1617-257 Wed Mar 8, hour 60: More on d

February 15, 2017 12:58 PM

A word on intuitive / Familiar.

HW16 due. Read along: Sec 30-32.
TT: Tue March 14 5PM-7PM @ EX 300. Extra OH: Dror Mon March 13 5-8PM BA 6178, Jeff Tue March 14 11-2 Huron 215 10th floor. } on board

Approximate details - same as before:

- Material: Everything from last TT / HW11 / Sec 22 until Friday, roughly proportional to time spent + around 20% from older material.
- Roughly choose 4/5, some questions multi-part.
- About 1/3 "prove as in class", 1/3 "solve as in HW", 1/3 "solve fresh".

} write as I speak

Reminders: $\Omega^0\mathbb{R}^3 \xrightarrow{d} \Omega^1\mathbb{R}^3 \xrightarrow{d} \Omega^2\mathbb{R}^3 \xrightarrow{d} \Omega^3\mathbb{R}^3$

$\quad f. \xrightarrow{grad} v.f. \xrightarrow{curl} v.f \xrightarrow{div} f$

Theorem. $\exists!$ linear operator $d: \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^{k+1}(\mathbb{R}^n)$ s.t.

1. If f is a 0-form, $df(\xi) = D_\xi f$. $\left[\text{so } df = \sum \frac{\partial f}{\partial x_i} dx_i\right]$
2. $\omega \in \Omega^k, \eta \in \Omega^l \Rightarrow d(\omega\wedge\eta) = (d\omega)\wedge\eta + (-1)^k \omega\wedge d\eta$
3. $d^2 = 0$; more precisely, $d(d\omega) = 0$.

Pf 1-3 imply uniqueness

$$d(\sum \underbrace{a_I}_{\omega} dx_I) = \sum_{j=1}^n \sum_I \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I \overset{\text{loosely}}{=} \sum_{j=1}^n dx_j \wedge \frac{\partial \omega}{\partial x_j}$$

done the

Finish proof of Thm.

Then: if $\phi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, have $\phi^*: \Omega^k\mathbb{R}^m \longrightarrow \Omega^k\mathbb{R}^n$

Claims 1. ϕ^* is linear

2. $(\phi\circ\psi)^* = \psi^*\circ\phi^*$
3. $\phi^*(\omega\wedge\eta) = \phi^*(\omega)\wedge\phi^*(\eta)$

} old

4. $\phi^*(d\omega) = d\phi^*(\omega)$ } new. } not proven.

Example $\mathbb{R}^2_{r,\theta} \xrightarrow[\binom{r\cos\theta}{r\sin\theta}]{\phi} \mathbb{R}^2_{x,y}$ $\quad \omega = \frac{x\,dy - y\,dx}{x^2+y^2} \in \Omega^2(\mathbb{R}^2_{x,y})$. } only $\phi^*\omega$ computed.

Compute $\underbrace{\phi^*(d\omega)}_{\text{second}}$ & $\underbrace{d\phi^*\omega}_{\text{first}}$

Pf of 4 For Functions

$$d(\phi^* f)(\xi) = D_\xi \phi^* f$$
$$\phi^*(df)(\xi) = (df)(\phi_*\xi) = D_{\phi_*\xi} f$$

For general forms:

$$d(\phi^* \sum a_I dx_I) = d(\sum \phi^* a_I\, \phi^*(dx_{i_1})\wedge\phi^*(dx_{i_2})\dots)$$
$$= d\left(\sum (\phi^* a_I) \bigwedge_{\alpha=1}^k \phi^* dx_{i_\alpha}\right) = \sum d\left(\phi^* a_I \bigwedge_{\alpha=1}^k d\phi^* x_{i_\alpha}\right)$$
$$= \sum d\phi^* a_I \wedge \bigwedge_{\alpha=1}^k d\phi^* x_{i_\alpha}$$

$\phi^*(d\sum a_I dx_I) = \phi^*(\sum da_I \wedge \bigwedge_{\alpha=1}^k dx_{i_\alpha}) = \sum \phi^*(da_I \wedge \bigwedge_{\alpha=1}^k dx_{i_\alpha})$

<u>Claim</u> If $\phi: \mathbb{R}^n_{x_i} \to \mathbb{R}^n_{y_j}$ & $\omega = f\, dy_{\underline{n}} \in \Omega^{top}(\mathbb{R}^n)$, then $\phi^*(\omega) = \det(D\phi)\cdot \phi^* f \cdot dx_{\underline{n}}$

<u>proof</u> use $\psi_I(x_1 \dots x_k) = \det(X_I)$, w/ X_I = rows I of $X = (x_1|\dots|x_k)$ from a while ago:

$\Rightarrow dx_{\underline{n}}(v_1 \dots v_n) = \det(v_1|\dots|v_n)$ so

$\phi^*(\omega)(e_1 \dots e_n) = \omega(\phi_* e_1 \dots \phi_* e_n) = f(x)\cdot \det(D\phi)$. This is also the r.h.s.

<u>Claim</u> If $\phi: \mathbb{R}^n_{x_i} \to \mathbb{R}^m_{y_j}$ & $\omega = \sum a_I dy_I \in \Omega^k(\mathbb{R}^m)$, then

$$\phi^*(\omega) = \sum_{I \in \binom{m}{k}} \sum_{J \in \binom{n}{k}} \phi^*(a_I)\cdot \det(D\phi(x)_{J,I})\cdot dx_J$$

The J rows & I col's of $D\phi(x)$.