Dror Bar-Natan: Classes: 2017: MAT 1350 AKT:
Handout for January 31, 2017:

Γ-Calculus

Derived from Cheat Sheet Meta-Calculi, in http://drorbn.net/AcademicPensieve/Projects/MetaCalculi/.

σ-calculus. $\sigma_1 * \sigma_2 = \sigma_1 \cup \sigma_2$, $\quad m^{ab}_c(\sigma) = (\sigma \backslash \{a,b\}) \cup (c \to \sigma_a\sigma_b) \,/\, (T_a, T_b \to T_c)$, $\quad \text{tr}_c(\sigma) = \sigma \backslash c$, $\quad R^\pm_{ab} \mapsto (a \to 1,\, b \to T_a^{\pm 1})$

Gassner calculus / Γ-calculus. Preserves $C_1 :=$ [col sum = 1] ($\Leftrightarrow$OC) and ✓ $C_2 := [\forall a, b,\ (T_a - 1) \mid (A_{ab} - \delta_{ab}\sigma_b)]$

• Except under tr_c, at $T_* = 1$, $\omega = 1$ and $A = I$.

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[\substack{\mu := 1-\beta \\ T_a, T_b \to T_c}]{m^{ab}_c} \begin{array}{c|cc} \mu\omega & c & S \\ \hline c & \gamma + \alpha\delta/\mu & \epsilon + \delta\theta/\mu \\ S & \phi + \alpha\psi/\mu & \Xi + \psi\theta/\mu \end{array} \qquad \begin{array}{c|cc} \omega & c & S \\ \hline c & \alpha & \theta \\ S & \psi & \Xi \end{array} \xrightarrow[\mu := 1-\alpha]{\text{tr}_c} \begin{array}{c|c} \mu\omega & S \\ \hline S & \Xi + \psi\theta/\mu \end{array} \qquad R^\pm_{ab} \underset{\Gamma}{=} \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1 - T_a^{\pm 1} \\ b & 0 & T_a^{\pm 1} \end{array}$$

$$\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array} \xrightarrow[\substack{\mu := T_a - 1 \\ \nu := \alpha - \sigma_a}]{\Delta^a_{bc}} \left(\begin{array}{c|ccc} \omega & b & c & S \\ \hline b & (\sigma_a - \alpha T_a - \nu T_c)/\mu & (T_b - 1)T_c\nu/\mu & (T_b - 1)T_c\theta/\mu \\ c & (T_c - 1)\nu/\mu & (\alpha - \sigma_a T_a - \nu T_c)/\mu & (T_c - 1)\theta/\mu \\ S & \phi & \phi & \Xi \end{array} \right)_{T_a \mapsto T_b T_c}$$

Satisfies: ✓ $R^+_{13} /\!\!/ \Delta^1_{12} = R^+_{23} \# R^+_{13}$.
✓ $R^-_{13} /\!\!/ \Delta^1_{12} = R^-_{13} \# R^-_{23}$.
✓ $\Delta^a_{a_1a_2} /\!\!/ \Delta^b_{b_1b_2} /\!\!/ m^{a_1b_1}_{c_1} /\!\!/ m^{a_2b_2}_{c_2} = m^{ab}_c /\!\!/ \Delta^c_{c_1c_2}$.

$$\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array} \xrightarrow{S^a} \left(\begin{array}{c|cc} \alpha\omega/\sigma_a & a & S \\ \hline a & 1/\alpha & \theta/\alpha \\ S & -\phi/\alpha & (\alpha\Xi - \phi\theta)/\alpha \end{array} \right)_{T_a \to T_a^{-1}}$$

Satisfies: ✓ $R^\pm_{12} /\!\!/ S^{1 \text{ or } 2} = R^\mp_{12}$. ✓ $m^{ab}_c /\!\!/ S^c = S^a /\!\!/ S^b /\!\!/ m^{ba}_c$.
✓ $S^a /\!\!/ S^a = I$. ✓ $\Delta^a_{bc} /\!\!/ S^b /\!\!/ S^c = S^a /\!\!/ \Delta^a_{cb}$.
✓ Assuming C_2, $\eta^a /\!\!/ \epsilon_a = \Delta^a_{bc} /\!\!/ S^c /\!\!/ m^{bc}_a$ (also 3 variants).

The map (tangle $T \mapsto$ matrix A) is anti-multiplicative. The MVA mod units: $L \mapsto (\omega, A) \mapsto \omega \det'(A - I)/(1 - T')$ ✓
