1617-257 Wed Mar 29, hour 68: Stokes' theorem in low dimensions

February 15, 2017 12:58 PM

Read along: Sec 37-38. Eval rate: 28/93. HW18 due; HW19 (last!) on web by midnight.
Makeup class tomorrow at 5PM at MP 134, video will be available

Thm If M^k is compact and oriented and $\omega\in\Omega^{k-1}(M)$, then $\int_M d\omega=\int_{\partial M}\omega$

"de-Rham complex" $\quad \Omega^0(\mathbb{R}^3)\xrightarrow{d}\Omega^1(\mathbb{R}^3)\xrightarrow{d}\Omega^2(\mathbb{R}^3)\longrightarrow\Omega^3(\mathbb{R}^3)$

$\{f\}\qquad \sum a_i dx_i \qquad b_1 dx_2\wedge dx_3 + \text{cyc perms} \qquad c\,dx_3$

$\updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow$

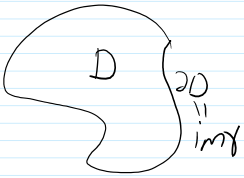
vector calculus $\quad \{f\}\xrightarrow{\text{grad}}\left\{\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\right\}\xrightarrow{\text{curl}}\left\{\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}\right\}\xrightarrow{\text{div}}\{c\}$

leave full BB length on board

Example. $M=[0,1]_x$, $\omega=f \rightsquigarrow \int_0^1 f'=f(1)-f(0)$

Example

$M=D\subset\mathbb{R}^2 \qquad \omega=P\,dx+Q\,dy \qquad d\omega=\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\wedge dy$



$$\int_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=\int_{\partial D}P\,dx+Q\,dy=\int_a^b\left(P(\gamma(t))\dot\gamma_1+Q(\gamma(t))\dot\gamma_2\right)dt$$

$\gamma:[a,b]\to\mathbb{R}^2=\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}$ "Green's theorem"

Let $F=\begin{pmatrix}Q\\-P\end{pmatrix}$; get $\int_D \operatorname{div}(F)=\int_a^b F_1\dot\gamma_2-F_2\dot\gamma_1=\int_a^b\begin{pmatrix}F_1\\F_2\end{pmatrix}\cdot\begin{pmatrix}\dot\gamma_2\\-\dot\gamma_1\end{pmatrix}=$

$=\int F\cdot\begin{pmatrix}0&1\\-1&0\end{pmatrix}\dot\gamma=\int F\cdot\vec n$ The divergence theorem.

Example: Use for $\omega=\frac12(x\,dy-y\,dx)$

Example: The Planimeter.

Continue in $\mathbb{R}^3$:

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$\{f\}\qquad \sum a_i dx_i \qquad b_1 dx_2\wedge dx_3 + \text{cyc perms} \qquad c\,dx_3$

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vector calculus $\quad \{f\}\xrightarrow{\text{grad}}\left\{\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\right\}\xrightarrow{\text{curl}}\left\{\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}\right\}\xrightarrow{\text{div}}\{c\}$

manifolds $\quad$ points w/signs $\cup(s_i,p_i)$ $\quad$ curves $\Gamma=\operatorname{im}(\gamma:[0,1]\to\mathbb{R}^3)$ $\quad$ surfaces $S=\operatorname{im}(\sigma:D^2\to\mathbb{R}^3)$ $\quad$ solids/domains $D\subset\mathbb{R}^3$

done line

integration $\quad \int_M\omega \qquad \sum s_i f(p_i) \qquad \int_0^1\vec a\cdot\dot\gamma=\int_\Gamma \vec a\cdot\vec T\,dV \qquad \int_S b\cdot\vec n\,dV \qquad \int_D c$

Stokes: $\qquad$ Stokes: $\qquad$ Stokes:

Stokes:

$$\int_0^1 (\operatorname{grad} f)\cdot\dot\gamma = f(\gamma(1)) - f(\gamma(0))$$

Stoke's:

$$\int_S (\operatorname{curl} a)\cdot\vec{n}\,dV = \int_{\partial S} \vec{a}\cdot\vec{T}\,dV$$

Stokes:

$$\int_D \operatorname{div} b = \int_{\partial D} b\cdot\vec{n}\,dV$$

Aside

$$\sigma^*(b_1 dx_{23} + \text{cyc perms}) = \Big(b_1 dx_{23}\big(\underbrace{\sigma_* e_1}_{\substack{\partial_x\sigma_1\\ \partial_x\sigma_2\\ \partial_x\sigma_3}}, \underbrace{\sigma_* e_2}_{\substack{\partial_y\sigma_1\\ \partial_y\sigma_2\\ \partial_y\sigma_3}}\big) + cp\Big)\,dx\wedge dy$$

$$= \big(b_1(\partial_x\sigma_2 - \partial_y\sigma_3) + cp\big)\,dx\wedge dy = b\cdot\big[(\partial_x\sigma)\times(\partial_y\sigma)\big]\,dx\wedge dy = b\cdot\vec{n}\cdot\operatorname{Vol}(\partial_x\sigma, \partial_y\sigma)$$