5-4t^2-3t$	3 / ✘ 3 / ✘
	8_{20}^n	t^2-2t+3 $4t-4$	2 / ✔ 1 / ✘		8_{21}^n	$-t^2+4t-5$ $t^3-8t^2+16t-20$	2 / ✘ 1 / ✘
	9_1^a	$t^4-t^3+t^2-t+1$ $4t^7+7t^5+9t^3+10t$	4 / ✘ 4 / ✘		9_2^a	$4t-7$ $30t-40$	1 / ✘ 1 / ✘
	9_3^a	$2t^3-3t^2+3t-3$ $-13t^5+12t^4-25t^3+20t^2-32t+24$	3 / ✘ 3 / ✘		9_4^a	$3t^2-5t+5$ $23t^3-28t^2+46t-44$	2 / ✘ 2 / ✘
	9_5^a	$6t-11$ $100-65t$	1 / ✘ 2 / ✘		9_6^a	$2t^3-4t^2+5t-5$ $13t^5-24t^4+45t^3-52t^2+68t-64$	3 / ✘ 3 / ✘
	9_7^a	$3t^2-7t+9$ $23t^3-56t^2+99t-108$	2 / ✘ 2 / ✘		9_8^a	$-2t^2+8t-11$ $3t^3-16t^2+29t-28$	2 / ✘ 2 / ✘
	9_9^a	$2t^3-4t^2+6t-7$ $13t^5-24t^4+55t^3-72t^2+98t-96$	3 / ✘ 3 / ✘		9_{10}^a	$4t^2-8t+9$ $-40t^3+72t^2-114t+120$	2 / ✘ 2, 3 / ✘
	9_{11}^a	$-t^3+5t^2-7t+7$ $-2t^5+16t^4-41t^3+52t^2-66t+64$	3 / ✘ 2 / ✘		9_{12}^a	$-2t^2+9t-13$ $5t^3-36t^2+84t-100$	2 / ✘ 1 / ✘
	9_{13}^a	$4t^2-9t+11$ $-40t^3+92t^2-154t+168$	2 / ✘ 2, 3 / ✘		9_{14}^a	$2t^2-9t+15$ $-t^3+8t^2-35t+60$	2 / ✘ 1 / ✘
	9_{15}^a	$-2t^2+10t-15$ $-5t^3+40t^2-108t+136$	2 / ✘ 2 / ✘		9_{16}^a	$2t^3-5t^2+8t-9$ $-13t^5+36t^4-80t^3+120t^2-161t+168$	3 / ✘ 3 / ✘
	9_{17}^a	t^3-5t^2+9t-9 $t^5-8t^4+23t^3-32t^2+28t-24$	3 / ✘ 2 / ✘		9_{18}^a	$4t^2-10t+13$ $40t^3-108t^2+193t-220$	2 / ✘ 2 / ✘
	9_{19}^a	$2t^2-10t+17$ $t^3-8t^2+20t-24$	2 / ✘ 1 / ✘		9_{20}^a	$-t^3+5t^2-9t+11$ $2t^5-16t^4+47t^3-84t^2+117t-124$	3 / ✘ 2 / ✘
	9_{21}^a	$-2t^2+11t-17$ $-5t^3+44t^2-127t+164$	2 / ✘ 1 / ✘		9_{22}^a	$t^3-5t^2+10t-11$ $-t^5+8t^4-24t^3+38t^2-40t+36$	3 / ✘ 1 / ✘
	9_{23}^a	$4t^2-11t+15$ $40t^3-128t^2+243t-288$	2 / ✘ 2 / ✘		9_{24}^a	$-t^3+5t^2-10t+13$ $-4t^2+16t-20$	3 / ✘ 1 / ✘
	9_{25}^a	$-3t^2+12t-17$ $12t^3-70t^2+153t-188$	2 / ✘ 2 / ✘		9_{26}^a	$t^3-5t^2+11t-13$ $-t^5+8t^4-31t^3+64t^2-85t+92$	3 / ✘ 1 / ✘
	9_{27}^a	$-t^3+5t^2-11t+15$ $t^3-8t^2+24t-32$	3 / ✔ 1 / ✘		9_{28}^a	$t^3-5t^2+12t-15$ $t^5-8t^4+30t^3-68t^2+105t-120$	3 / ✘ 1 / ✘
	9_{29}^a	$t^3-5t^2+12t-15$ $t^5-8t^4+26t^3-48t^2+59t-56$	3 / ✘ 2 / ✘		9_{30}^a	$-t^3+5t^2-12t+17$ $2t^3-10t^2+25t-32$	3 / ✘ 1 / ✘